Quasi-polynomials, asymptotics and $[Q, R] = 0$

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CMS Summer meeting, Regina, June 2019

Quasi-polynomials

Γ a lattice in a finite dimensional $\mathbb{R}$ vector space W (lattice basis $\rightsquigarrow \mathbb{Z}^d \subset \mathbb{R}^d$)

Definition

$q\colon \Gamma \to \mathbb{C}$ is *quasi-polynomial* if there is a sublattice Γ' with Γ/Γ' finite such that restriction of q to each coset of Γ' is polynomial. (More generally allow q to be only partially defined.)

$$q(\gamma) = \sum_g g^{-\gamma} q_g(\gamma)$$

q_g polynomial, sum over $g \in \mathrm{Hom}(\Gamma, U(1))$ of finite order

Ehrhart's theorem

- V finite dimensional $\mathbb{R}$ vector space, dimension d
- Λ rank d lattice in V
- P d-dimensional closed polytope with vertices in $\Lambda \otimes \mathbb{Q}$

$$N_P(k) = \#(kP \cap \Lambda) = \#(P \cap \tfrac{1}{k}\Lambda)$$

Theorem (Ehrhart)

$N_P(k)$ is a quasi-polynomial function of k.

Leading term $\mathrm{vol}_\Lambda(P)k^d$

Around $[Q, R] = 0$

- G compact connected Lie group, max torus T, weight lattice $\Lambda \subset \mathfrak{t}^*$, Λ_+ dominant weights
- (M, ω, μ) compact Hamiltonian G-space, compatible a-$\mathbb{C}$ structure J, prequantum line bundle L
- $D = \overline{\partial} + \overline{\partial}^*$ Dolbeault-Dirac operator, D_{L^k} twist by $L^{\otimes k}$

$$\text{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda_+} m(k, \lambda)\chi_\lambda$$

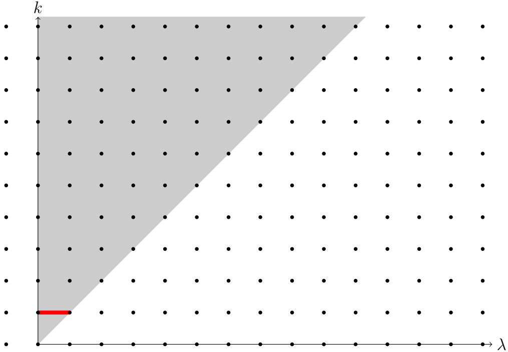
$$m \colon \mathbb{Z}_{>0} \times \Lambda_+ \to \mathbb{Z}$$

Theorem (Meinrenken-Sjamaar, '99)

$\exists$ a finite set of closed polytopes $P \subset \mathfrak{t}^*$ such that $m|_{C_P}$ is quasi-polynomial, where

$$C_P = \{(t, tv) | t > 0, v \in P\} \subset \mathbb{R} \times \mathfrak{t}^*$$

Example of C_P



The cone C_P for $P = [0, 1] \subset \mathfrak{t}^* = \mathbb{R}$, $\Lambda = \mathbb{Z}$.

Asymptotics of m

G a torus. (M^{2d}, ω, μ, L) prequantized Hamiltonian G-space

$$\mathrm{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda} m(k,\lambda)\chi_\lambda$$

$$\Theta(m;k) = \sum_{\lambda \in \Lambda} m(k,\lambda)\delta_{\lambda/k}$$

Theorem

$$\Theta(m;k) \sim k^d \sum_{n=0}^{\infty} \frac{1}{k^n} DH_M(\mathrm{Td}_n(M))$$

