Norm-Square Localization for Hamiltonian LG-Spaces

by

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Abstract

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In this thesis we prove norm-square localization formulas for two invariants of Hamiltonian loop group spaces: twisted Duistermaat-Heckman distributions and a K-theoretic 'quantization'. The terms in the formulas are indexed by the components of the critical set of the norm-square $||\Psi||^2$ of the moment map $\Psi: \mathcal{M} \to L\mathfrak{g}^*$. These formulas are analogous to results proved by Paradan ([48], [49], [50]) in the case of Hamiltonian *G*-spaces. An important application of the norm-square localization formula (for the quantization) is to prove that the multiplicity of the minimal level *k* representation in the quantization $Q(\mathcal{M}, L^k)$ is a quasi-polynomial function of *k*. This is closely related to the [Q, R] = 0 Theorem of Alekseev-Meinrenken-Woodward ([4]) for Hamiltonian loop group spaces. For my sister, Eleni.

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Chapter 1

Introduction

In this thesis we use a method called norm-square localization to study two invariants of Hamiltonian loop group spaces: twisted Duistermaat-Heckman distributions and a K-theoretic 'quantization'. A major reason for studying these invariants is that they encode information about the symplectic quotients, which are usually difficult to study directly. In the case of twisted Duistermaat-Heckman distributions, this information consists of intersection pairings; while the K-theoretic 'quantization' encodes indices of certain Dirac operators on the quotients.

An important class of examples of Hamiltonian loop group spaces are spaces of flat connections on compact Riemann surfaces with boundary, modulo gauge transformations which are trivial along the boundary. The moment map in these examples is simply restriction of the connection to the boundary. Taking symplectic quotients recovers the well-known finite-dimensional moduli spaces of flat connections on Riemann surfaces. The Duistermaat-Heckman measure that we study here encodes the Liouville volumes of the moduli spaces (and more generally, intersection pairings). The K-theoretic quantization encodes (via vanishing theorems for higher cohomology groups) the dimensions of spaces of holomorphic sections of certain line bundles on the moduli spaces (given by the Verlinde formulas).

By norm-square localization (also often called non-abelian localization) we mean that certain geometric data becomes localized near the set of critical points of the norm-square of the moment map $\|\phi\|^2$. The usefulness of studying the norm-square of the moment map became apparent in the early 80's with the work of Atiyah-Bott [6] and the thesis of Kirwan [31]; in these applications, the norm-square of the moment map was used as a Morse function to prove theorems about the cohomology of symplectic quotients.

A paper of Witten [63] from the early 90's used the norm-square of the moment map in a new way: Witten studied pairings of Duistermaat-Heckman distributions with Gaussians on \mathfrak{g}^* , and argued that the resulting integral could be localized to a small open neighbourhood of $\operatorname{Crit}(||\phi||^2)$. Moreover, the dominant contribution to these integrals was from the component $\phi^{-1}(0) \subset \operatorname{Crit}(||\phi||^2)$, with the other components giving lower order 'corrections'. This picture was then worked out rigorously and in a quite general setting by Paradan [48], [49]. Paradan derived a detailed norm-square localization formula for twisted Duistermaat-Heckman distributions of proper Hamiltonian *G*-spaces (*G* a compact connected Lie group); in this formula the contribution from $\phi^{-1}(0)$ captures the behaviour of the Duistermaat-Heckman distribution near $0 \in \mathfrak{g}^*$, with the other components of $\operatorname{Crit}(||\phi||^2)$ giving 'corrections' supported away from the origin. This could be thought of as a manifestation of the Morse stratification, with the Duistermaat-Heckman measure being assembled from the main $\phi^{-1}(0)$ contribution plus 'corrections', analogous to the way the manifold itself decomposes into the open dense Morse stratum which flows to the minimum $\phi^{-1}(0)$ together with the higher Morse strata. Another approach to these normsquare localization formulas was developed by Woodward [65], and more recently an approach based on Hamiltonian cobordism techniques was introduced by Harada and Karshon [26].

Paradan [50] also studied a K-theoretic version of the problem: the G-equivariant index Q(M) of a Dirac operator on a compact Hamiltonian G-space M. The idea in this case was to use the Hamiltonian vector field of the function $\|\phi\|^2$ to deform the symbol of the Dirac operator, in the space of transversally elliptic symbols. The vanishing set of this vector field coincides with the set of critical points of $\|\phi\|^2$. This leads to a norm-square localization formula for the index Q(M), as a sum of contributions from the components of $\operatorname{Crit}(\|\phi\|^2)$. The resulting formula for the multiplicities is a 'discrete' version of the formula for Duistermaat-Heckman measures. It shares the key feature that the contribution from $\phi^{-1}(0)$ captures the multiplicities for dominant weights near $0 \in \mathfrak{g}^*$, with the other components of $\operatorname{Crit}(\|\phi\|^2)$ giving 'corrections' supported away from the origin. This remarkable feature of the formula leads to a proof of the quantization-commutes-with-reduction theorem, which asserts that the multiplicity of the trivial representation in Q(M) equals the index for the reduced space Q(M//G) (suitably interpreted when the quotient is singular). The principal challenge in proving this theorem is to explain why the multiplicity of the trivial representation only seems to depend on a small neighbourhood of $\phi^{-1}(0)$ this is not at all clear from the Atiyah-Segal-Singer fixed-point formula for example. By contrast the norm-square localization formula of Paradan is very well-suited to this purpose.

Many well-known results on compact Hamiltonian G-spaces have analogs for proper Hamiltonian LG-spaces \mathcal{M} (where LG is the infinite-dimensional loop group). These similarities are not entirely surprising given the strong analogies between the geometry and representation theory of a compact Lie group G and its loop group LG (c.f. [55]), combined with the existence of finite-dimensional symplectic cross-sections ([46]). Examples of parallel results include the convexity theorem, Duistermaat-Heckman formulas, the classification of multiplicity-free spaces, fixed-point formulas for an index-theoretic quantization, and a [Q, R] = 0 theorem, c.f. [46], [2], [5], [44], [32]. A further example, related to the norm-square of the moment map, is the Kirwan surjectivity theorem for Hamiltonian LG-spaces of Bott-Tolman-Weitsman ([14]). The Bott-Tolman-Weitsman proof proceeded roughly along the same lines as Kirwan's proof, applying ideas from Morse theory (with extra complications because \mathcal{M} is infinite-dimensional) to the norm-square of the moment map. The goal of this thesis is to derive versions of the formulas of Paradan discussed above for Hamiltonian LG-spaces.

Important in studying Hamiltonian LG-spaces are certain finite-dimensional quotients, obtained by dividing out the (necessarily free) action of the based loop group ΩG . These finite-dimensional quotients are known as *quasi-Hamiltonian* G-spaces ([2]), and by now have an extensive theory (c.f. [43] for an introductory overview). This theory plays a key role both in the definition and study of the invariants considered below.

1.1 Summary of the thesis.

This introductory chapter contains some background material on Hamiltonian LG-spaces, quasi-Hamiltonian G-spaces, the norm-square of the moment map, and notation. We now give a summary of the main results contained in Chapters 2, 3, 4.



Figure 1.1: The left-most image shows a single contribution to the norm-square formula for a certain multiplicity-free Hamiltonian LSU(3)-space. The next two images show the sum of the first 6 (resp. 12) contributions. Light gray indicates regions where the sum is +1, while black indicates regions where it is -1.

Chapter 2.

The results of this chapter appeared in the paper [35]. Let $\Psi : \mathcal{M} \to L\mathfrak{g}^*$ be a proper Hamiltonian LG-space. We prove a norm-square localization formula for twisted Duistermaat-Heckman distributions of a Hamiltonian LG-space. The formula expresses a twisted Duistermaat-Heckman distribution \mathfrak{m} on \mathfrak{t} as a sum of contributions:

$$\mathfrak{m} = \sum_{\beta \in W \cdot \mathcal{B}} \mathfrak{m}_{\beta}, \tag{1.1}$$

where \mathcal{B} indexes components of the critical set of $\|\Psi\|^2$, and $W = N_G(T)/T$ is the Weyl group. The sum is infinite but locally finite, in the sense that the supports of only finitely many terms intersect each bounded set. The terms of (1.1) consist of a central contribution (from the critical value 0), and correction terms supported in half-spaces not containing the origin.

The distribution \mathfrak{m} is defined by means of a certain finite-dimensional quotient $M = \mathcal{M}/\Omega G$. The space M is a quasi-Hamiltonian G-space ([2]). We prove (1.1) by applying Hamiltonian cobordism methods to a covering space of the *abelianization* of this quasi-Hamiltonian space.

The Hamiltonian cobordism methods we use were first introduced by Ginzburg-Guillemin-Karshon ([21]), and extended to other types of localization formulas (including norm-square localization) by Harada-Karshon ([26]). In the Harada-Karshon approach, the Hamiltonian G-space is found to be cobordant (in a suitable sense) to a small open neighbourhood of the critical set, once the moment map on the neighbourhood has been suitably *polarized* and *completed*. A norm-square localization formula for twisted Duistermaat-Heckman measures then follows from Stokes' theorem.

Using the terminology of Harada-Karshon [26], the term \mathfrak{m}_{β} of (1.1) can itself be described as a twisted DH distribution for a *polarized completion* of a small finite dimensional submanifold near the part of the critical set indexed by β . Going further, we describe local models for the 'polarized completions' appearing in the Harada-Karshon Theorem, and use these to derive formulas similar to those of Paradan ([49]), expressing the contributions \mathfrak{m}_{β} in terms of integrals of differential forms over submanifolds near the corresponding part of the critical set; the final result is equation (2.26) below. We do not make any large claim of originality in our derivation of these detailed formulas; they follow also from applying the results of Paradan or Woodward to the abelianization of the quasi-Hamiltonian space.

Figure 1.1 shows an example of the decomposition (1.1) for the DH distribution of a multiplicity-free Hamiltonian LSU(3)-space (example due to Chris Woodward). In this example the contribution from 0 vanishes, and the remaining terms in (1.1) are constant multiples (± 1 relative to suitably normalized Lebesgue measure) of indicator functions for half-spaces (see Chapter 2 for further discussion).

Chapter 3.

In this chapter we give a second proof of the norm-square localization formula proved in Chapter 2. This second proof is much more combinatorial, and is inspired by recent work of Szenes-Vergne ([60]), which explained how Paradan's formulas could be obtained from combinatorial results on multi-spline functions and partition functions. Key ingredients in this second proof are the abelian localization formula for quasi-Hamiltonian G-spaces due to Alekseev-Meinrenken-Woodward ([5]) and a combinatorial formula for multiple Bernoulli series due to Boysal-Vergne ([15]). The main purpose of this chapter is as a 'warm-up' for the more complex 'quantum version' of this argument presented in Chapter 4.

The main result is a formula

$$\mathfrak{m} = \sum_{\Delta} \mathfrak{m}_{\Delta}^{\mathrm{pol}},\tag{1.2}$$

where the sum is over an infinite collection of affine subspaces (the so-called *admissible* subspaces). The subspaces Δ are obtained by taking affine spans of images of fixed-point sets $\mathcal{M}^{\mathfrak{t}_{\Delta}}$ in \mathfrak{t} , where $\mathfrak{t}_{\Delta} \subset \mathfrak{t}$ is the subspace orthogonal to Δ . The contribution $\mathfrak{m}_{\Delta}^{\mathrm{pol}}$ vanishes unless the point $\beta = \mathrm{pr}_{\Delta}(0)$ (the orthogonal projection of 0 onto Δ) corresponds to a component of the critical set of the norm-square of the moment map.

For $\Delta \neq \mathfrak{t}$, the support $\mathfrak{m}_{\Delta}^{\text{pol}}$ is contained in a half-space, and coincide with those appearing in (1.1), although the description of them which emerges is somewhat different. The contribution $\mathfrak{m}_{\Delta}^{\text{pol}}$ is given by the formula

$$\mathfrak{m}_{\Delta}^{\mathrm{pol}} = \mathrm{Eul}(\mathfrak{g}/\mathfrak{t}, \partial_{\mu}) \sum_{C \subset \mathcal{M}^{\mathfrak{t}_{\Delta}}} \sum_{F \subset C} \int_{F} e^{\omega} \mathrm{Ber}(\nu_{F,C}; \mathrm{pr}_{\Delta}(\gamma)) * H(\nu_{C}; \gamma_{\Delta}^{+}).$$
(1.3)

In simple cases, Ber(-) is a Bernoulli polynomial viewed as a distribution supported on the subspace Δ , and H(-) is a convolution of Heaviside distributions.

We introduce a closely related distribution \mathfrak{m}_{Δ} , such that $\mathfrak{m}_{\Delta}^{\text{pol}}$ is obtained from \mathfrak{m}_{Δ} by 'taking polynomial germs along Δ '. The formula for the inverse Fourier transform of \mathfrak{m}_{Δ} (equation (3.17)) can be identified with the abelian localization formula applied to submanifolds $C \subset M^{\mathfrak{t}_{\Delta}}$ (or perhaps better: the total space of the normal bundle ν_C).

Chapter 4.

This chapter describes joint work with E. Meinrenken. We now take G to be (in addition) simple and simply connected. The quantization $Q(\mathcal{M}, L)$ of a Hamiltonian LG-space \mathcal{M} with level $k_1 \geq 1$ prequantum line bundle L is studied using methods analogous to those of Chapter 3, replacing the abelian localization formula with an Atiyah-Segal-Singer-type fixed-point formula, and Bernoulli series with Verlinde series (also known as rational trigonometric series). The strategy employed is inspired by the already-mentioned work of Szenes-Vergne ([60]). A key aspect of this approach is to study the behaviour of the quantization $Q(\mathcal{M}, L^n)$ as the power $n \in \mathbb{N}$ of the prequantum line bundle is varied.



Figure 1.2: The left image shows a single contribution to the norm-square localization formula for a multiplicity-free Hamiltonian LSU(3)-space (at level k = 2). The right image shows the sum of the first 6 contributions.

We begin by introducing and studying Verlinde series: these are combinatorial sums which appear in the fixed-point formula for $Q(\mathcal{M}, L)$. (They appeared originally in the famous Verlinde formula [62] for the dimensions of spaces of holomorphic sections of line bundles over the moduli spaces of flat connections on Riemann surfaces; we comment on this briefly below.) We provide a proof for an unpublished result of Boysal-Vergne on Verlinde series. This result is applied to a fixed-point formula for the quantization due to Alekseev-Meinrenken-Woodward ([44], [4]), leading eventually to the 'norm-square localization' formula for the multiplicity function $m : \Lambda^* \times \mathbb{N} \to \mathbb{Z}$ of the quantization $Q(\mathcal{M}, L^n)$, which is our main result:

Theorem 1.1.1. The multiplicity function $m : \Lambda^* \times \mathbb{N} \to \mathbb{Z}$ for $Q(\mathcal{M}, L^n)$ has a locally finite decomposition

$$m = \sum_{\Delta} m_{\Delta}^{\rm qpol}.$$

where Δ ranges over a certain infinite collection of affine subspaces of \mathfrak{t}^* . The contribution m_{Δ}^{qpol} vanishes unless $\mathcal{M}^{\beta} \cap \Psi^{-1}(\beta) \neq \emptyset$, where β is the nearest point in Δ to the origin. Furthermore m_{Δ}^{qpol} is obtained by taking 'quasi-polynomial germs along Δ ' of a multiplicity function $m_{\Delta} = \mathcal{F}(Q_{\Delta})$, where

$$Q_{\Delta}(t,k) = \sum_{C \subset N^{T_{\Delta}}} \sum_{F \subset C^{t}} \int_{F} \frac{\hat{A}(F) \mathrm{Ch}(\mathcal{L}_{k,\Delta,C},t)^{1/2} \mathrm{Ch}(\wedge_{\mathbb{C}} \mathfrak{n}_{-} \otimes S\nu_{C}(\gamma_{\Delta}^{+}),t)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} \delta_{T_{\ell}T_{\Delta}}(t)$$

 $Q_{\Delta}(-,k) = q_{\Delta}(-,k)\delta_{T_{\ell}T_{\Delta}}$ is a distribution supported on $T_{\ell}T_{\Delta} \subset T$; the generalized function $q_{\Delta}(-,k)$ on $T_{\ell}T_{\Delta}$ takes the form of fixed-point contributions for $T_{\ell}T_{\Delta}$ -equivariant spin-c structures $S_{k,\Delta,C}$ on submanifolds $C \subset N^{T_{\Delta}}$, tensored with the \mathbb{Z}_2 -graded bundle $\wedge_{\mathbb{C}}\mathfrak{n}_- \otimes S\nu_C(\gamma_{\Delta}^+)$.

In this theorem, $M = \mathcal{M}/\Omega G$ is the associated quasi-Hamiltonian *G*-space, which has a group-valued moment map $\Phi : M \to G$, and $N = \Phi^{-1}(U)$ where *U* is a tubular neighbourhood of the maximal torus. T_{ℓ} is a certain finite subgroup of the maximal torus *T* depending on the level $k = nk_1$, and $T_{\Delta} = \exp(\mathfrak{t}_{\Delta}) \subset T$ where \mathfrak{t}_{Δ} is the subspace of \mathfrak{t} orthogonal to Δ . $\gamma \in \mathfrak{t}^*$ is a generic point near 0, and $\gamma_{\Delta}^+ = \operatorname{pr}_{\Delta}(\gamma) - \gamma$. $S\nu_C(\gamma_{\Delta}^+)$ denotes the symmetric algebra of the normal bundle ν_C , where the latter is equipped with a complex structure such that the weights of the T_{Δ} action are γ_{Δ}^+ -polarized. The notation, the properties of the spin-c structure $S_{k,\Delta,C}$, and the meaning of taking 'quasi-polynomial germs' along Δ will be explained in detail in Chapter 4. Note that near the beginning of Chapter 4, we have included a list of symbols used in that chapter.

Our proof is modelled on the approaches of Paradan and Szenes-Vergne described above, and the formula is analogous to Paradan's formula for a compact Hamiltonian G-space.

Figure 1.2 shows an example of the norm-square localization formula for a certain multiplicity-free LSU(3)-space (the same example as in Figure 1.1). The right image shows the result of summing the first 6 non-trivial terms. We discuss this example in detail in Chapter 4, Section 4.4. This simple example already shows an interesting feature: there are 3 fixed-point contributions, each of which is non-zero around $0 \in \Lambda^*$. Passing to the norm-square localization formula, there is some *cancellation* between these, and the contribution of $\Psi^{-1}(0)$ turns out to be 0. This example suggests that the norm-square localization formula is a more powerful tool (than the Atiyah-Segal-Singer fixed-point formula) for detecting the local behaviour of the multiplicity function m.

Together with a Lie-theoretic inequality that we prove, Theorem 1.1.1 implies that the multiplicity of the minimal level k positive energy representation of LG in $Q(\mathcal{M}, L^n)$ is a quasi-polynomial function of k (or n). The latter result is closely related to the [Q, R] = 0 Theorem for Hamiltonian LG-spaces.

The needed Lie-theoretic inequality is proved in Section 4.5 of Chapter 4:

Theorem 1.1.2. Let G be simple and simply connected. Let $\kappa \in \mathfrak{t}_+$ be a vertex of the Stiefel diagram. Let ρ be the half-sum of the positive roots of \mathfrak{g} and ρ_{κ} the half-sum of the positive roots of $G_{\exp(\kappa)}$. Then

$$\mathbf{h}^{\vee} \|\kappa\|^2 - \langle \rho + \rho_{\kappa}, \kappa \rangle \ge 0, \tag{1.4}$$

where $\|\cdot\|^2$ is the norm defined by the basic inner product, and \mathbf{h}^{\vee} is the dual Coxeter number.

This inequality (and our use for it) is analogous to Paradan-Vergne's 'magic inequality' ([51]), which they used to prove a spin-c [Q, R] = 0 theorem ([52]).