Equivariant Todd class, expanded in homogeneous terms:

$$\mathrm{Td}(M)(X) = \sum_{n=0}^{\infty} \mathrm{Td}_n(M)(X)$$

Piecewise quasi-polynomial functions

Fix lattice $\Lambda \subset V$

P rational polyhedron, $\sigma \in V_{\mathbb{Q}}$

$$C_{P,\sigma} = \{(t, tv + \sigma) | t > 0, v \in P\}$$

characteristic function: $[C_{P,\sigma}]$

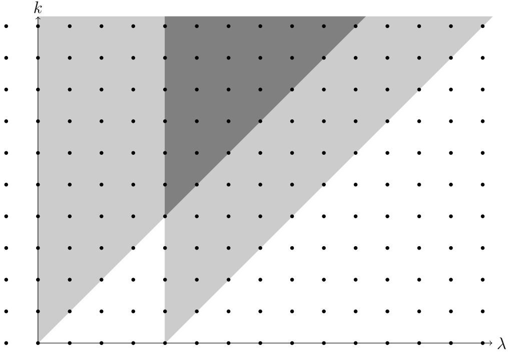
Definition

$m \colon \mathbb{Z}_{>0} \times \Lambda \to \mathbb{C}$ is *piecewise quasi-polynomial* if

$$m = \sum_{P,\sigma} q_{P,\sigma}[C_{P,\sigma}]$$

where $q_{P,\sigma}(k, \lambda)$ quasi-polynomial and $\{P + [0,1]\sigma\}$ is locally finite.

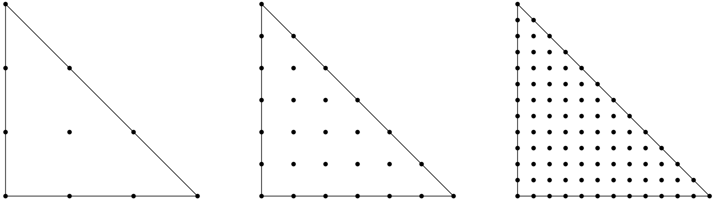
Example of m



The function $m = [C_P] + [C_{P,\sigma}]$ for $\Lambda = \mathbb{Z} \subset \mathbb{R} = V$, $P = [0, 1]$, and $\sigma = 4$. On the light gray region $m = 1$, and on the dark gray region $m = 2$.

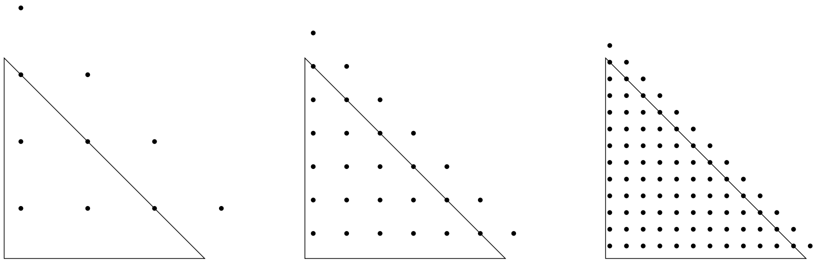
Family of measures

$$\Theta(m;k) = \sum_{\lambda \in \Lambda} m(k,\lambda)\delta_{\lambda/k}$$



Support of $\Theta(q[C_P];k)$, $k = 3, 6, 12$, with $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2 = V$, and $P = \{x \geq 0, y \geq 0, x + y \leq 1\}$.

Example, with shifts



Support of $\Theta(q[C_{P,\sigma}]; k)$, $k = 3, 6, 12$, with $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2 = V$, $\sigma = \left(\frac{3}{2}, \frac{9}{2}\right)$, and $P = \{x \geq 0, y \geq 0, x + y \leq 1\}$.

$\Theta(k)$ a sequence of distributions $k = 1, 2, 3, \ldots$

Definition

$\Theta(k)$ *admits an asymptotic expansion* if there is a $j \in \mathbb{Z}$ and distributions $u_n(k)$, *periodic in* k, such that for all N and all test functions φ,

$$\langle \Theta(k), \varphi \rangle = k^j \sum_{n=0}^{N} \frac{1}{k^n} \langle u_n(k), \varphi \rangle + o(k^{j-N})$$

$$\Theta(k) \sim \mathcal{A}(k), \quad \mathcal{A}(k) = k^j \sum_{n=0}^{\infty} \frac{1}{k^n} u_n(k)$$

Asymptotic expansions and piecewise quasi-polynomials

Theorem (LPV)

For any piecewise quasi-polynomial m, the sequence $\Theta(m;k)$ admits an asymptotic expansion $\mathcal{A}(m;k)$.

If m has compact support, asymptotic expansion becomes exact when both sides are paired with polynomials. (Ehrhart's theorem on counting lattice points in kP is a special case.)

- Series of reductions: general case $\rightsquigarrow$ single polyhedron $\rightsquigarrow$ single cone $\rightsquigarrow$ 1-dimensional cone.
- 1-dimensional case is the Euler-Maclaurin formula

Example with $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $P = [0,1]$

$$m(k,\lambda) = \begin{cases} 1 & \text{if } 0 \leq \lambda \leq k \\ 0 & \text{else.} \end{cases}$$

$$\Theta(m;k) = \sum_{\lambda=0}^{k} \delta_{\lambda/k}$$

$$\mathcal{A}(m;k) = k\mu_{[0,1]} + \frac{1}{2}(\delta_0 + \delta_1) + k \sum_{n=2}^{\infty} \frac{B_n}{n!\,k^n}(-1)^{n-1}(\delta_1^{(n-1)} - \delta_0^{(n-1)})$$

$\mu_{[0,1]}$ Lebesgue on $[0,1]$, B_n Bernoulli numbers,
$\delta_x^{(r)}$ r^{th} derivative of δ_x

Example with $V = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

$$P = \{(\lambda_1, \lambda_2) | \lambda_i \geq 0, \lambda_1 + \lambda_2 \leq 1\}$$

$$m(k, \lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } (\lambda_1, \lambda_2) \in kP \\ 0 & \text{else.} \end{cases}$$

Let $\partial_0 P = P \cap \{\lambda_1 + \lambda_2 = 1\}$, $\partial_1 P = P \cap \{\lambda_2 = 0\}$, $\partial_2 P = P \cap \{\lambda_1 = 0\}$ closed 1-dim faces.

Leading and sub-leading terms of $\mathcal{A}(m; k)$:

$$k^2 \mu_P, \qquad \frac{1}{2} k(\mu_{\partial_1 P} + \mu_{\partial_2 P} + \mu_{\partial_0 P}).$$

Kernel of $m \mapsto \mathcal{A}(m)$

How much information does $\mathcal{A}(m)$ contain?

Not perfect. Example: $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $m(k, \lambda) = (-1)^\lambda \Rightarrow$ $\Theta(m; k) \sim 0$.

With a slightly stronger local finiteness condition, we can prove that obvious generalizations of the above example are responsible for the whole kernel.

Corollary: $\mathcal{A}$ is injective on subspace of m's given by finite sums only involving compact P's.

Unicity

$g \in \mathrm{Hom}(\Lambda, U(1))$ of finite order

$$(g \cdot m)(k, \lambda) = g^{\lambda} m(k, \lambda)$$

Theorem (LPV)

If $\mathcal{A}(g \cdot m) = 0$ for all g then $m = 0$.

Application [PV]: functoriality of quantization (symplectic or spin-c) under restriction to subgroups in the non-compact setting. ("Proof": restriction is easy for (twisted) DH distributions. Use above theorem (also Berline-Vergne and Kirillov formulas) to prove result.)

Comments on the proof

Step 1: piecewise quasi-polynomials have 'germs': if $m = \sum q_{P,\sigma}[C_{P,\sigma}]$, the 'germ' at $v \in V$:

$$T_v m = \sum q_{P,\sigma}[C_{T_v P,\sigma}]$$

$T_v P$ the *tangent cone* to P at v, and $m = 0 \Leftrightarrow T_v m = 0 \forall v \in V$

Step 2: Say $T_0 P$ is a single pointed cone (for simplicity). Fourier transform $\mathcal{F}(\Theta(T_0 m; k))$ is the boundary value of a meromorphic function, e.g.

$$\frac{1}{1 - e^{\mathrm{i}z}} \quad \text{where} \quad z = x/k + \mathrm{i}\epsilon, \quad \epsilon \to 0^+$$

Step 3: $\mathcal{F}(\mathcal{A}(T_0 m; k))$ obtained by Laurent expansion at 0. If this vanishes then original function vanishes.

Outlook

- Modest generalizations with auxiliary vector bundles
- Simplify aspects of the proof of $[Q, R] = 0$ in the singular case
- Re-examine spin-c $[Q, R] = 0$ (due to Paradan-Vergne)? Relation with symplectic $[Q, R] = 0$ unclear. Good setting encompassing both?

Thanks!