As motivation, let us comment briefly on the connection of these results with the [Q, R] = 0 Theorem and the special case of moduli spaces of flat connections (the Verlinde formula). For simplicity, assume 0 is a regular value of the moment map: in this case, the Alekseev-Meinrenken-Woodward [Q, R] = 0Theorem for Hamiltonian *LG*-spaces says that the multiplicity of the minimal level *k* positive energy representation in $Q(\mathcal{M}, L^n)$ equals the (orbifold) index of the Dolbeault-Dirac operator on the reduced space $\Psi^{-1}(0)/G$ twisted by the (orbifold) line bundle $L^n|_{\Psi^{-1}(0)}/G$. Using the (orbifold) index formula it follows that the latter is a (quasi-)polynomial function of *n*. Thus the [Q, R] = 0 Theorem follows from Theorems 1.1.1, 1.1.2 combined with an 'asymptotic' (k >> 0) [Q, R] = 0 Theorem. The 'asymptotic' [Q, R] = 0 Theorem is considerably less difficult to prove; one method (c.f. [37] for the Hamiltonian case, with 0 a regular value) involves applying the principal of stationary phase to the fixed-point formula for the index.

Let G be compact, connected, simply connected and simple. Fix a maximal torus T and positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$. The Weyl denominator is a function on T given by

$$J(t) = \sum_{w \in W} (-1)^{|w|} t^{w\rho} = t^{\rho} \prod_{\alpha \in \mathcal{R}_{+}} 1 - t^{-\alpha}$$

where W is the Weyl group, ρ is the half-sum of the positive roots \mathcal{R}_+ . For λ a dominant weight, let χ_{λ} be the character of the corresponding irreducible representation of G. Fix $k \geq 1$ and let Λ_k^* denote the set of dominant weights μ such that $B(\mu, \theta) \leq k$, where B is the basic inner product, and θ is the highest root. Use the basic inner product to identity $\mathfrak{g} = \mathfrak{g}^*$. For each $\lambda \in \Lambda_k^*$ set

$$t_{\lambda} = \exp\left(\frac{\lambda + \rho}{k + \mathbf{h}^{\vee}}\right) \in T.$$

Let $\Sigma_{g,b}$ be a compact Riemann surface of genus g and with b boundary circles. Fix $\mu_1, ..., \mu_b \in \Lambda_k^*$. The data $(\Sigma_{g,b}, \mu), \mu = (\mu_1, ..., \mu_b)$ define a symplectic space (possibly singular): the moduli space $\mathcal{M}(\Sigma_{g,b}, \mu)$ of flat connections on the trivial G-bundle over $\Sigma_{g,b}$, with holonomy around the i^{th} boundary circle in the conjugacy class of $\exp(\mu_i/k)$. The Verlinde formula gives the dimension $Q(\mathcal{M}(\Sigma_{g,b}, \mu))$ of the space of holomorphic sections of a prequantum line bundle on this space:

$$Q(\mathcal{M}(\Sigma_{g,b},\mu)) = |T_{k+\mathbf{h}^{\vee}}|^{g-1} \sum_{\lambda \in \Lambda_k^*} |J(t_{\lambda})|^{2-2g} \chi_{\mu_1}(t_{\lambda}) \cdots \chi_{\mu_b}(t_{\lambda}).$$

The left side of this expression turns out to equal the index-theoretic quantization of $\mathcal{M}(\Sigma_{g,b},\mu)$ that we have been discussing (this is a consequence of a non-trivial 'vanishing theorem' for the higher cohomology groups, due to C. Teleman). On the other hand, the right side can be obtained from the fixed-point formula for the quantization of a certain quasi-Hamiltonian *G*-space (equivalently, Hamiltonian *LG*space). This quasi-Hamiltonian *G*-space is relatively simple: it is the product

$$M = G^{2g} \times \mathcal{C}_1 \times \cdots \mathcal{C}_b, \qquad \mathcal{C}_i = G \cdot \exp(\mu_i/k)$$

The G-action is the diagonal G-action by conjugation, and the moment map is a product of g group commutators (for G^{2g}) with inclusion maps for the conjugacy classes C_i . (For some hint as to how the right side of the Verlinde formula is related to a fixed-point formula on M, note that the factor $|J(t_{\lambda})|^{-2g}$ is similar to the denominator of the Atiyah-Bott fixed-point formula, for the normal bundle $\nu(T^{2g}, G^{2g}) \simeq T^{2g} \times (\mathfrak{g}/\mathfrak{t})$ to the fixed-point set T^{2g} of t_{λ} in G^{2g} .) Thus the Verlinde formula follows from the fixed-point formula for the quantization of the quasi-Hamiltonian G-space M, together with the [Q, R] = 0 Theorem. For more details see [44], [4].

Finally, a few remarks about some confusing aspects of the proof of Theorem 1.1.1. It is important to consider the behaviour of the quantization $Q(\mathcal{M}, L^n)$ as a function of the power n of the prequantum line bundle. L^n is a level $k = nk_1$ line bundle, which means that the infinitesimal action of \widehat{LG} (the basic central extension of LG) on L^n is related to $k = nk_1$ times the moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$ by the usual Kostant condition. This implies that, for example if $m \in \mathcal{M}^T$, then a phase factor of the form $t^{k\Psi(m)}$ appears in the index formula. On the other hand, for the Verlinde series that appear in the index formula, the polyhedral cones on which they coincide with quasi-polynomials are of the form

$$\bigcup_{\ell\in\mathbb{N}}\ell\mathfrak{c}\times\{\ell\}\subset\mathfrak{t}^*\times\mathbb{N},\qquad \ell=k+\mathbf{h}^\vee,$$

where h^{\vee} is the dual Coxeter number, and \mathfrak{c} is a polyhedral subset of \mathfrak{t}^* . The appearance of the shift by the dual Coxeter number is fundamental, and can be traced back to the fact that h^{\vee} is the level of the spin representation of \widehat{LG} .

The 'discrepancy' between k and $\ell = k + h^{\vee}$ initially appears to be a problem for implementing the Szenes-Vergne strategy ([60]); it seems possible that for small k, this discrepancy could lead to extra wall-crossing that would destroy quasi-polynomial behaviour. There are also some extra phases in the index formula, coming from the spin-c structure being used. It took some time to understand how one could proceed: it was useful to think of these differences as additional shifts (similar to the shifts by negative roots, which are the main source of difficulty for the case of Hamiltonian *G*-spaces, c.f. [60]). For example, the 'discrepancy' between k and $\ell = k + h^{\vee}$ could be thought of as an extra shift by $-h^{\vee}\Psi$. This change in viewpoint helps, and one finds in the end a somewhat complicated expression involving all the various shifts, which nevertheless satisfies the needed inequality.

Another initially confusing aspect is the presence of fixed-point submanifolds $F \subset M$ whose stabilizer $\tilde{T}_F \subset T$ is disconnected. In this case, phase factors appear in the index formula which cannot be expressed in terms of the moment map(s). Which terms appear in the sum for a given level k depend on the intersection $T_\ell \cap \tilde{T}_F$, which will also vary with k. We keep careful track of these effects, but they turn out to make little difference to the argument, essentially because these sources of variation (with k) are quasi-polynomial already. For example, in case the stabilizer \tilde{T}_F is discrete, the corresponding contribution $m_F(\lambda, k), \lambda \in \Lambda^*, k \in \mathbb{N}$ to the multiplicity function is a quasi-polynomial function of λ and k, and so ends up being grouped with the contribution from $\Psi^{-1}(0)$ in the norm-square localization formula.

1.2 Notation in the thesis

Throughout, G will denote a compact, connected Lie group with Lie algebra \mathfrak{g} . Fix an Ad-invariant inner product (denoted \cdot or B(-,-)) on \mathfrak{g} . Let T be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, and $W = N_G(T)/T$ the Weyl group. We say that a function f on \mathfrak{t} is W-alternating or W-anti-symmetric if for $w \in W$, $w \cdot f = (-1)^{|w|} f$, where |w| is the length of the element $w \in W$. If $H \supset T$ is a maximal rank subgroup with Lie algebra \mathfrak{h} , there is a unique H-invariant complement \mathfrak{h}^{\perp} , the orthogonal complement to \mathfrak{h} for any invariant inner product on \mathfrak{g} . Let $\Lambda \subset \mathfrak{t}$ denote the kernel of the exponential map $\exp : \mathfrak{t} \to T$. The dual lattice $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$ is the real weight lattice. The real roots form a subset $\mathcal{R} \subset \Lambda^*$. Fix a positive Weyl chamber \mathfrak{t}_+ , and let \mathcal{R}_+ denote the positive roots. This choice determines a complex structure on \mathfrak{t}^{\perp} (hence also an orientation). Using the inner product, $\mathfrak{t}^{\perp} \simeq \mathfrak{g}/\mathfrak{t}$.

Let M be a G-manifold. Let $\mathcal{D}'(M)$ denote the space of distributions on M, that is, the dual of the space of compactly supported smooth functions $C^{\infty}_{\text{comp}}(M)$. A superscript G will denote the G-invariant elements, for example, $C^{\infty}_{\text{comp}}(M)^G$, $\mathcal{D}'(M)^G$, etc. Given $X \in \mathfrak{g}$, define a vector field X_M on M by

$$X_M(m) = \frac{d}{dt} \bigg|_{t=0} \exp(-tX) \cdot m.$$

The map $X \mapsto X_M$ is a Lie algebra homomorphism.

Throughout this thesis, a (possibly degenerate) Hamiltonian G-space (M, ω, ϕ) is a G-manifold M equipped with a closed 2-form ω , and a G-equivariant map $\phi : M \to \mathfrak{g}^*$ satisfying the moment map condition:

$$\iota(X_M)\omega = -d\langle\phi, X\rangle, \qquad X \in \mathfrak{g}.$$
(1.5)

We will make extensive use of the Cartan model for equivariant cohomology, which we briefly recall.

Let $\Omega_G(M) = (\Omega(M) \otimes \operatorname{Pol}(\mathfrak{g}))^G$, where $\operatorname{Pol}(\mathfrak{g}) = S\mathfrak{g}^*$ denotes the polynomial algebra on \mathfrak{g} (with the coadjoint action of G). Let $H_G(M) = \ker(d_G)/\operatorname{im}(d_G)$ where the equivariant differential d_G is defined by

$$(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X)$$

The two conditions (1) ω is a closed 2-form, and (2) the moment map condition (1.5), can be combined into a single equation $d_G \omega_G = 0$ where

$$\omega_G(X) = \omega - \langle \phi, X \rangle$$

is an equivariant 2-form.

Apart from equivariant forms with polynomial dependence on \mathfrak{g} (described above), in some places we use forms with more general dependence on \mathfrak{g} . An equivariant differential form with C^{∞} -coefficients is a smooth *G*-equivariant map defined on some neighbourhood of $0 \in \mathfrak{g}$:

$$\alpha: U \to \Omega(M), \qquad 0 \in U \subset \mathfrak{g}.$$

Usually α will be either polynomial (the equivariant symplectic form $\omega_G(X)$) or analytic (equivariant extensions of characteristic classes appearing in the Atiyah-Singer index theorem). The equivariant differential d_G is defined as above.

In a few places we use forms α with generalized coefficients (respectively, tempered generalized coefficients), meaning that α defines a *G*-equivariant map $C^{\infty}_{\text{comp}}(\mathfrak{g}) \to \Omega(M)$ (respectively, from the Schwartz space for \mathfrak{g} to $\Omega(M)$). We do not really use any deeper results about forms with generalized coefficients (c.f. [33] for a development of this topic), but as convenient means for expressing the norm-square localization formulas and related formulas. Various operations defined on certain classes of functions/distributions extend in a natural way to the corresponding classes of equivariant differential forms. The most important example for this thesis is the Fourier transform, defined on the class of equivariant differential forms with tempered generalized coefficients by duality:

$$\langle \mathcal{F}(\alpha), f \rangle := \langle \alpha, \mathcal{F}(f) \rangle \in \Omega(M),$$

where α is a form with tempered generalized coefficients, and f is a Schwarz function.

1.3 Hamiltonian LG-spaces and q-Hamiltonian G-spaces

Let LG denote the Banach Lie group consisting of maps $S^1 \to G$ of a fixed Sobolev level $s \ge 1$, with group operation given by pointwise multiplication, and let $L\mathfrak{g} = \Omega_s^0(S^1, \mathfrak{g})$ denote its Lie algebra (c.f. [55]). We define $L\mathfrak{g}^*$ to be the space $\Omega_{s-1}^1(S^1, \mathfrak{g})$ of \mathfrak{g} -valued 1-forms of Sobolev level s - 1, where the pairing with $L\mathfrak{g}$ is defined using the inner product on \mathfrak{g} , and integration over the circle. Identifying $L\mathfrak{g}^*$ with the space of connections on the trivial G-bundle over the circle, we have the affine-linear action of LG by gauge transformations

$$g \cdot \xi = \mathrm{Ad}_g \xi - dgg^{-1}.$$

Definition 1.3.1. A Hamiltonian LG-space $(\mathcal{M}, \omega, \Psi)$ is a Banach manifold \mathcal{M} with a smooth LGaction, equipped with a weakly non-degenerate, LG-invariant 2-form ω , and an LG-equivariant moment map $\Psi: \mathcal{M} \to L\mathfrak{g}^*$ satisfying

$$\iota(\xi_{\mathcal{M}})\omega = -d\langle \Psi, \xi \rangle, \qquad \xi \in L\mathfrak{g}.$$

Let $\Omega G \subset LG$ denote the subgroup consisting of loops based at the identity $e \in G$. This subgroup acts freely on $L\mathfrak{g}^*$. Hence, by equivariance of Ψ , ΩG also acts freely on \mathcal{M} .

1.3.1 The norm-square of the moment map.

For $\xi \in L\mathfrak{g}^*$, let

$$\|\xi\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\xi(\theta)|^2 d\theta,$$

where $|\xi(\theta)|$ denotes the norm on \mathfrak{g} . Given a Hamiltonian LG-space $(\mathcal{M}, \omega, \Psi)$, the norm-square of the moment map is the G-invariant function

$$\|\Psi\|^2: \mathcal{M} \to \mathbb{R}.$$

The following result gives a description of the critical set of $\|\Psi\|^2$.

Proposition 1.3.2 (c.f. [14]). Let $\Psi : \mathcal{M} \to L\mathfrak{g}^*$ be a proper Hamiltonian LG-space. We have a decomposition

$$\operatorname{Crit}(\|\Psi\|^2) = G \cdot \bigcup_{\beta \in \mathcal{B}} \mathcal{M}^\beta \cap \Psi^{-1}(\beta)$$
(1.6)

where $\mathcal{B} = \{\beta \in \mathfrak{t}^*_+ | \mathcal{M}^\beta \cap \Psi^{-1}(\beta) \neq \emptyset\} \subset \mathfrak{t}^*_+$ is discrete.

We make some brief remarks to motivate this result. If $m \in \mathcal{M}$ is a critical point of $\|\Psi\|^2$, then, by equivariance of Ψ , $\xi := \Psi(m)$ must be a critical point of the restriction of $\|\cdot\|^2$ to the ΩG orbit $\Omega G \cdot \xi \subset L\mathfrak{g}^*$. Let $g \in G$ be the holonomy of the connection ξ , and let $P_{e,g}G$ denote the space of paths $\gamma : [0,1] \to G$ of Sobolev class s having fixed endpoints $\gamma(0) = e, \gamma(1) = g$. ΩG acts on $P_{e,g}G$ by left multiplication. For $\gamma \in P_{e,g}G$, let $E(\gamma)$ denote the *energy* of the path:

$$E(\gamma) = \int_0^1 |\gamma(t)^{-1} \gamma'(t)|^2 dt$$

There is an ΩG -equivariant map

$$\operatorname{Hol}: \Omega G \cdot \xi \to P_{e,g}G,$$

the holonomy of the connection (using the trivialization $S^1 \times G$ of the bundle), and under this map

$$\|\zeta\|^2 = E(\operatorname{Hol}(\zeta)), \qquad \zeta \in \Omega G \cdot \xi \subset L\mathfrak{g}^*$$

The critical points of the energy functional on $P_{e,g}G$ are smooth geodesics (for the bi-invariant metric) from e to g, c.f. [13]. It follows that $\operatorname{Hol}(\xi) = \exp(t\xi)$ with $\xi \in \mathfrak{g} \subset L\mathfrak{g}^*$ a constant connection.

We have

$$0 = d_m \|\Psi\|^2 = 2\langle d_m \Psi, \xi \rangle = 2\iota(\xi_{\mathcal{M}})\omega_m$$

By weak non-degeneracy of ω , $\xi_{\mathcal{M}}(m) = 0$, which shows that $m \in \mathcal{M}^{\xi} \cap \Psi^{-1}(\xi)$. Since $\|\Psi\|^2$ is *G*-invariant, the decomposition (1.6) in terms of a subset $\mathcal{B} \subset \mathfrak{t}^*_+$ follows. Using properness of Ψ , it can be shown that \mathcal{B} is discrete.

1.3.2 Q-Hamiltonian G-spaces and the Equivalence Theorem.

The left (resp. right) invariant Maurer-Cartan forms on G will be denoted θ^L (resp. θ^R). The Cartan 3-form η is a bi-invariant closed 3-form on G given by

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L].$$

It has an equivariant extension (for G acting on itself by conjugation)

$$\eta_G(X) = \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot X, \qquad X \in \mathfrak{g}.$$

Definition 1.3.3 ([2]). A quasi-Hamiltonian (q-Hamiltonian) G space is a G-manifold M, together with a G-invariant 2-form ω and a smooth G-equivariant map $\Phi: M \to G$ satisfying

$$d_G \omega = -\Phi^* \eta_G, \tag{1.7}$$

and such that $\ker(\omega_m) \cap \ker(d_m \Phi) = \{0\}$ for all $m \in M$. If this last minimal degeneracy condition is not satisfied, then we refer to M as a degenerate q-Hamiltonian G-space.

Remark 1.3.4. If $\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha$, the half-sum of the positive roots, is a weight of the group G, then M is automatically even-dimensional and orientable [5] (for example, this happens if G is the product of a simply connected group and a torus). In any case, from now on we take M to be a connected, oriented, even-dimensional q-Hamiltonian G-space.

According to the Equivalence Theorem 8.3 in [2], there is a 1-1 correspondence between compact q-Hamiltonian G-spaces and proper Hamiltonian LG-spaces. Let $\Psi : \mathcal{M} \to L\mathfrak{g}^*$ be a proper Hamiltonian LG-space with 2-form $\omega_{\mathcal{M}}$. The corresponding q-Hamiltonian space $\Phi : \mathcal{M} \to G$ fits into a pullback diagram

$$\begin{array}{cccc}
\mathcal{M} & \stackrel{\Psi}{\longrightarrow} & L\mathfrak{g}^* \\
\downarrow_{\mathrm{Hol}} & \downarrow_{\mathrm{Hol}} \\
\mathcal{M} & \stackrel{\Phi}{\longrightarrow} & G
\end{array}$$
(1.8)

The vertical maps are quotients by the free action of the group of based loops ΩG . In [2], a primitive $\varpi \in \Omega^2(L\mathfrak{g}^*)^{LG}$ of $\operatorname{Hol}^*\eta$ is constructed, such that

$$\omega := \omega_{\mathcal{M}} + \Psi^* \varpi \tag{1.9}$$

is ΩG -basic. Thus ω hence descends to a G-invariant 2-form on the quotient M. By construction it satisfies

$$d_G\omega = -\Phi^*\eta_G$$

and (M, ω, Φ) is a q-Hamiltonian G-space.

For later use, define the (possibly singular) T-invariant closed subsets

$$X = \Phi^{-1}(T) \subset M, \qquad \mathcal{X} = \Psi^{-1}(\mathfrak{t}) \subset \mathcal{M}.$$

Restricting the pullback diagram (1.8) shows that \mathcal{X} is the fibre product of X with $\mathfrak{t} \subset L\mathfrak{g}^*$.

1.3.3 The Cross-section Theorem.

Let $g \in T$, and let G_g be the centralizer of g, with Lie algebra \mathfrak{g}_g . Since $G_g \supset T$, \mathfrak{g}_g has a unique G_g -invariant complement \mathfrak{g}_g^{\perp} . There exists a *slice* for the conjugation action near g, that is, an $\mathrm{Ad}(G_g)$ invariant open neighbourhood U_g of g in G_g such that there is a G-equivariant diffeomorphism onto an open subset of G:

$$G \times_{G_a} U_a \xrightarrow{\sim} \operatorname{Ad}(G) \cdot U_a, \qquad [h, x] \mapsto hxh^{-1}$$

Let $\Phi: M \to G$ be a q-Hamiltonian G-space, and let

$$Y_g = \Phi^{-1}(U_g).$$

This is a smooth G_g -invariant submanifold. Moreover, $G \cdot Y_g$ is an open subset of M and

$$G \cdot Y_g \simeq G \times_{G_g} Y_g, \tag{1.10}$$

as G-spaces. By Proposition 7.1 of [2], Y_g is a q-Hamiltonian G_g -space, with the pullback of the 2-form and moment map.

In fact, the cross-section Y_g can be made into a Hamiltonian G_g -space. If the chosen open neighbourhood U_g is sufficiently small, then the image of the map $g^{-1}\Phi|_{Y_g}$ is contained in a neighbourhood of $e \in G$ on which the exponential map has a smooth inverse log sending e to $0 \in \mathfrak{g}$. Let

$$\phi_g := \log(g^{-1}\Phi|_{Y_q})$$

The pullback of the Cartan 3-form η to U_g is exact. The standard de Rham homotopy operator for a vector bundle V is given, up to a sign, by pulling back the form under the scalar multiplication map $[0,1] \times V \to V$, followed by integration over [0,1]. Applying the standard homotopy operator for the vector space \mathfrak{g}_q to $(g \exp)^* \eta$ one obtains a G_q -invariant primitive ϖ of $(g \exp)^* \eta$. Then

$$\omega_g = \omega - \phi_g^* \varpi$$

satisfies $d\omega_q = 0$.

Theorem 1.3.5 ([2]). (Y_g, ω_g, ϕ_g) is a non-degenerate Hamiltonian G_g -space.

For use later, note that it follows from the construction and the fact that $i_T^* \eta = 0$, that

$$\iota_t^* \varpi = 0. \tag{1.11}$$

In the important special case when G is simple and simply connected, maximal slices for the G-action on itself by conjugation are easy to describe explicitly (we will use this in Chapter 4). Let σ be an open face of the fundamental alcove. The stabilizer G_g of g under the adjoint action is the same for all $g \in \exp(\sigma)$ and is denoted G_{σ} . The corresponding Lie algebra is \mathfrak{g}_{σ} . Since $G_{\sigma} \supset T$, \mathfrak{g}_{σ} has a unique G_{σ} -invariant complement $\mathfrak{g}_{\sigma}^{\perp}$. The set

$$U_{\sigma} = \operatorname{Ad}(G_{\sigma}) \cdot \bigcup_{\overline{\tau} \supset \sigma} \exp(\tau),$$

where the union is over the set of faces of the fundamental alcove whose closure contains σ , is a slice for the conjugation action. The corresponding cross-section is denoted $Y_{\sigma} = \Phi^{-1}(U_{\sigma})$. Similarly, for each element w of the Weyl group, one defines $G_{w\sigma}$ to be the stabilizer of any point in $\exp(w\sigma)$. One has a slice $U_{w\sigma}$ defined as above, and a corresponding cross-section $Y_{w\sigma} = \Phi^{-1}(U_{w\sigma})$, which becomes a non-degenerate Hamiltonian $G_{w\sigma}$ -space.

As a result of equation (1.10), the normal bundle

$$\nu(Y_g, M) \simeq Y_g \times \mathfrak{g}_g^{\perp}.$$

We orient the even-dimensional subspace \mathfrak{g}_g^{\perp} compatibly with the orientation provided by the Liouville form on the symplectic submanifold Y_g , and the orientation on M. Note in particular that when $\mathfrak{g}_g^{\perp} = \{0\}$, an orientation is just a sign ± 1 . Let $\mathfrak{g}_g/\mathfrak{t}$ denote the unique *T*-invariant complement of \mathfrak{t} in \mathfrak{g}_g .¹ Hence

$$\mathfrak{t}^{\perp} = (\mathfrak{g}_g/\mathfrak{t}) \oplus \mathfrak{g}_q^{\perp}.$$

Orient $\mathfrak{g}_g/\mathfrak{t}$ compatibly with the orientation on \mathfrak{g}_g^{\perp} and the orientation on \mathfrak{t}^{\perp} determined by the positive roots. Let $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t}, X) \in \operatorname{Pol}(\mathfrak{t})$ denote the *T*-equivariant Euler class for the oriented *T*-equivariant vector bundle $\mathfrak{g}_g/\mathfrak{t} \to \operatorname{pt}$. It is given by

$$\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t}, X) = \operatorname{sgn}(g)(-1)^{|\mathcal{R}_{g,+}|} \prod_{\alpha \in \mathcal{R}_{g,+}} \langle \alpha, X \rangle,$$

where $\mathcal{R}_{g,+} \subset \mathcal{R}_+$ is a set of positive roots of \mathfrak{g}_g , and $\operatorname{sgn}(g) = \pm 1$ is a sign depending on g (determined from the orientations).

1.3.4 Abelianization.

This subsection describes the 'abelianization' of a q-Hamiltonian G-space $\Phi : M \to G$, a construction introduced in [39] Section 3.2. This is a rough analog, for q-Hamiltonian spaces, of converting a Hamiltonian G-space into a Hamiltonian T-space by *projecting* the moment map to \mathfrak{t}^* . Consider the Ad(T)-equivariant smooth map $r: T \times \mathfrak{t}^{\perp} \to G$

$$r(t, X) = t \exp(X).$$

For a sufficiently small ball $B = B_{\epsilon}(\mathfrak{t}^{\perp})$ around the origin in \mathfrak{t}^{\perp} , the map r restricts to a T-equivariant diffeomorphism

$$r: T \times B \xrightarrow{\sim} U,$$

onto a tubular neighbourhood U of T in G. Let $\pi_T : U \simeq T \times B \to T$ (respectively $\pi_{t^{\perp}}$) denote the projection onto the first factor (respectively, second factor). Let $N = \Phi^{-1}(U) \subset M$, and define a smooth map $\Phi_a : N \to T$ by

$$\Phi_a = \pi_T \circ \Phi.$$

The pullback of the Cartan 3-form η to U is exact (its pullback to T vanishes). Applying the standard de Rham homotopy operator for the trivial bundle $U \simeq T \times \mathfrak{t}^{\perp} \to T$ to the pullback of η to U, one

¹We avoid using \mathfrak{t}^{\perp} since we use this to denote the *T*-invariant complement in \mathfrak{g} .

obtains a T-invariant primitive γ_a . Let $N := \Phi^{-1}(U) \subset M$. By construction, the 2-form

$$\omega_a = \omega - \Phi^* \gamma_a,$$

satisfies $d\omega_a = 0$.

Theorem 1.3.6 ([39] Proposition 3.4). (N, ω_a, Φ_a) is a (degenerate) q-Hamiltonian T-space, called the abelianization of M.

For later use, note that it follows from the construction and the fact that $\iota_T^*\eta = 0$ that

$$\iota_T^* \gamma_a = 0. \tag{1.12}$$

Let \mathcal{N} be the fibre product $N \times_T \mathfrak{t}$, and $\phi : \mathcal{N} \to \mathfrak{t}$ the corresponding map:

$$\begin{array}{cccc}
\mathcal{N} & \stackrel{\phi}{\longrightarrow} & \mathfrak{t} \\
\downarrow \exp & & \downarrow \exp \\
\mathcal{N} & \stackrel{\Phi_a}{\longrightarrow} & T
\end{array}$$
(1.13)

The covering map $\exp : \mathcal{N} \to N$ is the quotient by the action of the integral lattice Λ . Pulling ω_a back to \mathcal{N} , we obtain a (degenerate) Hamiltonian *T*-space ($\mathcal{N}, \exp^* \omega_a, \phi$). Since *N* is an open subset of *M*, *N* and \mathcal{N} both inherit an orientation. For use later, note that we have an $N_G(T)$ -equivariant map

$$\phi_{\mathfrak{g}/\mathfrak{t}} = \pi_{\mathfrak{t}^{\perp}} \circ \Phi \circ \exp : \mathcal{N} \to \mathfrak{g}/\mathfrak{t} \simeq \mathfrak{t}^{\perp}.$$

Comparing to pullback diagram (1.8) shows that the subset $\mathcal{X} = \Psi^{-1}(\mathfrak{t}) \subset \mathcal{M}$ of the corresponding Hamiltonian *LG*-space, can be identified with the subset $\exp^{-1}(X) \subset \mathcal{N}$, thus \mathcal{X} is the fibre product of X with \mathfrak{t} .

Chapter 2

Duistermaat-Heckman measures and Hamiltonian cobordism

In this chapter we prove a norm-square localization formula for twisted Duistermaat-Heckman distributions of a Hamiltonian LG-space.

We briefly outline the contents of the sections. In Section 1 we summarize results of Harada-Karshon, with small modifications needed for our purposes. As we rely on these results heavily, for completeness we have included brief outlines of the main proofs at the end of the chapter, in Section 2.5. In Section 1 we apply the Harada-Karshon Theorem to the (degenerate) Hamiltonian T-space described in Chapter 1: the covering space of the abelianization of the quasi-Hamiltonian G. This yields a (rather inexplicit) norm-square localization formula. To obtain more explicit formulas as in [48], [49], we explain how to compute the contributions in cross-sections. This takes some work, which is carried out in Section 2.2. Section 2.3 contains examples. The final three sections contain additional background material, including proof of an abelian localization theorem for total spaces of vector bundles used in Section 2.2, and some comments related to (non-)smoothness of the critical set.

2.1 Harada-Karshon approach to norm-square localization

In this section we describe a version of the Harada-Karshon (H-K) Theorem [26]. We restrict ourselves to the case that the localizing set Z is a smooth submanifold. This avoids some complications and has the appealing feature that the contribution from a component $Z_i \subset Z$ is especially simple to describe: it is the twisted DH distribution of any *polarized completion* of a tubular neighbourhood of Z_i . Only the smooth case is used in Section 2.2, as we eventually use a perturbation to ensure a smooth localizing set (some discussion of the situation when Z is not smooth is included in Section 2.6). A small difference needed later on in Section 2.2.2—between our setting and that in [26], is that we work with a moment map which might only be proper when restricted to the support of the equivariant cocycle α used to twist the Duistermaat-Heckman distribution. This explains why additional conditions on the support of α appear in the statements. Because of these differences and also for completeness, the proofs are outlined in Section 2.5; the interested reader should also consult the original paper [26] for a detailed treatment.

For $\xi \in \mathfrak{g}^*$, we write $\xi(\partial)$ for the directional derivative on \mathfrak{g}^* in the direction ξ . This extends to a map

 $p \in \operatorname{Pol}(\mathfrak{g}) = S\mathfrak{g}^* \mapsto p(\partial)$ from polynomials on \mathfrak{g} to constant-coefficient differential operators on \mathfrak{g}^* . An equivariant differential form $\alpha \in \Omega_G(M)$ can be decomposed as a sum $\alpha = \sum_k \alpha_k p_k$ where $\alpha_k \in \Omega(M)$, $p_k \in \operatorname{Pol}(\mathfrak{g})$. Let $\phi: M \to \mathfrak{g}^*$ be a smooth map. We define a linear map $C^{\infty}(\mathfrak{g}^*) \to \Omega(M)$ by

$$f \mapsto \alpha(-\partial) f \circ \phi := \sum_k \alpha_k \phi^*(p_k(-\partial) f).$$

For a couple of arguments it will be convenient to use equivariant currents. Let $\mathcal{C}(M)$ denote the space of currents on M, that is, the dual to the space $\Omega_{\text{comp}}(M)$ of compactly supported smooth differential forms. The space of equivariant currents is $\mathcal{C}_G(M) = (\mathcal{C}(M) \otimes \text{Pol}(\mathfrak{g}))^G$. Assuming M is oriented, $\Omega(M)$ can be identified with a subspace of $\mathcal{C}(M)$, and the above definitions extend naturally to $\mathcal{C}_G(M)$.

2.1.1 Twisted Duistermaat-Heckman measures.

Throughout this section, (N, ω, ϕ) will be an *oriented*, possibly degenerate, Hamiltonian G-space, that is, an oriented G-manifold N, equipped with a closed, equivariant 2-form

$$\omega_G(X) = \omega - \langle \phi, X \rangle, \qquad X \in \mathfrak{g}.$$

Let $\alpha \in \Omega_G(N)$ be a closed equivariant differential form, or more generally, a closed equivariant current.

We are interested in an invariant of the 4-tuple $(N, \omega, \phi, \alpha)$, the α -twisted Duistermaat-Heckman distribution, $DH(N, \omega, \phi, \alpha) \in \mathcal{D}'(\mathfrak{g}^*)$. These distributions were introduced by Jeffrey-Kirwan [28], and contain information about cohomology pairings on symplectic quotients. In the case that α is compactly supported, $DH(N, \omega, \phi, \alpha)$ can be defined in terms of its Fourier coefficients. For $X \in \mathfrak{g}$,

$$\langle \mathrm{DH}(N,\omega,\phi,\alpha), e^{-2\pi i \langle \cdot,X \rangle} \rangle := \int_{N} e^{\omega - 2\pi i \langle \phi,X \rangle} \alpha(2\pi iX).$$
 (2.1)

Note that the integrand is closed for the differential $d_{2\pi iX} := d - 2\pi i \iota(X_M)$. More generally we define

$$\langle \mathrm{DH}(N,\omega,\phi,\alpha), f \rangle = \int_N e^{\omega} \alpha(-\partial) f \circ \phi,$$
 (2.2)

for $f \in C^{\infty}_{\text{comp}}(\mathfrak{g}^*)$. To ensure that the integral exists, we require that ϕ be proper on the support of α . When the integral (compare to (2.1))

$$\varphi(X) = \int_{N} e^{\omega - 2\pi i \langle \phi, X \rangle} \alpha(2\pi i X)$$

defines a tempered generalized function, then $DH(N, \omega, \phi, \alpha)$ is the inverse Fourier transform of $\varphi(X)$.

In the case that $\alpha = \sum_k \alpha_k p_k$ is an equivariant current, the above integral (2.2) should be interpreted as a sum of pairings between currents α_k and forms $e^{\omega}\phi^*(p_k(-\partial)f)$. The DH distribution $DH(N, \omega, \phi, \alpha)$ can also be expressed in terms of push-forwards of currents:

$$DH(N, \omega, \phi, \alpha) = \sum_{k} p_k(\partial) \phi_*(e^{\omega} \alpha_k)^{[top]}.$$

Theorem 2.1.1. Let (N, ω, ϕ) be an oriented Hamiltonian G-space. Let α_1, α_2 be closed equivariant currents, and suppose $\alpha_1 - \alpha_2 = d_G \beta$ for some equivariant current β . Assume that ϕ is proper on the

support of $\alpha_1, \alpha_2, \beta$. Then

$$DH(N, \omega, \phi, \alpha_1) = DH(N, \omega, \phi, \alpha_2)$$

Proof. We must show $DH(N, \omega, \phi, d_G\beta) = 0$. If β is not compactly supported, let $\{\rho_k\}$ be a smooth G-invariant (locally finite) partition of unity on N with ρ_k compactly supported. For $f \in C^{\infty}_{\text{comp}}(\mathfrak{g}^*)$, $\operatorname{supp}(\beta) \cap \operatorname{supp}(\phi^* f)$ is compact, so intersects the supports of only finitely many of the functions ρ_k . Therefore for all but finitely many k

$$(d_G(\rho_k\beta))(-\partial)f \circ \phi = 0,$$

so that the sum can be brought outside the integral:

$$\langle \mathrm{DH}(N,\omega,\phi,d_G\beta),f\rangle = \sum_k \langle \mathrm{DH}(N,\omega,\phi,d_G(\rho_k\beta)),f\rangle.$$

Since this holds for any $f \in C^{\infty}_{\text{comp}}(\mathfrak{g}^*)$ we conclude

$$\mathrm{DH}(N,\omega,\phi,d_G\beta) = \sum_k \mathrm{DH}(N,\omega,\phi,d_G(\rho_k\beta)),$$

and each $\rho_k \beta$ is compactly supported.

So assume β is compactly supported, in which case $DH(N, \omega, \phi, d_G\beta)$ is a compactly supported distribution. Let $X \in \mathfrak{g}$. Using the formula for the Fourier coefficients (2.1)

$$\langle \mathrm{DH}(N,\omega,\phi,d_G\beta), e^{-2\pi i \langle \cdot,X \rangle} \rangle = \int_N e^{\omega - 2\pi i \langle \phi,X \rangle} d_{2\pi iX} \beta(2\pi iX)$$
$$= \int_N d_{2\pi iX} (\beta(2\pi iX) e^{\omega - 2\pi i \langle \phi,X \rangle})$$
$$= 0.$$

where the last line follows by Stokes' theorem.

Remark 2.1.2. It is not true in general that if $[\omega_1 - \phi_1] = [\omega_2 - \phi_2]$ in $H_G(N)$ then the corresponding Duistermaat-Heckman measures are equal (even for $N = \mathbb{R}^2$). Theorem 2.5.4 gives a sufficient condition. Remark 2.1.3. Later we will consider Duistermaat-Heckman distributions with the twisting cocycle involving a *G*-equivariant Thom form for the normal bundle to a submanifold. One use of Theorem 2.1.1 will be to show that the resulting distribution is independent of the choice of Thom form. Let $\iota: S \hookrightarrow N$ be an oriented *G*-invariant submanifold (without boundary), which is closed as a subset of *N*. Let $\alpha \in \Omega_G(N)$ be an equivariantly-closed differential form. Let *D* be a tubular neighbourhood of *S*, and suppose ϕ is proper on $\operatorname{supp}(\alpha) \cap \overline{D}$. Let τ_S , τ'_S be *G*-equivariant Thom forms for the normal bundle $\nu(S, N) \simeq D$, viewed as forms on *N* with support contained in *D*. By the Thom isomorphism theorem between cohomology of *S* and cohomology of *D* with compact vertical supports,

$$\tau_S - \tau'_S = d_G \beta_S$$

for some β with compact vertical support in D. Theorem 2.1.1 gives

$$\mathrm{DH}(N,\omega,\phi,\tau_S\alpha) = \mathrm{DH}(N,\omega,\phi,\tau'_S\alpha).$$

In the limit as the support of τ'_S is concentrated to S we obtain a current. Let δ_S be the closed equivariant current defined by S,

$$\langle \delta_S, \eta \rangle = \int_S \iota^* \eta, \qquad \eta \in \Omega_{\text{comp}}(N).$$

The difference $\delta_S - \tau_S$ is again exact (in $\mathcal{C}_G(N)$). For example, a d_G -primitive is

$$\beta = \int_0^\infty \Phi_t^* \big(\iota(\mathcal{E}) \tau_S \big) dt,$$

where \mathcal{E} is the Euler vector field on $\nu(S, N)$ with flow Φ_t , and $\iota(\mathcal{E})$ denotes contraction. From the formula one sees again supp $(\beta) \subset \overline{D}$, hence by Theorem 2.1.1

$$DH(N, \omega, \phi, \tau_S \alpha) = DH(N, \omega, \phi, \delta_S \alpha).$$

The right-side can be identified with $DH(S, \iota^*\omega, \iota^*\phi, \iota^*\alpha)$.

2.1.2 Taming maps and polarized completions.

Definition 2.1.4. A taming map $v : N \to \mathfrak{g}$ is a smooth *G*-equivariant map. A bounded taming map is one for which the map v is bounded. A taming map v defines a vector field v_N on N by

$$v_N(p) := (v(p))_N(p), \qquad p \in N.$$

The *localizing set* or *critical set* of v is

$$Z := \{ p \in N | v_N(p) = 0 \}.$$

We make the convention that the elements of $\phi(Z) \subset \mathfrak{g}^*$ will be referred to as *critical values*.

An example of a taming map is obtained from a G-invariant inner product \cdot on \mathfrak{g} . Using the inner product to identify \mathfrak{g} and \mathfrak{g}^* , we can take v to be the moment map itself. The localizing set for $v = \phi$ is

$$Z = G \cdot \bigcup_{\beta \in \mathfrak{t}_+^*} N^{\beta} \cap \phi^{-1}(\beta).$$

If ω is non-degenerate, the localizing set coincides with the set of critical points for the norm-square of the moment map [31].

An important reason for introducing taming maps is that they are a convenient way to prove that a map is *proper*, as a result of the following simple Lemma.

Lemma 2.1.5 ([26], Lemma 2.9). Let v be a bounded taming map on a G-manifold N, and let $\phi : N \to \mathfrak{g}^*$ be a continuous function. Suppose $\langle \phi, v \rangle$ is proper on a closed subset S. Then ϕ is proper on S.

Definition 2.1.6. Let α be a closed equivariant differential form on a Hamiltonian *G*-space (U, ω, ϕ) . Let v be a taming map with smooth localizing set $\iota_Z : Z \hookrightarrow U$. Suppose that $\langle \phi, v \rangle$ is proper and bounded below on $Z \cap \text{supp}(\alpha)$. A *v*-polarized completion of $(U, \omega, \phi, \alpha)$ is a Hamiltonian *G*-space $(U, \tilde{\omega}, \tilde{\phi})$ with $\iota_Z^*(\omega - \phi) = \iota_Z^*(\tilde{\omega} - \tilde{\phi})$, and such that $\langle \tilde{\phi}, v \rangle$ is proper and bounded below on the support of α .

In other words, $\tilde{\phi}$ is an *extension* of $\phi|_Z$ such that $\langle \tilde{\phi}, v \rangle$ is proper and bounded below on the support of α .

Definition 2.1.7. Let N be a G-manifold, Z a G-invariant submanifold, and U a G-invariant open set in N. We say that U can be *smoothly collapsed to part of* Z if there is a smooth G-invariant submanifold $Z' \subset Z \cap U$ and smooth G-equivariant map $p: [0, 1] \times U \to U$ such that

- 1. p_0 is the identity map on U,
- 2. $p_1(U) \subset Z'$,
- 3. $p_t(Z') \subset Z'$ for all $t \in [0, 1]$.

Mainly we will use G-invariant tubular neighbourhoods $U \supset Z$, which are clearly examples of the definition above. But for an argument in Section 2.2.5 it is convenient to have this slightly more flexible definition.

Remark 2.1.8. The definition says that the identity map $p_0 = \text{Id}_U$ and the map p_1 (which factors through Z'), are connected by a smooth, G-equivariant homotopy p. In particular, if α_0 , α_1 are closed equivariant differential forms on U whose pullbacks to Z' are equal, then α_0 , α_1 are cohomologous. H-K [26] work with a similar condition.

The next result is the main result on *existence* of polarized completions, and the *uniqueness* (under conditions) of the resulting DH distribution. For completeness we have included a proof in Section 2.5.

Lemma 2.1.9 ([26] Proposition 3.4, Lemmas 4.12, 4.17). Let α be a closed equivariant differential form on an oriented Hamiltonian G-space (N, ω, ϕ) . Let $v : N \to \mathfrak{g}$ be a bounded taming map with smooth localizing set Z. Let $U \supset Z \cap \operatorname{supp}(\alpha)$ be an open subset that can be smoothly collapsed to part of Z. Suppose that $\langle \phi, v \rangle$ is proper and bounded below on $Z \cap \operatorname{supp}(\alpha)$. Then

- 1. There exists a v-polarized completion $(U, \tilde{\omega}, \tilde{\phi})$ of the restriction $(U, \omega|_U, \phi|_U, \alpha|_U)$.
- 2. Suppose $v_i, Z_i, (U_i, \tilde{\omega}_i, \tilde{\phi}_i), i = 0, 1$ are two sets of data satisfying the above conditions, and such that v_0, v_1 agree on the support of α . Then

$$\mathrm{DH}(U_1, \tilde{\omega}_1, \tilde{\phi}_1, \alpha) = \mathrm{DH}(U_0, \tilde{\omega}_0, \tilde{\phi}_0, \alpha).$$

Note that Lemma 2.1.5 implies that $\tilde{\phi}$ is proper on the support of $\alpha|_U$, hence $DH(U, \tilde{\omega}, \tilde{\phi}, \alpha)$ is defined.

2.1.3 The Harada-Karshon Theorem.

Definition 2.1.10. Following [26], in the setting of Lemma 2.1.9, we define

$$DH_Z^v(N,\omega,\phi,\alpha) = DH(U,\tilde{\omega},\tilde{\phi},\alpha).$$
(2.3)

If $Z = \sqcup Z_i$ and $U = \sqcup U_i$, where U_i is an open G-invariant subset that can be smoothly collapsed to part of Z_i , define

$$\mathrm{DH}_{Z_i}^v(N,\omega,\phi,\alpha) = \mathrm{DH}(U_i,\tilde{\omega},\tilde{\phi},\alpha).$$

We have

$$\mathrm{DH}^v_Z(N,\omega,\phi,\alpha) = \sum_i \mathrm{DH}^v_{Z_i}(N,\omega,\phi,\alpha)$$

The right-hand-side of (2.3) is stable under various changes (which justifies the notation): according to Lemma 2.1.9, it is independent of the choice of open U and the choice of v-polarized completion. There is flexibility to modify the taming map v away from the support of α . Also, if the properness condition in Theorem 2.1.1 holds, then α can be replaced by a cohomologous form α' .

We can now state the main theorem of [26]. For completeness we have included a proof in Section 2.5.

Theorem 2.1.11 ([26], Theorem 5.20). Consider the setting of Lemma 2.1.9. Suppose further that $\langle \phi, v \rangle$ is proper and bounded below on the support of α . Then

$$DH(N,\omega,\phi,\alpha) = DH_Z^v(N,\omega,\phi,\alpha).$$
(2.4)

Remark 2.1.12. This result is related to earlier work on the subject, in particular the work of Paradan [48], [49] and Woodward [65]. As examples, one might look to Proposition 3.3 in [48] (a general localization result) and Proposition 2.9 in [49] (a formula for the contribution of a smooth component of the localizing set, in terms of a twisted DH distribution for a tubular neighbourhood $U \supset Z$). Theorem 5.1 in [65] is a norm-square localization formula, expressed in terms of twisted DH distributions of small submanifolds around the critical set.

The condition that v be bounded can be relaxed. If v is unbounded, but is bounded on a neighbourhood $\iota_i : N_i \hookrightarrow N$ of each component Z_i of Z, then we can extend the definition:

$$DH_{Z_i}^v(N,\omega,\phi,\alpha) := DH_{Z_i}^v(N_i,\iota_i^*\omega,\iota_i^*\phi,\iota_i^*\alpha)$$

The flexibility noted in Definition 2.1.10 holds in this setting. It is also often possible to apply Theorem 2.1.11 (notably for the case $v = \phi$, which can be unbounded):

Corollary 2.1.13. Consider the setting of Theorem 2.1.11, except that v may be unbounded. Suppose however that the subset $||v||^2(Z) \subset \mathbb{R}$ is discrete, and set $Z_r = Z \cap (||v||^2)^{-1}(r)$. Then

$$DH(N,\omega,\phi,\alpha) = \sum_{r} DH_{Z_r}^v(N,\omega,\phi,\alpha).$$
(2.5)

Proof. Since $||v||^2(Z) \subset \mathbb{R}$ is discrete, it is possible to find a smooth, positive, non-increasing function $f \in C^{\infty}(\mathbb{R}_{\geq 0})$, constant near each $r \in \mathbb{R}_{\geq 0}$ such that $Z_r \neq \emptyset$, and vanishing sufficiently rapidly at infinity, such that the modified taming map $v' = f(||v||^2)v$ is bounded. The localizing sets for v, v' are the same, and Theorem 2.1.11 can be applied, which gives the right-hand-side of (2.5) except for v'. Since $f(||v||^2)$ is constant near each $Z_r \neq \emptyset$, the condition that $(U_r, \tilde{\omega}_r, \tilde{\phi}_r)$ is a v'-polarized completion is equivalent to it being a v-polarized completion, since we can take U_r sufficiently small that $f(||v||^2)$ is constant on U_r . Thus $\mathrm{DH}_{Z_r}^{v'}(N, \omega, \phi, \alpha) = \mathrm{DH}_{Z_r}^v(N, \omega, \phi, \alpha)$.

The following application of Remark 2.1.3 will be useful in Section 2.

Lemma 2.1.14. Let (N', ω, ϕ) be an oriented Hamiltonian G-space, and α a closed equivariant differential form. Let $\iota: S \hookrightarrow N'$ be an oriented G-invariant submanifold (without boundary), which is closed as a subset of N'. Let D be a tubular neighbourhood of S, and suppose ϕ is proper on $\operatorname{supp}(\alpha) \cap \overline{D}$. Let τ_S be a G-equivariant Thom form for the normal bundle $\nu(S, N') \simeq D$, viewed as a form on N' with

support contained in D. Let $v : N' \to \mathfrak{g}$ be a bounded taming map with smooth localizing set Z, such that $Z_S := S \cap Z$ is also smooth. Suppose $\langle \phi, v \rangle$ is proper and bounded below on $Z \cap \operatorname{supp}(\alpha) \cap \overline{D}$. Then

$$DH_Z^v(N',\omega,\phi,\tau_S\alpha) = DH_{Z_S}^v(S,\iota^*\omega,\iota^*\phi,\iota^*\alpha).$$

Proof. The left-side is given by

$$\mathrm{DH}_{Z}^{v}(N',\omega,\phi,\tau_{S}\alpha) = \mathrm{DH}(U',\tilde{\omega},\tilde{\phi},\tau_{S}\alpha),$$

for a suitable v-polarized completion $(U', \tilde{\omega}, \tilde{\phi})$. Since ϕ is proper and bounded below on $Z \cap \text{supp}(\alpha) \cap \overline{D}$, and v is bounded, $\tilde{\phi}$ can be taken to be proper and bounded below on $\text{supp}(\alpha) \cap \overline{D}$ (c.f. construction of $\tilde{\phi}$ in Section 2.5). Using Remark 2.1.3, we can replace τ_S with the current δ_S :

$$DH_Z^v(N',\omega,\phi,\tau_S\alpha) = DH(U',\tilde{\omega},\tilde{\phi},\delta_S\alpha) = DH_Z^v(N',\omega,\phi,\delta_S\alpha).$$
(2.6)

Let U_S be a *G*-invariant tubular neighbourhood of Z_S in *S*, and let *U* be a *G*-invariant tubular neighbourhood of U_S in *N'*. Note that *U* contains $Z \cap \text{supp}(\delta_S \alpha) = Z_S \cap \text{supp}(\alpha)$. Moreover, *U* can be smoothly collapsed to part of *Z* (first collapse *U* to U_S , then collapse U_S to Z_S). Let $(U, \tilde{\omega}, \tilde{\phi})$ be a *v*-polarized completion of $(U, \omega, \phi, \delta_S \alpha)$. Thus

$$DH_{Z}^{v}(N',\omega,\phi,\delta_{S}\alpha) = DH(U,\tilde{\omega},\tilde{\phi},\delta_{S}\alpha) = DH(U_{S},\iota^{*}\tilde{\omega},\iota^{*}\tilde{\phi},\iota^{*}\alpha),$$
(2.7)

where the second equality follows because δ_S is the current defined by S. As U_S can be smoothly collapsed to S and $(U_S, \iota^* \tilde{\omega}, \iota^* \tilde{\phi})$ is a $v|_S$ -polarized completion of $(S, \iota^* \omega, \iota^* \phi, \iota^* \alpha)$ we have

$$DH(U_S, \iota^* \tilde{\omega}, \iota^* \tilde{\phi}, \iota^* \alpha) = DH_{Z_S}^v(S, \iota^* \omega, \iota^* \phi, \iota^* \alpha).$$
(2.8)

Combining (2.6), (2.7), (2.8) gives the result.

2.2 Norm-square localization formula

In this section we apply the Harada-Karshon Theorem to $(\mathcal{N}, \omega_a, \phi)$ to obtain a norm-square localization result for Hamiltonian *LG*-spaces (or equivalently, for q-Hamiltonian *G*-spaces). In the later subsections, local models for the polarized completions are described, and these are used to derive more explicit formulas for the contributions. Through most of this section, we identify \mathfrak{t} and \mathfrak{t}^* using an invariant inner product on \mathfrak{g} . We use notation introduced in Section 3 of Chapter 1 for the q-Hamiltonian space M, its abelianization N, the Hamiltonian covering space $(\mathcal{N}, \exp^* \omega_a, \phi)$, and so on.

2.2.1 Twisted DH distributions for a q-Hamiltonian space.

The twisted DH distributions that we study in this paper are not the same as those defined in [5] (which are conjugation-invariant distributions on the group G), but are closely related to them by an induction-type map. We briefly indicate the relation in remark 2.2.1 below. The type of DH distribution considered here was first introduced for q-Hamiltonian spaces in [39], where further results about DH distributions of this kind can be found. The idea of using a smooth cut-off form, playing the role of

a Poincare dual to a possibly singular subset, appeared in a paper of Jeffrey-Kirwan (see Section 5 in [27]).

To a *G*-equivariant cocycle $\alpha \in \Omega_G(M)$, we will associate a Λ -periodic, *W*-anti-symmetric distribution \mathfrak{m}^{α} on $\mathfrak{t} \simeq \mathfrak{t}^*$. The distribution \mathfrak{m}^{α} is intended to represent the α -twisted DH distribution of the *T*-space $\mathcal{X} = \Psi^{-1}(\mathfrak{t}) \subset \mathcal{M}$, the only problem being that the latter might not be smooth. To deal with this, we work instead with the slightly larger space $\mathcal{N} \supset \mathcal{X}$, together with a cut-off form $\tau_{\mathcal{N}}$ supported near \mathcal{X} that plays the role of a Poincare dual to \mathcal{X} .

Recall from Chapter 1, Section 1.3.4, U is a T-invariant tubular neighbourhood of T in G. Without loss of generality, we can take U to be $N_G(T)$ -invariant, where $N_G(T)$ is the normalizer of T in G. Let τ denote a T-equivariant Thom form for the vector bundle $\pi_T : U \simeq T \times \mathfrak{t}^{\perp} \to T$, with compact support $\operatorname{supp}(\tau) \subset D$ where D is a smaller tubular neighbourhood of T such that $\overline{D} \subset U$. Let $\tau_{\mathcal{N}} = \exp^* \Phi^* \tau$. The twisted DH distribution that we will study for the remainder of the chapter is

$$\mathfrak{m}^{\alpha} := \mathrm{DH}(\mathcal{N}, \exp^* \omega_a, \phi, \tau_{\mathcal{N}} \cdot \exp^* \alpha)$$

(In the future, pullbacks by exp will be omitted from the notation.) This is well-defined since ϕ is proper on $\exp^{-1}(\Phi^{-1}(\overline{D}))$ which contains the support of τ_N . It is independent of the choice of tubular neighbourhood $U \supset T$ and of the choice of τ by Theorem 2.1.1. The distribution \mathfrak{m}^{α} is alternating under the action of the affine Weyl group $W_{\text{aff}} = W \ltimes \Lambda$; this follows from the *G*-equivariance of the cocycle α , and the behaviour of the Thom form τ under the action of the normalizer $N_G(T)$ of *T* in *G* (for g_w a representative of $w \in W$, the pullback $g_w^* \tau(X)$ is cohomologous to $(-1)^{|w|} \tau(w^{-1} \cdot X)$).

If Ψ is transverse to \mathfrak{t} so that \mathcal{X} is smooth, then $\tau_{\mathcal{N}}$ is Poincare dual to \mathcal{X} , and it follows (similar to Lemma 2.1.14) that $\mathfrak{m}^{\alpha} = \mathrm{DH}(\mathcal{X}, \exp^* \omega_a, \phi, \exp^* \alpha)$, where the orientation of \mathcal{X} is determined by the orientations of \mathcal{N} and \mathfrak{t}^{\perp} .

As already mentioned, the main reason for introducing the twisted DH distributions \mathfrak{m}^{α} is that they encode certain cohomology pairings on reduced spaces—see [39] for the exact relation. As an example, consider the distribution \mathfrak{m} ($\alpha = 1$), and assume Φ has a non-trivial regular value. In this case $\mathfrak{m} = f d \operatorname{vol}_{\mathfrak{t}}$ for a Λ -periodic, W-anti-symmetric function f, which is polynomial on the chambers of an affine hyperplane arrangement in \mathfrak{t} (and $d \operatorname{vol}_{\mathfrak{t}}$ is the measure induced on \mathfrak{t} by the inner product). If $\xi \in \mathfrak{t}$ is a regular value of ϕ and $g = \exp(\xi)$ is a regular element of T (i.e. $G_q = T$), then

$$\operatorname{vol}(\Phi^{-1}(\exp(\xi))/T) = \frac{d}{\operatorname{vol}(T)} |f(\xi)|,$$

where d is the size of the generic stabilizer of the fibre $\Phi^{-1}(\exp(\xi))$ and $\operatorname{vol}(T)$ is the volume of T with respect to the metric induced by the inner product.

Remark 2.2.1. We briefly indicate the relation of \mathfrak{m}^{α} to the more usual *G*-invariant twisted Duistermaat-Heckman distributions; see [39] sections 4, 5 for a more detailed discussion. Consider first the case of a *G*-equivariant cocycle α on a proper *Hamiltonian G*-space (N, ω, ϕ) . Let $\phi_{\mathfrak{t}}$ and $\phi_{\mathfrak{t}^{\perp}}$ be the components of ϕ with respect to the orthogonal direct sum $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^{\perp}$. Let $U \subset N$ be the inverse image under $\phi_{\mathfrak{t}^{\perp}}$ of a small ball *B* around the origin in \mathfrak{t}^{\perp} , and let τ_U be the pullback to *U* of a *T*-equivariant Thom form with support contained in *B*. We have the following relation between twisted Duistermaat-Heckman distributions

$$DH(U,\omega,\phi_{\mathfrak{t}},\tau_U\cdot\alpha) = R_{\mathfrak{g}}(DH(N,\omega,\phi,\alpha)), \qquad (2.9)$$

where $R_{\mathfrak{g}}$ is an *isomorphism*

$$R_{\mathfrak{g}}: \mathcal{D}'(\mathfrak{g}^*)^G \to \mathcal{D}'(\mathfrak{t}^*)^{W-\text{anti}}$$

Up to a constant normalization, this map is the inverse of the dual map to the isomorphism

$$f \in C^{\infty}_{\operatorname{comp}}(\mathfrak{g}^{*})^{G} \mapsto f|_{\mathfrak{t}^{*}} \cdot \prod_{\alpha > 0} \langle \alpha, - \rangle \in C^{\infty}_{\operatorname{comp}}(\mathfrak{t}^{*})^{W-\operatorname{anti}}.$$

(Any W-anti-symmetric smooth function can be divided by $\prod_{\alpha>0} \langle \alpha, - \rangle$, and the result extends to a smooth G-invariant function.) The distribution $DH(U, \omega, \phi, \tau_U \cdot \alpha)$ therefore carries the same information. In the case of the Hamiltonian T-space \mathcal{N} above, we are in a situation in which the left side of (2.9) is defined, but the right side is not, as the G-space N does not exist.

The relation between \mathfrak{m}^{α} , viewed as a *W*-anti-symmetric distribution on *T*, and the conjugationinvariant distributions studied in [5] is similar. Assuming (for simplicity) that the half-sum of the positive roots ρ is a weight of *G*, the relation is given by an isomorphism:

$$R_G: \mathcal{D}'(G)^G \to \mathcal{D}'(T)^{W-\text{anti}}.$$

Up to a constant normalization, this map is the inverse of the dual map to the isomorphism

$$f \in C^{\infty}(G)^G \mapsto \left(f\big|_T\right) \cdot \sum_{w \in W} (-1)^{\ell(w)} e_{w\rho} \in C^{\infty}(T)^{W-\text{anti}}$$

given by restriction to T, followed by multiplication by the Weyl denominator.

2.2.2 Application of the Harada-Karshon theorem.

The map $\phi : \mathcal{N} \to \mathfrak{t}^*$ introduced in the previous section is *T*-equivariant, hence defines a taming map. However, the localizing set for this taming map need not be smooth. To get around this problem, we perturb the taming map by a small 'generic' vector $\gamma \in \mathfrak{t}$, letting

$$v = \phi - \gamma.$$

Then $\langle \phi, v \rangle = \|\phi\|^2 - \langle \phi, \gamma \rangle$ is proper and bounded below on the support of $\tau_{\mathcal{N}}$ (its behavior at infinity is dominated by the $\|\phi\|^2$ term). We will explain what we mean by 'generic' in the next section, but it will imply that the localizing set

$$\mathcal{Z} = \{v_{\mathcal{N}} = 0\} \subset \mathcal{N}$$

has some desirable properties: Z will be smooth, and each component $Z_{\beta} = \mathcal{N}^{\beta-\gamma} \cap \phi^{-1}(\beta)$ of Z will intersect the corresponding cross-section Y_g , $g = \exp(\beta)$ in a smooth compact submanifold Z_{β} . This will lead (eventually, in Section 2.2.7) to a more explicit formula for the contribution, in terms of an integral over Z_{β} and not involving the Thom form τ .

We have shown that the 4-tuple $(\mathcal{N}, \omega_a, \phi, \tau_{\mathcal{N}} \alpha)$ equipped with the taming map v satisfy all the conditions required in the Harada-Karshon Theorem 2.1.11 (see also Corollary 2.1.13), which gives the following 'norm-square localization formula'.

Theorem 2.2.2. Let $\Phi : M \to G$ be a q-Hamiltonian G-space, and let $\phi : \mathcal{N} \to \mathfrak{t}$ be the covering space of its abelianization, as above. Let $v = \phi - \gamma$ for a generic vector γ . Let $\tau_{\mathcal{N}}$ be the pullback of the Thom

form. Then

$$DH(\mathcal{N}, \omega_a, \phi, \tau_{\mathcal{N}} \alpha) = DH^v_{\mathcal{Z}}(\mathcal{N}, \omega_a, \phi, \tau_{\mathcal{N}} \alpha).$$
(2.10)

It will take some work to extract a more explicit description of the right-hand-side of this equation. As a first step, note that

$$\mathcal{Z} = \bigcup_{\beta \in \mathfrak{t}} \mathcal{N}^{\overline{\beta}} \cap \phi^{-1}(\beta),$$

where

$$\beta := \beta - \gamma.$$

We will refer to the set

$$\mathcal{Z}_{\beta} = \mathcal{N}^{\overline{\beta}} \cap \phi^{-1}(\beta)$$

as the β -component of the localizing set (it might not be connected).

2.2.3 Relation to $\|\Psi\|^2$.

By taking the support of the Thom form τ to be sufficiently small in the 'vertical' (\mathfrak{t}^{\perp}) direction, we can assume that $\tau_{\mathcal{N}}$ vanishes identically on each component of \mathcal{Z}_{β} which does not intersect $\mathcal{X} = \exp^{-1}(\Phi^{-1}(T))$, so that these components give trivial contribution to (2.10). Recall that \mathcal{X} can be identified with the set $\Psi^{-1}(\mathfrak{t})$, inside the corresponding Hamiltonian *LG*-space \mathcal{M} , and $\phi|_{\mathcal{X}} = \Psi|_{\mathcal{X}}$. Under this identification

$$\mathcal{Z}_{\beta} \cap \mathcal{X} = \mathcal{M}^{\beta} \cap \Psi^{-1}(\beta). \tag{2.11}$$

If $\gamma = 0$, this simplifies to

$$\mathcal{Z}_{\beta} \cap \mathcal{X} = \mathcal{M}^{\beta} \cap \Psi^{-1}(\beta).$$

By Proposition 1.3.2, the latter is non-empty iff $\beta \in W \cdot \mathcal{B}$, where \mathcal{B} indexes components of the critical set of $\|\Psi\|^2$. Thus if $\gamma = 0$ then the non-trivial contributions to the norm-square localization formula (2.10) are indexed by the set $W \cdot \mathcal{B}$. More generally if $\gamma \neq 0$ is small and generic, then the localizing set \mathcal{Z} can be viewed as a kind of desingularization (c.f. [48], [49]) of the intersection of \mathcal{X} with the critical set of $\|\Psi\|^2$. We next explain this in more detail.

Let $\mathfrak{t}_s, s \in \mathcal{S}$ index the sub-algebras of \mathfrak{t} which arise as infinitesimal stabilizers of points in N.¹ Each connected component of $\mathcal{N}^{\mathfrak{t}_s}$ is mapped by ϕ into an affine subspace $\Delta \subset \mathfrak{t}$ which is a translate of the orthogonal complement to \mathfrak{t}_s .

In this way, we obtain a periodic collection \mathcal{W} of affine subspaces Δ . For example, for $\mathfrak{t}_s = 0$ we obtain $\Delta = \mathfrak{t}$. For $\Delta \in \mathcal{W}$, let \mathfrak{t}_{Δ} be the subspace orthogonal to Δ (this is one of the sub-algebras $\mathfrak{t}_s, s \in \mathcal{S}$); this is a rational subalgebra, in the sense that $T_{\Delta} := \exp(\mathfrak{t}_{\Delta})$ is closed. Let $\mathfrak{t}_{\Delta}^{\perp}$ denote the orthogonal complement to \mathfrak{t}_{Δ} ; the affine subspace Δ is a translate of $\mathfrak{t}_{\Delta}^{\perp}$.

For an affine subspace Δ and point $x \in \mathfrak{t}$, let $\operatorname{pr}_{\Delta}(x) \in \Delta$ denote the orthogonal projection of x onto Δ . Since \mathfrak{t}_{Δ} , Δ are orthogonal, $\operatorname{pr}_{\Delta}(\gamma) = \beta' \Rightarrow \overline{\beta'} = \beta' - \gamma \in \mathfrak{t}_{\Delta}$. Thus there is a decomposition

$$\mathcal{Z}_{\beta'} = \bigcup_{\mathrm{pr}_{\Delta}(\gamma) = \beta'} \mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta'), \qquad (2.12)$$

¹In our case, \mathcal{S} is finite, since M is compact. The discussion here goes through more generally however, see [48], [49].

the union being indexed by a subset of \mathcal{W} . Taking the intersection with \mathcal{X} we obtain

$$\mathcal{Z}_{\beta'} \cap \mathcal{X} = \bigcup_{\mathrm{pr}_{\Delta}(\gamma) = \beta'} \mathcal{M}^{\mathfrak{t}_{\Delta}} \cap \Psi^{-1}(\beta').$$
(2.13)

In particular, the critical values of $v = \phi - \gamma$ are contained in the set

$$\{\operatorname{pr}_{\Delta}(\gamma)|\Delta \in \mathcal{W}\}.$$

Likewise, $W \cdot \mathcal{B}$ is contained in the set

$$\{\operatorname{pr}_{\Delta}(0)|\Delta \in \mathcal{W}\}.$$

For each $\beta \in W \cdot \mathcal{B}$, introduce the set

$$S(\beta) := \{ \operatorname{pr}_{\Delta}(\gamma) | \Delta \in \mathcal{W}, \operatorname{pr}_{\Delta}(0) = \beta \}.$$

Since $W \cdot \mathcal{B}$ is discrete, if γ is sufficiently small, the finite sets $S(\beta)$ are disjoint.

Proposition 2.2.3. If γ is sufficiently small and $\beta' \in \mathfrak{t}$ is such that $\mathcal{Z}_{\beta'} \cap \mathcal{X} \neq \emptyset$, then $\beta' \in S(\beta)$ for some $\beta \in W \cdot \mathcal{B}$.

Proof. By (2.13), $\mathcal{Z}_{\beta'} \cap X$ is non-empty iff there is a $\Delta \in \mathcal{W}$ such that $\operatorname{pr}_{\Delta}(\gamma) = \beta'$ and

$$\mathcal{M}^{\mathfrak{t}_{\Delta}} \cap \Psi^{-1}(\beta') \neq \emptyset.$$

Let $\beta = \operatorname{pr}_{\Delta}(0)$. Each component $C \subset \mathcal{M}^{\mathfrak{t}_{\Delta}} \cap \Psi^{-1}(\Delta)$ is mapped by Ψ onto a closed set in Δ . By adjusting γ , we can make $\beta' = \operatorname{pr}_{\Delta}(\gamma)$ as close as we like to $\beta = \operatorname{pr}_{\Delta}(0)$, and so ensure that $\beta' \in \Psi(C)$ $\Rightarrow \beta \in \Psi(C)$. It follows that for γ sufficiently small,

$$\mathcal{M}^{\mathfrak{t}_{\Delta}} \cap \Psi^{-1}(\beta') \neq \emptyset \Rightarrow \mathcal{M}^{\mathfrak{t}_{\Delta}} \cap \Psi^{-1}(\beta) \neq \emptyset.$$

And thus $\beta \in W \cdot \mathcal{B}$.

The non-trivial contributions in (2.10) are indexed by the set of $\beta' \in \mathfrak{t}$ for which $\mathcal{Z}_{\beta'} \cap \mathcal{X} \neq \emptyset$. The Proposition shows that, for γ sufficiently small, this set is contained in the disjoint union of the finite sets $S(\beta), \beta \in W \cdot \mathcal{B}$. If we define

$$\mathfrak{m}_{\beta} = \sum_{\beta' \in S(\beta)} \mathrm{DH}^{v}_{\mathcal{Z}_{\beta'}}(\mathcal{N}, \omega_{a}, \phi, \tau_{\mathcal{N}}\alpha),$$

then (2.10) yields the norm-square decomposition described in the introduction:

$$DH(\mathcal{N}, \omega_a, \phi, \tau_{\mathcal{N}} \alpha) = \sum_{\beta \in W \cdot \mathcal{B}} \mathfrak{m}_{\beta}.$$
(2.14)

Remark 2.2.4. The H-K results can also be applied to $v_1 = \phi$ directly (without perturbing by γ), with $Z_1 = \{(v_1)_{\mathcal{N}} = 0\}$ possibly singular. See the discussion in Section 2.6, where we explain that the norm-square contributions for v_1 are the terms \mathfrak{m}_{β} in the formula above. In particular this implies that the terms are W-anti-symmetric: $w_*\mathfrak{m}_{\beta} = (-1)^{\ell(w)}\mathfrak{m}_{w\beta}$.

2.2.4 Genericity conditions.

Let $\beta \in \mathfrak{t}$, and $g = \exp(\beta)$. The root space decomposition for the infinitesimal stabilizer \mathfrak{g}_q is

$$\mathfrak{g}_g^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\mathcal{R}_g} \mathfrak{g}_\alpha, \tag{2.15}$$

where \mathcal{R}_g consists of those roots $\alpha \in \mathcal{R}$ such that $g^{\alpha} = e^{2\pi i \langle \alpha, \beta \rangle} = 1$, i.e. such that $\langle \alpha, \beta \rangle \in \mathbb{Z}$. The *Stieffel diagram* is the affine hyperplane arrangement in \mathfrak{t} consisting of all $H_{\alpha,n} = \{\xi \in \mathfrak{t} | \langle \alpha, \xi \rangle = n\}$ where $\alpha \in \mathcal{R}, n \in \mathbb{Z}$. Let σ be an open face of the Stieffel diagram. It follows from (2.15) that for $\beta \in \sigma$, the infinitesimal stabilizer $\mathfrak{g}_{\exp(\beta)} \supset \mathfrak{t}$ is independent of β ; we denote it by \mathfrak{g}_{σ} .

In order to derive a more explicit formula for the contribution, it will be convenient to choose γ (the center of the decomposition) sufficiently generic, similar to [48], [49].

Proposition 2.2.5. There is an open dense set (the complement of a periodic hyperplane arrangement in \mathfrak{t}) of points $\gamma \in \mathfrak{t}$ such that for all $\Delta, \Delta' \in \mathcal{W}$ and all open faces σ of the Stieffel diagram we have

- 1. $\Delta' \subsetneq \Delta \Rightarrow \operatorname{pr}_{\Delta}(\gamma) \neq \operatorname{pr}_{\Delta'}(\gamma)$.
- 2. $\operatorname{pr}_{\Delta}(\gamma) \in \sigma \Rightarrow \Delta \subset \langle \sigma \rangle$.

Here $\langle \sigma \rangle$ denotes the affine extension of σ .

Remark 2.2.6. The first condition was introduced by Paradan ([48], around Proposition 6.9; see there for more discussion); we will see that it ensures that the localizing set for $v = \phi - \gamma$ is smooth. The second condition is an additional one, see Proposition 2.2.10. The second condition will ensure that the intersection of each component of the localizing set with a cross-section is compact: let $\mathcal{Z}_{\beta} = \mathcal{N}^{\overline{\beta}} \cap \phi^{-1}(\beta)$ (where $\overline{\beta} = \beta - \gamma$), we will show that

$$Z_{\beta} := \mathcal{Z}_{\beta} \cap Y_q, \qquad g = \exp(\beta),$$

is contained in $\mathcal{X} = \Psi^{-1}(\mathfrak{t}^*)$, hence is compact.

Proof. Since Δ' has lower dimension than Δ , one can perturb γ to ensure $\operatorname{pr}_{\Delta}(\gamma) \neq \operatorname{pr}_{\Delta'}(\gamma)$. Likewise in the second condition, if Δ is not contained in $\langle \sigma \rangle$, then $\Delta \cap \langle \sigma \rangle$ has strictly smaller dimension than Δ , and so a small perturbation of γ will ensure that $\operatorname{pr}_{\Delta}(\gamma) \notin \sigma$.

Definition 2.2.7. We say that $\gamma \in \mathfrak{t}$ is *generic* if it is in the open dense set described in the previous proposition.

Proposition 2.2.8. Let $\gamma \in \mathfrak{t}$ be generic. Then the union (2.12) is disjoint. Equivalently, for all $p \in \mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$, the stabilizer of p in \mathfrak{t} is exactly \mathfrak{t}_{Δ} . Thus $\mathfrak{t}_{\Delta}^{\perp}$ acts locally freely on $\mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$.

Proof. If $p \in \mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$ had larger stabilizer, then $\beta = \phi(p)$ would be contained in some $\Delta' \in \mathcal{W}$ with $\mathfrak{t}_{\Delta'} \supset \mathfrak{t}_{\Delta} \Rightarrow \Delta' \subset \Delta$. In this case, $\operatorname{pr}_{\Delta}(\gamma) = \operatorname{pr}_{\Delta'}(\gamma) = \beta$, which contradicts the genericity assumption.

Corollary 2.2.9. Let $\gamma \in \mathfrak{t}$ be generic, $\beta = \mathrm{pr}_{\Delta}(\gamma)$. Then $\overline{\beta}$ acts with non-zero weights on the normal bundle $\nu(\mathcal{N}^{\mathfrak{t}_{\Delta}}, \mathcal{N})|_{\mathcal{Z}_{\beta}}$.

Proof. Without loss of generality assume \mathcal{Z}_{β} is connected and let C be the component of $\mathcal{N}^{\mathfrak{t}_{\Delta}}$ containing \mathcal{Z}_{β} . Suppose $\overline{\beta}$ acts trivially on a non-trivial subbundle ν' of $\nu(C, \mathcal{N})$. Then a slightly enlarged submanifold $C' \subset \mathcal{N}^{\overline{\beta}}$ containing C can be found, with $TC'|_{C} = TC \oplus \nu'$. Since \mathfrak{t}_{Δ} acts non-trivially on ν' , the generic infinitesimal stabilizer of C' is strictly smaller than \mathfrak{t}_{Δ} , and thus there is a $\Delta' \in \mathcal{W}$ properly containing Δ . Since $\overline{\beta} \in \mathfrak{t}_{\Delta'}$, $\operatorname{pr}_{\Delta'}(\gamma) = \beta$, and $\mathcal{N}^{\mathfrak{t}_{\Delta'}} \cap \phi^{-1}(\beta) \supset \mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$. This contradicts Proposition 2.2.8.

Proposition 2.2.10. Let $\gamma \in \mathfrak{t}$ be generic, $\Delta \in \mathcal{W}$, $\beta = \operatorname{pr}_{\Delta}(\gamma)$, and $g = \exp(\beta)$. Let $\sigma \subset \mathfrak{t}$ be the open face of the Stieffel diagram containing β , and let \mathfrak{g}_{σ} be the corresponding infinitesimal stabilizer. Then:

- 1. The centralizer of \mathfrak{t}_{Δ} in \mathfrak{g}_{σ} is \mathfrak{t} .
- 2. There is a slice U_g around g for the conjugation action of G on itself, such that $U_q^{\mathfrak{t}_{\Delta}} = U_g \cap T$.
- 3. Let $Y_g = \Phi^{-1}(U_g)$ be the corresponding cross-section. Then $Y_g^{\mathfrak{t}_{\Delta}} \subset X = \Phi^{-1}(T)$.

Proof. The root space decomposition for \mathfrak{g}_{σ} is:

$$\mathfrak{g}^{\mathbb{C}}_{\sigma} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha|_{\sigma} \in \mathbb{Z}} \mathfrak{g}_{\alpha}.$$

Suppose the root space $\mathfrak{g}_{\alpha} \subset \mathfrak{g}_{\sigma}^{\mathbb{C}}$ is fixed by \mathfrak{t}_{Δ} . Then:

$$\alpha|_{\sigma} \in \mathbb{Z}$$
 and $\alpha|_{\mathfrak{t}_{\Lambda}} = 0.$

Let σ_0 be the subspace parallel to σ which passes through 0. Then $\alpha|_{\sigma_0}$ is a fixed integer, hence must be 0. Thus

$$\alpha(\sigma_0 + \mathfrak{t}_{\Delta}) = 0$$

But Δ is contained in the affine extension of σ . This implies that the subspace \mathfrak{t}_{Δ} orthogonal to Δ contains σ_0^{\perp} . Therefore

$$\sigma_0 + \mathfrak{t}_\Delta = \mathfrak{t},$$

and $\alpha = 0$. We have thus shown that $\mathfrak{g}_{\sigma}^{\mathfrak{t}_{\Delta}} = \mathfrak{t}$ as desired.

Recall $\mathfrak{g}_g = \mathfrak{g}_\sigma$ is the Lie algebra of G_g . The first statement shows that the centralizer $C_{G_g}(T_\Delta) = G_g^{\mathfrak{t}_\Delta}$ has Lie algebra \mathfrak{t} , and thus its identity component is T. Since $g \in T$, we can choose a slice U_g around g sufficiently small that it only intersects the identity component T of $G_g^{\mathfrak{t}_\Delta}$, hence $U_g^{\mathfrak{t}_\Delta} = U_g \cap T$. That $Y_g^{\mathfrak{t}_\Delta} \subset X = \Phi^{-1}(T)$ follows from equivariance of the moment map Φ .

Remark 2.2.11. We will further assume that U_g is chosen sufficiently small that exp restricts to a diffeomorphism on each component of $\exp^{-1}(U_g \cap T) \subset \mathfrak{t}$. Considering the pullback diagram (1.13), we can identify $U_g \cap T$ with the unique component of $\exp^{-1}(U_g \cap T)$ containing β . Similarly, the cross-section $Y_g^{\mathfrak{t}_{\Delta}}$ is identified with the corresponding subset of $\mathcal{X} \subset \mathcal{N}$. By taking U_g smaller if necessary, we can assume that the only translate of $\mathfrak{t}_{\Delta}^{\perp}$ in \mathcal{W} which meets $U_g \cap T$ is $\Delta = \beta + \mathfrak{t}_{\Delta}^{\perp}$, and thus $\phi(Y_g^{\mathfrak{t}_{\Delta}}) \subset \Delta$.

Corollary 2.2.12. In the setting of Proposition 2.2.10 and making the identification in Remark 2.2.11, $(Y_a^{t_{\Delta}}, \omega_a, \phi)$ is a (non-degenerate) Hamiltonian T-space.

Proof. By Proposition 2.2.10 and Remark 2.2.11, $Y_g^{\mathfrak{t}_{\Delta}} \subset \mathcal{X}$. It follows that the restrictions of ϕ and $\phi_g + \beta$ agree, where $\phi_g = \log(\exp(-\beta)\Phi)$ is a Hamiltonian moment map for the cross-section. Moreover,

by equation (1.12), the pullback of ω_a to $Y_g^{\mathfrak{t}_{\Delta}}$ agrees with the pullback of ω . By equation (1.11) this also agrees with the pullback of ω_g (the 2-form for the cross-section). As ω_g is symplectic, $Y_g^{\mathfrak{t}_{\Delta}}$ is a non-degenerate Hamiltonian *T*-space.

Proposition 2.2.13. In the setting of Corollary 2.2.12 and Remark 2.2.11, β is a regular value for the restriction of ϕ , viewed as a map into Δ :

$$\phi: Y_q^{\mathfrak{t}_\Delta} \to \Delta.$$

The intersection $Z_{\beta} = \mathcal{Z}_{\beta} \cap Y_g = Y_g^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}$ is a smooth compact submanifold.

Proof. By Proposition 2.2.8, the stabilizer of each point in $Y_g^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta) \subset \mathcal{N}^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$ is exactly \mathfrak{t}_{Δ} . Since $Y_g^{\mathfrak{t}_{\Delta}}$ is a non-degenerate Hamiltonian *T*-space, it follows that β is a regular value of the restriction of ϕ , when the latter is viewed as a map into Δ . Since $Y_g^{\mathfrak{t}_{\Delta}} \subset X = \Phi^{-1}(T)$, the level set Z_{β} is compact.

Remark 2.2.14. Shrinking U if necessary, β is also a regular value for the restriction

$$\phi: \mathcal{N}^{\mathfrak{t}_\Delta} \cap \phi^{-1}(\Delta) \to \Delta$$

(This is an open condition.) This implies that the level set $\mathcal{N}^{t_{\Delta}} \cap \phi^{-1}(\beta)$ is smooth. Hence for generic γ , the critical set \mathcal{Z} is smooth.

2.2.5 Reduction to cross-sections.

Recall that the contribution of a critical value $\beta \in \mathfrak{t}$ to the norm-square localization formula (2.10) is

$$DH^{v}_{\mathcal{Z}_{\beta}}(\mathcal{N},\omega_{a},\phi,\tau_{\mathcal{N}}\alpha).$$
(2.16)

By definition, this is a twisted DH distribution for a small tubular neighbourhood of Z_{β} in \mathcal{N} . In this section we relate this to a twisted DH distribution for a small tubular neighbourhood of $Z_{\beta} := Z_{\beta} \cap Y_g$ inside a cross-section Y_g , $g = \exp(\beta)$. When γ is generic, the level set Z_{β} is smooth and compact by Proposition 2.2.13. This will lead to a more explicit formula for (2.16) in the next sections.

Let us briefly motivate the method used to reduce (2.16) to a computation inside a cross-section Y_g . Recall that τ_N is the pullback of an equivariant Thom form for the vector bundle $T \times \mathfrak{t}^{\perp}$. We can split \mathfrak{t}^{\perp} into a direct sum $\mathfrak{t}^{\perp} = \mathfrak{g}_g^{\perp} \oplus (\mathfrak{g}_g/\mathfrak{t})$. The normal bundle to Y_g is isomorphic to $Y_g \times \mathfrak{g}_g^{\perp}$ by taking tangents to the \mathfrak{g}_g^{\perp} -orbit directions. Using this observation, we replace τ_N with a product of forms, one of which is a Thom form for $\nu(Y_g, \mathcal{N})$, and then apply Lemma 2.1.14.

Let $g \in T$, let $U_g \subset G_g$ be a slice around g for the conjugation action, chosen sufficiently small as in Proposition 2.2.10. Recall that $\mathfrak{g}_g/\mathfrak{t}$ denotes the unique *T*-invariant complement to \mathfrak{t} in \mathfrak{g}_g , oriented as described in Chapter 1. The *T*-equivariant smooth map $r: (U_g \cap T) \times (\mathfrak{g}_g/\mathfrak{t}) \to G$ given by

$$r(h, X) = h \exp(X),$$

restricts to a T-equivariant diffeomorphism on a small neighbourhood of $(U_g \cap T) \times \{0\}$, and in particular induces an isomorphism

$$(U_g \cap T) \times (\mathfrak{g}_g/\mathfrak{t}) \simeq \nu(U_g \cap T, U_g)$$



Figure 2.1: Neighbourhood of U' in G. View the torus T as pointing perpendicular to the page. U' is a small open ball in T around g, $\mathcal{U} = \pi_T^{-1}(U')$ is the gray region. The region inside the dashed-line border is $\mathcal{U}_g \simeq U_g \times \mathfrak{g}_g^{\perp}$.

Let $\tau(\mathfrak{g}_g/\mathfrak{t}) \in \Omega_T(U_g)$ be a *T*-equivariant Thom form for $\nu(U_g \cap T, U_g)$ (viewed as a form on U_g), chosen such that its support is contained in *D* (recall $\operatorname{supp}(\tau) \subset D$ where *D* is a tubular neighbourhood with $\overline{D} \subset U$, and τ is the Thom form that we had chosen for $\pi_T : U \to T$). The pullback of $\tau(\mathfrak{g}_g/\mathfrak{t}, X)$ to *T* is $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t}, X) \in \operatorname{Pol}(\mathfrak{t})$.

Consider the map $c: U_g \times \mathfrak{g}_g^{\perp} \to G$ given by

$$c(u, X) = \exp(X)u\exp(-X).$$

This map is T-equivariant for the adjoint action of T on U_g , \mathfrak{g}_g^{\perp} , and G. It is a smooth surjection onto the open subset $\operatorname{Ad}_G(U_g)$. Restricting c to a sufficiently small neighbourhood of $U_g \times \{0\}$, we obtain a diffeomorphism onto an open tubular neighbourhood \mathcal{U}_g of U_g in G. This gives an identification of the normal bundle

$$\nu(U_q, G) \simeq U_q \times \mathfrak{g}_q^{\perp}.$$

We orient \mathfrak{g}_g^{\perp} as described in Chapter 1. Let

$$\pi_g: \mathcal{U}_g \to U_g$$

be the map obtained by inverting c on \mathcal{U}_g and projecting to the first factor. Let $\tau(\mathfrak{g}_g^{\perp}) \in \Omega_T(\mathcal{U}_g)$ be a T-equivariant Thom form (viewed as a form on \mathcal{U}_g using the map c), chosen such that $\operatorname{supp}(\tau(\mathfrak{g}_g^{\perp}) \cdot \pi_g^*\tau(\mathfrak{g}_g/\mathfrak{t})) \subset D$. The pullback of $\tau(\mathfrak{g}_g^{\perp})$ to T is $\operatorname{Eul}(\mathfrak{g}_g^{\perp}, X) \in \operatorname{Pol}(\mathfrak{t})$. Setting $\mathcal{Y}_g = \Phi^{-1}(\mathcal{U}_g)$ we have a pullback diagram:

By choosing a smaller ball $U' \subset U_g \cap T$ around g and shrinking the initial U (adjusting τ and D as well) if necessary, we can ensure that

$$\pi_T^{-1}(U') =: \mathcal{U} \subset \mathcal{U}_g.$$

See Figure 2.2.5. It then makes sense to restrict the equivariant forms τ , $\pi_g^* \tau(\mathfrak{g}_g/\mathfrak{t})$, $\tau(\mathfrak{g}_g^{\perp})$ to the common open set \mathcal{U} .

Lemma 2.2.15. The restrictions to \mathcal{U} of τ and $\tau(\mathfrak{g}_g^{\perp}) \cdot \pi_g^* \tau(\mathfrak{g}_g/\mathfrak{t})$ are *T*-equivariantly cohomologous. Moreover, a *T*-equivariant primitive $\beta \in \Omega_T(\mathcal{U})$ for the difference can be taken with support contained in $D \cap \mathcal{U}$.

Proof. The restriction $\tau(\mathfrak{g}_g^{\perp}) \cdot \pi_g^* \tau(\mathfrak{g}_g/\mathfrak{t})|_{\mathcal{U}}$ is in the same cohomology class with compact vertical supports as $\tau|_{\mathcal{U}}$ (both are 'Poincare dual' to $\mathcal{U} \cap T$ in \mathcal{U}). To see this, note that the *T*-equivariant cohomology of $\mathcal{U} \cap D$ with compact vertical supports is generated by τ (as a Pol(\mathfrak{t})-module), and pull-back to $\mathcal{U} \cap T$ is injective (the bundle $\pi_T : \mathcal{U} \cap D \to \mathcal{U} \cap T$ is *T*-equivariantly diffeomorphic to $(\mathcal{U} \cap T) \times \mathfrak{t}^{\perp}$, the base $\mathcal{U} \cap T$ is contractible). We have

$$\iota_T^* \big(\tau(\mathfrak{g}_g^{\perp}, X) \cdot \pi_g^* \tau(\mathfrak{g}_g/\mathfrak{t}, X) \big) = \operatorname{Eul}(\mathfrak{g}_g^{\perp}, X) \operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t}, X) = \operatorname{Eul}(\mathfrak{t}^{\perp}, X),$$

where the last equality follows because we chose the orientations on \mathfrak{g}_g^{\perp} , $\mathfrak{g}_g/\mathfrak{t}$ such that the product orientation agrees with the orientation on \mathfrak{t}^{\perp} determined by the positive roots. This is the same as the pull-back of $\tau(X)$, hence the two forms correspond to the same class in *T*-equivariant cohomology of $\mathcal{U} \cap D$ with compact vertical supports. \Box

Let $\Phi: M \to G$ be a q-Hamiltonian G-space, $Y_g = \Phi^{-1}(U_g)$ a cross-section, with U_g as in Proposition 2.2.10. Consider a component $\mathcal{Z}_{\beta} \subset \mathcal{Z}$. Via the covering map $\exp: \mathcal{N} \to N$ (see (1.13)), we make various identifications: \mathcal{Z}_{β} is identified with a subset of $\Phi^{-1}(\mathcal{U}) \subset N$, ϕ (hence also v) can be viewed as a map $\Phi^{-1}(\mathcal{U}) \to \mathfrak{t}$, and $\tau_{\mathcal{N}}$ is identified with $\Phi^*\tau$. By Lemma 2.2.15 and the flexibility in Definition 2.1.10, $\tau_{\mathcal{N}} = \Phi^*\tau$ can be replaced with the cohomologous (on $\Phi^{-1}(\mathcal{U})$) form $\Phi^*\tau(\mathfrak{g}_g^{\perp}) \cdot \pi_g^*\tau(\mathfrak{g}_g/\mathfrak{t})$. Therefore the contribution of β to the norm-square formula is given by:

$$DH^{v}_{\mathcal{Z}_{\beta}}(\mathcal{N},\omega_{a},\phi,\alpha\cdot\tau_{\mathcal{N}}) = DH^{v}_{\mathcal{Z}_{\beta}}(\mathcal{N},\omega_{a},\phi,\alpha\cdot\Phi^{*}\tau(\mathfrak{g}_{q}^{\perp})\cdot\tau(\mathfrak{g}_{g}/\mathfrak{t})).$$
(2.18)

Let

$$Z_{\beta} := \mathcal{Z}_{\beta} \cap Y_g = Y_g^{\overline{\beta}} \cap \phi^{-1}(\beta).$$

In the previous subsection we showed that for generic γ , Z_{β} is smooth.

Theorem 2.2.16. The contribution from $\beta \in \mathfrak{t}$ to the norm-square formula is equal to

$$\mathrm{DH}^{v}_{Z_{\mathscr{B}}}(Y_{g}\cap\mathcal{N},\omega_{a},\phi,\alpha\cdot\Phi^{*}\tau(\mathfrak{g}_{g}/\mathfrak{t})).$$

(We have omitted pull-backs to $Y_g \cap \mathcal{N}$ from the notation.)

Proof. By equations (2.17), (2.18), the contribution of β to the norm-square formula is

$$\mathrm{DH}^{v}_{\mathcal{Z}_{\beta}}(\mathcal{N},\omega_{a},\phi,\alpha\cdot\Phi^{*}\tau(\mathfrak{g}_{q}^{\perp})\cdot\tau(\mathfrak{g}_{q}/\mathfrak{t})).$$

By equivariance of Φ , the normal bundle to Y_g in M is isomorphic to $Y_g \times \mathfrak{g}_g^{\perp}$, and the pullback of the Thom form $\Phi^*\tau(\mathfrak{g}_g^{\perp})$ is a T-equivariant Thom form for this vector bundle. Let $N' = \Phi^{-1}(\mathcal{U})$, and note that v is bounded on N' (we are using the identification of N' with a subset of \mathcal{N} to view v as a map $N' \to \mathfrak{t}$). Applying Lemma 2.1.14 with $S = N' \cap Y_g$, $\tau_S = \Phi^*\tau(\mathfrak{g}_g^{\perp})$ (and with $\alpha \cdot \Phi^*\tau(\mathfrak{g}_g/\mathfrak{t})$ in place of α) gives the result.
2.2.6 Polarized completions.

Theorem 2.2.16 expressed the contribution of a critical value β to the norm-square localization formula as

$$\mathrm{DH}^{v}_{Z_{\beta}}(Y_{g}\cap\mathcal{N},\omega_{a},\phi,\alpha\cdot\Phi^{*}\tau(\mathfrak{g}_{g}/\mathfrak{t})),\qquad Z_{\beta}=Y_{g}^{\overline{\beta}}\cap\phi^{-1}(\beta)$$
(2.19)

where Y_g is a cross-section for the q-Hamiltonian space M near $g = \exp(\beta)$. Let U_β be a small tubular neighbourhood of Z_β inside $Y_g \cap \mathcal{N}$. To compute (2.19), we must construct a v-polarized completion $(U_\beta, \tilde{\omega}, \tilde{\phi})$ of $(U_\beta, \omega_a, \phi, \Phi^*\tau(\mathfrak{g}_g/\mathfrak{t})\alpha)$. In detail: it suffices to find a 2-form $\tilde{\omega}$ and moment map $\tilde{\phi}$ on U_β , which agree with ω_a and ϕ (respectively) on the localizing set Z_β , and such that

$$\langle \tilde{\phi}, v \rangle = \langle \tilde{\phi}, \phi - \gamma \rangle$$

is proper and bounded below.

To keep the notation simple, assume that a single sub-algebra \mathfrak{t}_{Δ} contributes to the disjoint union

$$Z_{\beta} = \bigcup_{\mathrm{pr}_{\Delta}(\gamma) = \beta} Y_g^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$$

(In the general case, the arguments here and in the next subsections are applied separately to each sub-algebra \mathfrak{t}_{Δ} which contributes.) The normal bundle splits into a direct sum

$$\nu(Z_{\beta}, Y_g) = \nu(Z_{\beta}, Y_g^{\mathfrak{t}_{\Delta}}) \oplus \nu(Y_g^{\mathfrak{t}_{\Delta}}, Y_g)|_{Z_{\beta}}.$$
(2.20)

By Proposition 2.2.13, $\phi(Y_g^{t_{\Delta}}) \subset \Delta$ and β is a regular value for the restriction of ϕ viewed as a map into the affine subspace $\Delta = \beta + (t_{\Delta}^{\perp})^*$. By the Ehresmann fibration theorem, a neighbourhood of the level set $Z_{\beta} = Y_g^{t_{\Delta}} \cap \phi^{-1}(\beta)$ inside $Y_g^{t_{\Delta}}$ is diffeomorphic to $B_{\epsilon} \times Z_{\beta}$, where B_{ϵ} denotes an ϵ -ball around $0 \in (t_{\Delta}^{\perp})^*$, and $\phi = \beta + pr_1$, where pr_1 denotes projection to the first factor B_{ϵ} . Hence, shrinking U_{β} if necessary, we obtain a local model for $U_{\beta}^{t_{\Delta}}$. This, together with the splitting of the normal bundle (2.20), give a local model for U_{β} .

Choose a connection $\theta \in \Omega^1(Z_\beta) \otimes \mathfrak{t}_{\Delta}^{\perp}$ for the $\mathfrak{t}_{\Delta}^{\perp}$ -action on Z_β (recall T_{Δ}^{\perp} acts locally freely on Z_β). Let

$$q: Z_{\beta} \to Z_{\beta}/T$$

be the quotient map, and let ω_{red} be the symplectic form on the reduced space $Y_g^{\mathfrak{t}_{\Delta}}//T = Z_{\beta}/T$. We equip $(\mathfrak{t}_{\Delta}^{\perp})^* \times Z_{\beta}$ with the 2-form

$$q^* \omega_{\rm red} + d \langle {\rm pr}_1, \theta \rangle. \tag{2.21}$$

Proposition 2.2.17. There is a T-equivariant diffeomorphism

$$\psi_0: B_\epsilon \times \mathcal{V} \xrightarrow{\sim} U_\beta,$$

where B_{ϵ} denotes an ϵ -ball around $0 \in (\mathfrak{t}_{\Delta}^{\perp})^*$, and

$$\mathcal{V} = \nu(Y_g^{\mathfrak{t}_\Delta}, Y_g)|_{Z_\beta}.$$

 ψ_0 is such that $\psi_0(B_\epsilon \times Z_\beta) = U_\beta^{\mathbf{t}_\Delta}$ and intertwines ϕ with $\mathrm{pr}_1 + \beta$ on this submanifold. Furthermore, the pullback of the 2-form ω to Z_β coincides with the pullback of the 2-form in equation (2.21) to Z_β .

Remark 2.2.18. Recall that by Corollary 2.2.12, $(Y_g^{\mathfrak{t}_{\Delta}}, \omega_a, \phi)$ is a non-degenerate Hamiltonian *T*-space. The coisotropic embedding theorem provides a description also of the symplectic form on a small neighbourhood of $Z_{\beta} = Y_g^{\mathfrak{t}_{\Delta}} \cap \phi^{-1}(\beta)$ inside $Y_g^{\mathfrak{t}_{\Delta}}$. It follows that ψ_0 can be chosen to also intertwine the 2-forms ω on $U_{\beta}^{\mathfrak{t}_{\Delta}}$ with the 2-form given in equation (2.21) (restricted to $B_{\epsilon} \times Z_{\beta}$). We will not need this fact, because the definition of polarized completions only requires that the 2-forms agree on the localizing set Z_{β} , which holds in any case.

Referring to Proposition 2.2.17, the construction of the polarized completion below is organized as follows:

- 1. Construct $\tilde{\omega}$, $\tilde{\phi}$ on $(\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V}$ (extending the 2-form (2.21) and moment map $\mathrm{pr}_1 + \beta$) using minimal coupling. The resulting $\tilde{\phi}$ will involve the 'polarized' weights of the \mathfrak{t}_{Δ} -action on \mathcal{V} .
- 2. Choose a T-equivariant diffeomorphism

$$\psi: (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V} \to U_{\beta}.$$

This diffeomorphism will agree with ψ_0 on $\{0\} \times \mathcal{V}$, but be such that $\psi((\mathfrak{t}_{\Delta}^{\perp})^* \times Z_{\beta}) = U_{\beta}^{\mathfrak{t}_{\Delta}}$ (instead of $\psi_0(B_{\epsilon} \times Z_{\beta}) = U_{\beta}^{\mathfrak{t}_{\Delta}}$). This is the 'completion' step.

3. Show that $\langle \tilde{\phi}, \psi^* v \rangle$ is proper and bounded below on $(\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V}$. Using ψ to identify $(\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V}$ with U_{β} , this will complete the argument.

Construction of $\tilde{\omega} - \tilde{\phi}$ on $(\mathfrak{t}^{\perp}_{\Lambda})^* \times \mathcal{V}$.

Let $\pi : \mathcal{V} \to Z_{\beta}$ denote the projection map. Recall that, as a consequence of the genericity assumptions, $\overline{\beta} \in \mathfrak{t}_{\Delta}$ acts with nonzero weights on \mathcal{V} . Let $\alpha_k^- \in \mathfrak{t}_{\Delta}^*$, k = 1, ..., n be the (distinct) weights with signs fixed by the condition

$$\langle \alpha_k^-, \overline{\beta} \rangle < 0$$

and let

$$\alpha_k^+ = -\alpha_k^-, \qquad k = 1, ..., n$$

 \mathcal{V} can be equipped with a complex structure such that the (distinct) weights of the \mathfrak{t}_{Δ} -action are $\alpha_1^-, ..., \alpha_n^-$. In fact, this condition determines the complex structure up to homotopy. Let $V \simeq \mathbb{C}^N$ be a fibre of \mathcal{V} . Choosing a hermitian inner product on \mathcal{V} , we obtain a reduction of the structure group of \mathcal{V} to U(V) = U(N). Let P denote the corresponding T-equivariant principal U(V)-bundle. Thus

$$\mathcal{V} = P \times_{U(V)} V.$$

Let ω_V be the standard symplectic form on V. The action of U(V) is Hamiltonian; let $\phi_V : V \to \mathfrak{u}(V)^*$ be the moment map. The T_{Δ} -action on V induces a linear map $a : \mathfrak{t}_{\Delta} \to \mathfrak{u}(V)$, and the composition $a^* \circ \phi_V$ is given by

$$a^* \circ \phi_V(z) = -\pi \sum_k |z_k|^2 \alpha_k^-,$$
(2.22)

where $z = \sum z_k$ is the weight-space decomposition.

At this stage, it is convenient to choose a complementary subtorus T'_{Δ} to T_{Δ} ($\mathfrak{t}^{\perp}_{\Delta}$ itself might not be integral). The exact choice is not important. We can, for example, choose T'_{Δ} such that $T = T_{\Delta} \times T'_{\Delta}$

 $(\Rightarrow T/T_{\Delta} \simeq T'_{\Delta})$. We thus have two splittings $\mathfrak{t} = \mathfrak{t}_{\Delta} \oplus \mathfrak{t}_{\Delta}^{\perp} = \mathfrak{t}_{\Delta} \oplus \mathfrak{t}'_{\Delta}$, and this canonically determines an isomorphism $(\mathfrak{t}_{\Delta}^{\perp})^* \simeq (\mathfrak{t}'_{\Delta})^*$ (both can be identified with the annihilator $\operatorname{ann}(\mathfrak{t}_{\Delta}) \subset \mathfrak{t}^*$).

Next choose a *T*-invariant connection σ on *P*. Since $\mathfrak{t}_{\Delta}^{\perp}$ acts locally freely on \mathcal{Z}_{β} , while \mathfrak{t}_{Δ} acts trivially, T'_{Δ} also acts locally freely on \mathcal{Z}_{β} . The connection can thus be chosen such that the induced vector fields X_P , $X \in \mathfrak{t}'_{\Delta}$ are horizontal; the connection 1-form $\sigma \in \Omega^1(P) \otimes \mathfrak{u}(V)$ is then \mathfrak{t}'_{Δ} -basic. The *T*-equivariant curvature of σ is by definition:

$$R_T = d_T \sigma + \frac{1}{2}[\sigma, \sigma] = R - \mu \in \Omega^2_T(P) \otimes \mathfrak{u}(V)$$

where $R = d\sigma + \frac{1}{2}[\sigma,\sigma]$ is the ordinary curvature, and $\mu(X) := \sigma(X_P)$ is the moment map of the connection σ (c.f. [40]). This vanishes for $X \in \mathfrak{t}_{\Delta}$, while for $X \in \mathfrak{t}_{\Delta}$, $\mu(X)$ is the constant function $-a(X) \in \Omega^0(P) \otimes \mathfrak{u}(V)$.

Consider the composition

$$\Omega_{U(V)}(V) \to \Omega_{T \times U(V)}(P \times V) \to \Omega_T(P \times_{U(V)} V) = \Omega_T(\mathcal{V}),$$

where the first map is simply pullback. The second map is the *Cartan map*, defined by substituting the *T*-equivariant curvature for the $\mathfrak{u}(V)$ variables, followed by using the connection σ to project the resulting form to its horizontal part (the result is U(V)-basic and descends to the quotient). The Cartan map is a homotopy inverse to the pullback map $\Omega_T(P \times_{U(V)} V) \to \Omega_{T \times U(V)}(P \times V)$ (c.f. [40]). Applying this composition to $\omega_V - \phi_V$ we obtain a ('minimal coupling') *T*-equivariant 2-form $\omega_{\min} - \phi_{\min}$ on the total space of \mathcal{V} :

$$\omega_{\min} = \Pi_{hor}(\omega_V) - \langle \phi_V, R \rangle, \qquad \phi_{\min} = -\langle \phi_V, \mu \rangle.$$

Here Π_{hor} denotes the horizontal projection operator for the connection σ . Since $\mu(X) = -a(X)$ for $X \in \mathfrak{t}_{\Delta}$ (and vanishes for $X \in \mathfrak{t}'_{\Delta}$), ϕ_{\min} is given by the same formula (2.22), where now $|\cdot|$ denotes the hermitian inner product on \mathcal{V} . Since σ is \mathfrak{t}'_{Δ} -basic, the 2-form ω_{\min} is \mathfrak{t}'_{Δ} -basic.

We define the 'total' T-equivariant 2-form $\omega_{\mathcal{V}} - \phi_{\mathcal{V}}$ on \mathcal{V} by

$$\omega_{\mathcal{V}} = \omega_{\rm red} + \omega_{\rm min}, \qquad \phi_{\mathcal{V}} = \beta + \phi_{\rm min}$$

We summarize with a Theorem.

Theorem 2.2.19. On the local model

$$U_{\text{mod}} := (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V},$$

there is a closed T-equivariant 2-form $\tilde{\omega} - \tilde{\phi}$,

$$\begin{split} \tilde{\omega} &:= \omega_{\mathcal{V}} + d \langle \mathrm{pr}_1, \theta \rangle = \omega_{\mathrm{red}} + \omega_{\mathrm{min}} + d \langle \mathrm{pr}_1, \theta \rangle, \\ \tilde{\phi} &:= \phi_{\mathcal{V}} + \mathrm{pr}_1 = \beta - \pi \sum_k |z_k|^2 \alpha_k^- + \mathrm{pr}_1, \end{split}$$

where $\omega_{\mathcal{V}} - \phi_{\mathcal{V}}$ is a closed T-equivariant 2-form on \mathcal{V} . Its basic properties are:

- The pullback to the base $(\mathfrak{t}_{\Delta}^{\perp})^* \times Z_{\beta} \subset U_{\mathrm{mod}}$ is $\omega_{\mathrm{red}} + d\langle \mathrm{pr}_1, \theta \rangle (\mathrm{pr}_1 + \beta)$.
- The pullback to \mathcal{V} is $\omega_{\mathcal{V}} \phi_{\mathcal{V}}$. Pulling back further to Z_{β} gives $\omega_{\text{red}} \beta$.

- $\omega_{\mathcal{V}}$ is T'_{Δ} -basic, $d\langle \mathrm{pr}_1, \theta \rangle$ is T_{Δ} -basic.
- $\phi_{\mathcal{V}}$ takes values in \mathfrak{t}^*_{Δ} , pr_1 takes values in $(\mathfrak{t}^{\perp}_{\Delta})^*$.

The diffeomorphism ψ .

Recall that the diffeomorphism ψ_0 identified U_β with a neighbourhood

$$B_{\epsilon} \times \mathcal{V} \subset (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V} = \mathrm{U}_{\mathrm{mod}}.$$

Under pullback

$$(\psi_0^*\phi)|_{B_\epsilon \times Z_\beta} = \mathrm{pr}_1 + \beta$$

and consequently

$$(\psi_0^* v)|_{B_\epsilon \times Z_\beta} = \mathrm{pr}_1 + \beta - \gamma = \mathrm{pr}_1 + \overline{\beta}.$$

Let $f: (\mathfrak{t}_{\Delta}^{\perp})^* \xrightarrow{\sim} B_{\epsilon}$ be the diffeomorphism

$$f(\xi) = \frac{\epsilon\xi}{\sqrt{1+|\xi|^2}}$$

and define

$$\psi := \psi_0 \circ (f \times \mathrm{Id}) : \mathrm{U}_{\mathrm{mod}} = (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V} \xrightarrow{\sim} U_{\beta}.$$

Then

$$(\psi^* v)|_{(\mathfrak{t}^{\perp}_{\Delta})^* \times Z_{\beta}} = f^* \mathrm{pr}_1 + \overline{\beta}.$$
(2.23)

Note that $|f^* \operatorname{pr}_1| < \epsilon$.

Proposition 2.2.20. With notation as above, and assuming the neighburhood U_{β} is chosen sufficiently small, $(U_{mod}, \tilde{\omega}, \tilde{\phi})$ is a $\psi^* v$ -polarized completion of $(U_{mod}, \psi^* \omega_a, \psi^* \phi)$.

Proof. By Proposition 2.2.17, Theorem 2.2.19, and using the fact that $(f \times \text{Id})$ restricts to the identity on Z_{β} , it follows that the *T*-equivariant 2-form $\tilde{\omega} - \tilde{\phi}$ agrees with $\omega_a - \phi$ on Z_{β} . It remains to check that $\langle \tilde{\phi}, \psi^* v \rangle$ is proper and bounded below. Write

$$\psi^* v = f^* \mathrm{pr}_1 + \overline{\beta} + v',$$

where $v'|_{(\mathfrak{t}^{\perp}_{\Delta})^* \times Z_{\beta}} = 0$ by (2.23) (v' is the error term).

For any $\epsilon' > 0$, we can ensure that $|v'| < \epsilon'$ by taking the tubular neighbourhood U_{β} of Z_{β} to be sufficiently small in the direction normal to $U_{\beta}^{\mathfrak{t}_{\Delta}} \simeq B_{\epsilon} \times Z_{\beta}$. Decompose v' into components,

$$v' = v'_0 + v'_1 \in \mathfrak{t}_\Delta \oplus \mathfrak{t}_\Delta^\perp$$

Let $z \in \mathcal{V}$ and $\xi \in (\mathfrak{t}_{\Delta}^{\perp})^*$. We have

$$\psi^* v(z,\xi) = f(\xi) + \overline{\beta} + v'_0(z,\xi) + v'_1(z,\xi),$$

$$\tilde{\phi}(z,\xi) = \xi + \beta - \pi \sum |z_k|^2 \alpha_k^-.$$

Thus

$$\langle \tilde{\phi}, \psi^* v \rangle (z, \xi) = \left[\frac{\epsilon |\xi|^2}{\sqrt{1 + |\xi|^2}} + \langle \xi, v_1' \rangle \right] + \langle \beta, \psi^* v \rangle - \pi \sum |z_k|^2 \langle \alpha_k^-, \overline{\beta} + v_0' \rangle$$

(Note that some terms vanish because they involve pairings of elements of \mathfrak{t}_{Δ} with elements of $(\mathfrak{t}_{\Delta}^{\perp})^*$ or the reverse.) We examine each of the three terms in turn:

1. If we take $\epsilon' < \frac{1}{2}\epsilon$, then

$$|\langle \xi, v_1' \rangle| \le \frac{1}{2}\epsilon |\xi|.$$

Thus the first term is dominated by $\frac{\epsilon|\xi|^2}{\sqrt{1+|\xi|^2}}$ as $|\xi|$ goes to infinity. It is therefore bounded below, and goes to infinity as $|\xi|$ goes to infinity.

2. The second term is bounded by a constant:

$$|\langle \beta, \psi^* v \rangle| \le |\beta|(\epsilon + |\overline{\beta}| + \epsilon').$$

3. Recall that $\langle \alpha_k^-, \overline{\beta} \rangle < 0$. For the third term, we take ϵ' sufficiently small that for each k = 1, ..., n,

$$-\pi \langle \alpha_k^-, \overline{\beta} + v_0' \rangle > \epsilon'' > 0,$$

for some constant $\epsilon'' > 0$. Then the third term is non-negative for all z, ξ and

$$-\pi \sum |z_k|^2 \langle \alpha_k^-, \overline{\beta} + v_0' \rangle > \epsilon'' |z|^2,$$

which goes to infinity as |z| goes to infinity.

Since $\langle \tilde{\phi}, \psi^* v \rangle$ is bounded below and goes to infinity as $|(z,\xi)| \to \infty$, this completes the proof. \Box

As explained at the beginning of this section, using ψ to identify U_{β} with $U_{\text{mod}} = (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V}$, we obtain a *v*-polarized completion of U_{β} .

2.2.7 Explicit formulas.

Recall that the contribution of β to the H-K formula for $DH(\mathcal{N}, \omega_a, \phi, \tau_{\mathcal{N}}\alpha)$ is the twisted Duistermaat-Heckman measure

$$\mathrm{DH}(\mathrm{U}_{\mathrm{mod}}, \tilde{\omega}, \tilde{\phi}, \Phi^* \tau(\mathfrak{g}_q/\mathfrak{t})\alpha),$$

where U_{mod} , $\tilde{\omega}$, $\tilde{\phi}$ are described in Theorem 2.2.19. In this section, we derive an explicit formula for this contribution. Paradan [48], [49] obtained the same type of formula using different methods. Woodward [65] also obtained similar formulas. The calculations in this section show how these expressions can be recovered from the Harada-Karshon Theorem.

Additional notation.

- X, ζ, Y, ξ will denote elements of $\mathfrak{t}_{\Delta}, \mathfrak{t}_{\Delta}^*, \mathfrak{t}_{\Delta}^\perp, (\mathfrak{t}_{\Delta}^\perp)^*$ respectively. (We distinguish between elements of \mathfrak{t} and \mathfrak{t}^* in this section, as this will make the presentation clearer.)
- The top degree part of the form $e^{\langle d\xi, \theta \rangle} \in \Omega(U_{\text{mod}})$ has a decomposition (determined up to scalar multiples) into a product $d\text{vol}(\xi) \cdot \nu$ where $d\text{vol}(\xi)$ is a volume form on $(\mathfrak{t}_{\Delta}^{\perp})^*$. In coordinates $d\text{vol}(\xi) =$

 $d\xi_1 \cdots d\xi_d$, $\nu = (-1)^{\frac{1}{2}d(d-1)}\theta^1 \cdots \theta^d$. It is convenient to fix such a decomposition, and let $\operatorname{vol}(T'_\Delta)$ denote the induced volume of T'_Δ (using the canonical isomorphism $(\mathfrak{t}^\perp_\Delta)^* \simeq (\mathfrak{t}'_\Delta)^*$).

• Let $\overline{\alpha} \in \Omega_{T_{\Delta}}(Z_{\beta}/T'_{\Delta})$ denote the image of a cocycle $\alpha \in \Omega_T(U_{\text{mod}})$ under the composition of restriction to Z_{β} followed by the *Cartan map* for $Z_{\beta} \to Z_{\beta}/T'_{\Delta}$ (c.f. [40], or the discussion in Section 2.2.6). This composition implements the Kirwan map at the level of cocycles.

Orientations. The reduced space Z_{β}/T'_{Δ} is a symplectic orbifold, which we orient using its Liouville form. We orient $(\mathfrak{t}_{\Delta}^{\perp})^*$ using $d\operatorname{vol}(\xi)$, and the fibres of $Z_{\beta} \to Z_{\beta}/T'_{\Delta}$ using ν . The product orientation on $(\mathfrak{t}_{\Delta}^{\perp})^* \times Z_{\beta}$ then agrees with the orientation induced by the 2-form on the base $(\mathfrak{t}_{\Delta}^{\perp})^* \times Z_{\beta}$ (which is symplectic near $Z_{\beta} \times \{0\}$ by the Coisotropic Embedding Theorem). The orientation of the vector bundle $\mathcal{V} \to Z_{\beta}$ is fixed by requiring the total orientation to agree with that of U_{mod} (an open subset of the symplectic manifold Y_g). The resulting orientation on \mathcal{V} might not agree with the orientation induced by the complex structure on \mathcal{V} for which the weights are α_k^- . The difference is a sign $\epsilon = \pm 1$.

Lemma 2.2.21 (c.f. [61]). We can replace $\Phi^*\tau(\mathfrak{g}_g/\mathfrak{t}) \cdot \alpha$ with the pull-back to U_{mod} of $\Phi^*\text{Eul}(\mathfrak{g}_g/\mathfrak{t}) \cdot \overline{\alpha}$ without changing the result, that is,

$$DH(U_{mod}, \tilde{\omega}, \phi, \Phi^*\tau(\mathfrak{g}_g/\mathfrak{t})\alpha) = Eul(\mathfrak{g}_g/\mathfrak{t}, \partial)DH(U_{mod}, \tilde{\omega}, \phi, \overline{\alpha}).$$

(Recall $g = \exp(\beta)$ and $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t},\xi)$ is a polynomial on \mathfrak{t} ; see Chapter 1.)

Proof. As shown in the previous section, $\tilde{\phi}$ is v-polarized. Since $v|_{U_{\beta}}$ is bounded, $\tilde{\phi}$ is proper. Therefore by Theorem 2.1.1, we can replace $\Phi^*\tau(\mathfrak{g}_g/\mathfrak{t})\alpha$ with any T-equivariantly cohomologous form. Since $\operatorname{pr}_{Z_{\beta}}: U_{\mathrm{mod}} \to Z_{\beta}$ is a vector bundle, the pullback map $\iota_{Z_{\beta}}^*$ is homotopy inverse to $\operatorname{pr}_{Z_{\beta}}^*$. Moreover, the Cartan map for the locally free T'_{Δ} action on Z_{β} is homotopy inverse to the pullback map $\Omega_{T_{\Delta}}(Z_{\beta}/T'_{\Delta}) \to \Omega_T(Z_{\beta})$. This shows that we can replace $\Phi^*\tau(\mathfrak{g}_g/\mathfrak{t}) \cdot \alpha$ with the pullback of its image under the map $\Omega(U_{\mathrm{mod}}) \to \Omega(Z_{\beta}/T'_{\Delta})$ given by pullback to Z_{β} followed by the Cartan map for $Z_{\beta} \to Z_{\beta}/T'_{\Delta}$. Recall that $\tau(\mathfrak{g}_g/\mathfrak{t})$ is a T-equivariant Thom form for the trivial vector bundle with fibre $\mathfrak{g}_g/\mathfrak{t}$ over an open subset U of T, its pullback to T being $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t})$. Since $Z_{\beta} \subset \Phi^{-1}(T)$, the pullback of $\Phi^*\tau(\mathfrak{g}_g/\mathfrak{t})$ to Z_{β} equals the pullback of $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t})$. Finally it is immediate from the definition of DH distributions that for any polynomial p,

$$DH(U_{mod}, \tilde{\omega}, \phi, p\overline{\alpha}) = p(\partial)DH(U_{mod}, \tilde{\omega}, \phi, \overline{\alpha}).$$

Let

$$\mathfrak{m} := \mathrm{DH}(\mathrm{U}_{\mathrm{mod}}, \tilde{\omega}, \tilde{\phi}, \overline{\alpha})$$

For the remainder of this section, we will focus on computing \mathfrak{m} , and replace the factor $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t},\partial)$ only at the very end.

Lemma 2.2.22. \mathfrak{m} is a tempered distribution. Its pairing with a Schwartz function f is given by

$$\langle \mathfrak{m}, f \rangle = \int_{(\mathfrak{t}_{\Delta}^{\perp})^*} d\mathrm{vol}(\xi) \int_{\mathcal{V}} \nu e^{\omega_{\mathcal{V}} + \langle \xi, F \rangle} \overline{\alpha}(-\partial_{\zeta}) f \circ \tilde{\phi}.$$

Proof. By definition,

$$\langle \mathfrak{m}, f \rangle = \int_{(\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V}} e^{\tilde{\omega}} \overline{\alpha}(-\partial_{\zeta}) f \circ \tilde{\phi}.$$
(2.24)

Recall that the $(\mathfrak{t}_{\Delta}^{\perp})^*$ component of $\tilde{\phi}$ is $\mathrm{pr}_1 : (\mathfrak{t}_{\Delta}^{\perp})^* \times \mathcal{V} \to (\mathfrak{t}_{\Delta}^{\perp})^*$, so that $|\tilde{\phi}(z,\xi)| \ge |\xi|$. Also, $\langle \tilde{\phi}(\xi,z), \overline{\beta} \rangle \ge C|z|^2 + D$ for some constants C > 0, D. Combining these facts shows that $|\tilde{\phi}(z,\xi)| \ge C'(|\xi| + |z|^2) + D'$ for some constants C' > 0, D'. It follows that the integrand is rapidly decreasing on U_{mod} for any Schwartz function f, and \mathfrak{m} is tempered.

We have

$$e^{\tilde{\omega}} = e^{\omega_{\mathcal{V}} + \langle \xi, F \rangle} e^{\langle d\xi, \theta \rangle}.$$

Only the top degree part $dvol(\xi) \nu$ of $e^{\langle d\xi, \theta \rangle}$ contributes to the integral over $(\mathfrak{t}_{\Delta}^{\perp})^*$. Then use Fubini's theorem to re-write (2.24) as an iterated integral.

Lemma 2.2.23. The \mathfrak{t}_{Δ} -Fourier transform of \mathfrak{m} is a generalized function on $\mathfrak{t}_{\Delta} \times (\mathfrak{t}_{\Delta}^{\perp})^*$ given by

$$\langle \mathcal{F}_{\mathfrak{t}_{\Delta}}(\mathfrak{m}), f dX \rangle = \int_{(\mathfrak{t}_{\Delta}^{\perp})^{*}} d\mathrm{vol}(\xi) \int_{\mathcal{V}} \int_{\mathfrak{t}_{\Delta}} \overline{\alpha}(2\pi i X) \nu e^{\omega_{\mathcal{V}}(2\pi i X) + \langle \xi, F \rangle} f(X, \xi) dX.$$

In this expression, $f(X,\xi)$ is a Schwartz function on $\mathfrak{t}_{\Delta} \times (\mathfrak{t}_{\Delta}^{\perp})^*$, and dX is a smooth translation-invariant measure on \mathfrak{t}_{Δ} .

Proof. Using $\tilde{\phi} = (\phi_{\mathcal{V}}, \mathrm{pr}_1) : \mathrm{U}_{\mathrm{mod}} \to \mathfrak{t}^*_{\Delta} \times (\mathfrak{t}^{\perp}_{\Delta})^*$,

$$\mathcal{F}_{\mathfrak{t}_{\Delta}}(f\,dX)\circ\tilde{\phi}=\int_{\mathfrak{t}_{\Delta}}f(X,\mathrm{pr}_{1})e^{-2\pi i\langle\phi_{\mathcal{V}},X\rangle}dX.$$

By the previous lemma

$$\begin{aligned} \langle \mathcal{F}_{\mathfrak{t}_{\Delta}}(\mathfrak{m}), f \, dX \rangle \\ &= \int_{(\mathfrak{t}_{\Delta}^{\perp})^{*}} d\mathrm{vol}(\xi) \int_{\mathcal{V}} \nu e^{\omega_{\mathcal{V}} + \langle \xi, F \rangle} \overline{\alpha}(-\partial_{\zeta}) \mathcal{F}_{\mathfrak{t}_{\Delta}}(f \, dX) \circ \tilde{\phi} \\ &= \int_{(\mathfrak{t}_{\Delta}^{\perp})^{*}} d\mathrm{vol}(\xi) \int_{\mathcal{V}} \nu e^{\omega_{\mathcal{V}} + \langle \xi, F \rangle} \int_{\mathfrak{t}_{\Delta}} \overline{\alpha}(2\pi i X) f(X, \xi) e^{-2\pi i \langle \phi_{\mathcal{V}}, X \rangle} dX. \end{aligned}$$

For the next result, let s_{Δ} denote the locally constant function on Z_{β} , whose value on a component is the number of elements in the stabilizer for the action of T'_{Δ} on that component. We will make use of a *T*-equivariant form with tempered generalized coefficients $\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V})$ introduced by Paradan [48], [49]—a definition is provided in Section 2.4.

Theorem 2.2.24 (compare [49] equation (4.57)). We have

$$\mathfrak{m} = \int_{Z_{\beta}/T} \frac{\operatorname{vol}(T_{\Delta}')}{s_{\Delta}} \delta_{\beta} * \overline{\alpha}(\partial_{\zeta}) \mathcal{F}_{\mathfrak{t}_{\Delta}}^{-1} \left(\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X) \right) e^{\omega_{\operatorname{red}} + \langle \xi, F \rangle} d\operatorname{vol}(\xi).$$

Proof. Lemma 2.2.23 can be written more concisely as the equality:

$$\mathcal{F}_{\mathfrak{t}_{\Delta}}(\mathfrak{m})(X,\xi) = \int_{\mathcal{V}} \overline{\alpha}(2\pi i X) \nu e^{\omega_{\mathcal{V}}(2\pi i X) + \langle \xi,F\rangle},$$

with the understanding that the integrand should be smeared with a Schwartz function $f(X,\xi)$ on $\mathfrak{t}_{\Delta} \times (\mathfrak{t}_{\Delta}^{\perp})^*$ times a smooth translation invariant measure dX on \mathfrak{t}_{Δ} , before the integral over \mathcal{V} is performed.

With the exception of ν , the forms in the integrand are $d_{T_{\Delta}}$ -cocycles: indeed, $\overline{\alpha}$ and $e^{(\omega_{\nu})_{T_{\Delta}}}$ are $d_{T_{\Delta}}$ cocycles; as F is the pullback of the closed form $d\theta$ on $Z_{\beta} \subset \mathcal{V}^{T_{\Delta}}$, $\langle \xi, F \rangle$ is also $d_{T_{\Delta}}$ -closed. Thus, to apply Corollary 2.4.3, we just need to check a certain polarization condition. This condition (with exponent a = 2) is satisfied because of the *polarization* of the weights $\langle \alpha_k^-, \overline{\beta} \rangle < 0, \ k = 1, ..., n$ used in the construction of ϕ . Therefore, the integral over the fibres $\mathcal{V} \to Z_{\beta}$ can be computed using Corollary 2.4.3:

$$\mathcal{F}_{\mathfrak{t}_{\Delta}}(\mathfrak{m})(X) = \int_{Z_{\beta}} d\mathrm{vol}(\xi) \,\nu \,\overline{\alpha}(2\pi i X) e^{\omega_{\mathrm{red}} - 2\pi i \langle \beta, X \rangle + \langle \xi, F \rangle} \mathrm{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X).$$

All the forms in the integrand are T'_{Δ} -basic except ν (the Euler form is T'_{Δ} -basic because the connection σ that we chose on \mathcal{V} was T'_{Δ} -basic). Integrating over the fibres of $q: Z_{\beta} \to Z_{\beta}/T'_{\Delta} = Z_{\beta}/T$ we have

$$q_*\nu = \frac{\operatorname{vol}(T'_\Delta)}{s_\Delta}.$$

(There is no sign, because the fibres are oriented using ν .) Hence,

$$\mathcal{F}_{\mathfrak{t}_{\Delta}}(\mathfrak{m})(X) = \int_{Z_{\beta}/T} \frac{\operatorname{vol}(T_{\Delta}')}{s_{\Delta}} \overline{\alpha}(2\pi i X) e^{\omega_{\operatorname{red}} - 2\pi i \langle \beta, X \rangle + \langle \xi, F \rangle} \operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X) d\operatorname{vol}(\xi).$$

Now take the \mathfrak{t}_{Δ} -inverse Fourier transform.

It follows from Theorem 2.2.24 that the contribution is *polynomial* in ξ (the exponential $e^{\langle \xi, F \rangle}$ truncates at finite degree). For $\alpha \in \mathfrak{t}^*_{\Delta}$, the *Heaviside distribution* H_{α} has support $\mathbb{R}_{\geq 0}\alpha$ and is defined by

$$\langle H_{\alpha}, f \rangle = \int_{0}^{\infty} f(t\alpha) dt.$$

In Section 2.4 we briefly describe how to compute the inverse Fourier transform of the form $\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V})$ (c.f. [23], [18], [48], [49] for further discussion). Recall that $\alpha_k^+ = -\alpha_k^-$, k = 1, ..., n denote the distinct weights of the \mathfrak{t}_{Δ} -action on \mathcal{V} , with signs such that $\langle \alpha_k^+, \overline{\beta} \rangle > 0$. The end result is

$$\mathfrak{m} = \epsilon \int_{Z_{\beta}/T} \frac{\operatorname{vol}(T_{\Delta}')}{s_{\Delta}} \overline{\alpha}(\partial_{\zeta}) \delta_{\beta} * \prod_{k=1}^{n} H_{\alpha_{k}^{+}}^{r_{k}} * \sum_{\ell \geq 0} \left(-\sum_{m=1}^{r_{k}} c_{m}(\mathcal{V}_{k}) H_{\alpha_{k}^{+}}^{m} \right)^{\ell} e^{\omega_{\operatorname{red}} + \langle \xi, F \rangle} d\operatorname{vol}(\xi).$$
(2.25)

where $c_m(\mathcal{V}_k)$ is the m^{th} Chern class of \mathcal{V}_k , r_k is the complex rank of \mathcal{V}_k . The product over k and exponents ℓ , m, r_k in this expression denote convolution. The sum over ℓ truncates after finitely many terms. The constant $\epsilon = \pm 1$ is +1 iff the orientation induced by the complex structure on \mathcal{V} for which the T_{Δ} weights are $\alpha_1^-, ..., \alpha_n^-$ agrees with the original orientation on \mathcal{V} .

Equation (2.25) is a finite sum of terms of the form

$$cp(\partial_{\zeta})\delta_{\beta} * \prod_{k=1}^{m} H^{N_k}_{\alpha_k^+} d\mathrm{vol}(\xi),$$

where c is a constant (obtained by integrating a differential form over Z_{β}/T), p is a polynomial, and $N_k \geq r_k$. We can see explicitly that \mathfrak{m} is supported in the closed cone $\beta + (\mathfrak{t}_{\Delta}^{\perp})^* + \mathbb{R}_{\geq 0}\alpha_1^+ + \cdots + \mathbb{R}_{\geq 0}\alpha_n^+$.

Finally, we put back the factor of $\operatorname{Eul}(\mathfrak{g}_q/\mathfrak{t},\partial)$ to obtain an explicit formula for the contribution from

 β to the norm-square localization formula:

$$\mathrm{DH}_{Z_{\beta}}^{v}(Y_{g}\cap\mathcal{N},\omega_{a},\phi,\tau_{\mathcal{N}}\alpha)=\mathrm{Eul}(\mathfrak{g}_{g}/\mathfrak{t},\partial)\mathfrak{m}.$$
(2.26)

with \mathfrak{m} given by equation (2.25) above.

2.3 Examples

Here we give two examples: the 4-sphere (a q-Hamiltonian SU(2)-space), and a certain multiplicity-free q-Hamiltonian SU(3)-space due to Chris Woodward.

2.3.1 The 4-sphere.

The q-Hamiltonian structure on S^4 can be constructed by gluing together two copies of a ball in \mathbb{C}^2 (a *Hamiltonian SU*(2)-space) along their boundaries, analogous to the way the 2-sphere (a Hamiltonian U(1)-space) can be glued together from two copies of a ball in \mathbb{R}^2 (also Hamiltonian U(1)-spaces). See [5] for details.

Identify $\mathfrak{t}^* \simeq \mathbb{R}$ in such a way that \mathbb{Z} is the weight lattice. Using the basic inner product to identify $\mathfrak{t} \simeq \mathfrak{t}^*$, the fundamental alcove is the closed interval [0,1]. We use notation as in the main part of the paper: Φ is the SU(2)-valued moment map, $\Phi_a : N \to T = U(1)$ the abelianization, \mathcal{N} the covering space of N with moment map $\phi : \mathcal{N} \to \mathfrak{t}^* \simeq \mathbb{R}$, etc. The covering space \mathcal{X} of $X = \Phi^{-1}(T)$ (inside \mathcal{N}) is (topologically) an infinite line of 2-spheres, with each sphere touching its two neighbours at the poles ("beads on a string"), the poles being sent by ϕ to $\mathbb{Z} \subset \mathbb{R}$. The poles are precisely the (isolated) T-fixed points (they correspond to points in S^4 which are fixed by all of SU(2)). The space \mathcal{N} is a 4-dimensional non-singular "thickening" of \mathcal{X} .

Let $\mathfrak{m} = \mathrm{DH}(\mathcal{N}, \omega_a, \phi, \tau) \in \mathcal{D}'(\mathfrak{t}^*)$; according to the discussion of Section 2.2.1, this can heuristically be thought of as the (untwisted) Duistermaat-Heckman measure of \mathcal{X} . Choose $\gamma \in (0, 1) =: \sigma$. The standard cross-section Y_{σ} is the finite cylinder $\mathcal{X} \cap \phi^{-1}(0, 1)$, which has Duistermaat-Heckman measure $\mathbf{1}_{[0,1]}dx$ where $\mathbf{1}_S$ denotes the indicator function of the subset S, and dx denotes Lebesgue measure. The work of Sections 2.3—2.6 implies that the central contribution to the norm-square localization formula is polynomial times Lebesgue measure and agrees with the DH measure of Y_{σ} near γ . Therefore, the central contribution is

$$\mathfrak{m}_{\gamma} = 1 dx.$$

(A second way to see this is to use the correspondence between \mathfrak{m} and volumes of reduced spaces, the reduced space $\Phi^{-1}(\exp(\gamma))/T$ being a point.) Note that this, together with the anti-symmetry of \mathfrak{m} under the affine Weyl group (generated by reflections in the lattice points \mathbb{Z}), determines \mathfrak{m} on $\mathbb{R} \setminus \mathbb{Z}$.

The correction terms \mathfrak{m}_{β} come from the *T*-fixed points (the maximal torus is 1-dimensional, so there are no other subtori). Consider, for example, the critical value $\beta = 0$. The corresponding part of the critical set is a single point $p := \mathcal{Z}_{\beta} = \phi^{-1}(0)$, and $\overline{\beta} = 0 - \gamma < 0$. The contribution \mathfrak{m}_{β} is supported in the half-space $\overline{\beta}x \ge 0 \Leftrightarrow x \le 0$. The local normal form constructed in Section 2.2.6 is $\mathcal{V} = T_p M$. By construction, the cross-section around $p \in M$ is isomorphic to an open ball in \mathbb{C}^2 , as a Hamiltonian SU(2)-space. Using the induced complex structure $T_pM \simeq \mathbb{C}^2$, the weights of the $U(1) \subset SU(2)$ action are +1, -1. To get the correct contribution, we flip the complex structure on the second copy of \mathbb{C} so



Figure 2.2: First 5 terms for the 4-sphere.

that the weights of the U(1) action are both +1 (they must have negative pairing with $\overline{\beta}$), and so the moment map is:

$$\tilde{\phi}(z_1, z_2) = -\pi(|z_1|^2 + |z_2|^2),$$

while the 2-form is the standard symplectic 2-form for \mathbb{C}^2 (since we reversed the complex structure on one of the copies of \mathbb{C} , this 2-form induces the opposite orientation from the 2-form of the cross-section). The DH measure of this local normal form is

$$-H_{-1} * H_{-1} = x \cdot \mathbf{1}_{(-\infty,0]} dx.$$

Here H_{-1} is the Heaviside distribution equal to 1 on $(-\infty, 0)$, and the factor of $\epsilon = -1$ appears because the orientation on $\mathcal{V} = T_p M$ induced by the complex structure for which the weights are both +1 is opposite the original orientation.

To obtain the norm-square contribution, the last step is to apply the differential operator $\operatorname{Eul}(\mathfrak{g}/\mathfrak{t},\partial)$ (this takes into account the effect of the Thom form), which in this case is $-2\frac{d}{dx}$ (since the unique positive root is $\alpha = 2$ with our normalization). And so the contribution is:

$$\mathfrak{m}_0 = -2 \cdot \mathbf{1}_{(-\infty,0]} dx.$$

Taking additional critical points into account produces new corrections in a similar way:

$$\frac{\mathfrak{m}}{dx} = 1 - 2 \cdot \mathbf{1}_{(-\infty,0]} - 2 \cdot \mathbf{1}_{[1,\infty)} + 2 \cdot \mathbf{1}_{(-\infty,-1]} + 2 \cdot \mathbf{1}_{[2,\infty)} - \cdots$$

2.3.2 A multiplicity-free q-Hamiltonian SU(3)-space.

Figure 2.3 shows the moment map image $\phi(\mathcal{N})$ corresponding to a certain multiplicity-free q-Hamiltonian SU(3)-space M. This example is due to Chris Woodward (c.f. [64], [65] where a similar multiplicity-



Figure 2.3: A multiplicity-free q-Hamiltonian SU(3)-space. On the right, the sum of the contributions from a,b,c is shown.

free Hamiltonian SU(3)-space is discussed). See also Chapter 4, Section 4.4.8, where we outline one construction of this example.

The triangle indicated with a bold border is the fundamental alcove, and the smaller shaded triangle inside is the moment polytope of M. The Duistermaat-Heckman measure is 0 away from the moment map image, while inside the image, its value (relative to a suitably normalized Lebesgue measure) alternates between ± 1 (it is anti-symmetric under the action of the affine Weyl group). The three facets of the moment polytope correspond to three submanifolds fixed by 1-dimensional subtori of T. Each of these fixed-point submanifolds is (topologically) a 2-torus. Note in particular that the vertices of the moment polytope do not correspond to T-fixed points—this is an example of a q-Hamiltonian space with no T-fixed points whatsoever.

Also shown in Figure 2.3 are the critical values which are closest to the origin. In this case we can choose $\gamma = 0$, and the central contribution is identically zero. We next determine the contribution from the critical value $\beta = a$ lying in the fundamental alcove. Let $\mathfrak{t}_{\Delta} \subset \mathfrak{t}$ be the subalgebra generated by β (the direction orthogonal to the wall Δ), and let $\mathfrak{t}_{\Delta}^{\perp}$ be its orthogonal complement. We have a direct sum decomposition $\mathfrak{t}^* = \mathfrak{t}_{\Delta}^* \oplus (\mathfrak{t}_{\Delta}^{\perp})^*$.

As usual, let N denote the abelianization. The critical set $Z_{\beta} = \Phi_a^{-1}(\exp(\beta)) \cap N^{\mathfrak{t}_{\Delta}}$ is (topologically) a circle inside the 2-torus $N^{\mathfrak{t}_{\Delta}}$. According to the discussion in Sections 2.3—2.6, the contribution can be computed using the local normal form: $\mathcal{V} \times (\mathfrak{t}_{\Delta}^{\perp})^*$. Since M is multiplicity-free, the cross-section is 4-dimensional, and so $\mathcal{V} \simeq Z_{\beta} \times \mathbb{C}$. Therefore the local normal form is

$$\mathcal{V} \times (\mathfrak{t}^{\perp}_{\Lambda})^* \simeq S^1 \times \mathbb{C} \times \mathbb{R}.$$

The weight of T_{Δ} on the rank 1 complex line bundle $S^1 \times \mathbb{C}$ is 1.

According to equation (2.26), the contribution \mathfrak{m}_{β} is

$$\mathfrak{m}_{\beta} = \delta_{\beta} * (H_1 \otimes d\mathrm{vol}(\xi)),$$

where $\xi \in (\mathfrak{t}_{\Delta}^{\perp})^*$, $dvol(\xi)$ is normalized Lebesgue measure on $(\mathfrak{t}_{\Delta}^{\perp})^*$, and H_1 is a Heaviside distribution on \mathfrak{t}_{Δ}^* . The contributions from the critical points b, c are Weyl reflections of the contribution from a, and also have the opposite sign (-1), coming from the factor $\operatorname{Eul}(\mathfrak{g}_g/\mathfrak{t},\partial)$. The sum of the contributions from a, b, c is shown in Figure 2.3. Adding the contributions from more critical values produces similar additional corrections.

2.4 Abelian localization

The abelian localization formula in equivariant cohomology goes back to the well-known papers of Berline-Vergne [11] and Atiyah-Bott [7]. In this section we give a proof of a version of abelian localization used in the main part of the paper. This version appeared first in [54], and a generalization was proved in [48], [49]. We only treat the special case of the total space of a vector bundle, although it is not difficult to combine the cobordism methods with the vector bundle case here to obtain more general results. Let H be a torus with Lie algebra \mathfrak{h} . Let $\pi : \mathcal{V} \to Z$ be an oriented, even-rank, H-equivariant vector bundle over a compact oriented base Z, such that $\mathcal{V}^H = Z$.

We will want basic bounds on the growth of differential forms along the fibres of $\pi : \mathcal{V} \to Z$. Choose an inner product on the fibres, a metric on the base Z and a connection on $\mathcal{V} \to Z$. The connection, inner product on the fibres, and metric on the base, induce a metric on \mathcal{V} . Let $|\cdot|$ denote the corresponding norm on the fibres of $T\mathcal{V}$, and we use the same notation for the induced norm on the fibres of $\wedge T^*\mathcal{V}$. We will say that $f \in C^{\infty}(\mathcal{V})$ is rapidly decreasing along the fibres if, for each $p \in Z$ and $0 \neq v \in \mathcal{V}_p$, the function $t \mapsto f(tv)$ is rapidly decreasing. Let dvol denote the Riemannian volume measure on \mathcal{V} ; it has the property that the volume of the disc bundle of radius r is a polynomial in r of degree equal to the real rank of the vector bundle \mathcal{V} . We will say that a form $\eta \in \Omega(\mathcal{V})$ is rapidly decreasing along the fibres if $|\eta|$ is rapidly decreasing along the fibres; in this case η is integrable, and

$$\left|\int_{\mathcal{V}} \eta\right| \leq \int_{\mathcal{V}} |\eta| d\text{vol.}$$

Proposition 2.4.1. Let H, \mathcal{V}, Z be as above. Let η be a differential form which is rapidly decreasing along the fibres of $\pi : \mathcal{V} \to Z$. Let $X = X_1 + iX_2 \in \mathfrak{h}_{\mathbb{C}}$ and write $W = 2\pi i X$. Suppose that $\mathcal{V}^{X_2} = Z$ and

$$d_W\eta := (d - \iota(W))\eta = 0.$$

Then we have the following abelian localization formula:

$$\int_{\mathcal{V}} \eta = \int_{Z} \frac{\eta}{\operatorname{Eul}(\mathcal{V}, W)}$$

Proof. We adapt the argument in [25]. Since $\mathcal{V}^{X_2} = Z$, the weights of the *H*-action on the fibres of \mathcal{V} do not vanish on X_2 , and so do not vanish on W. It follows that $\text{Eul}(\mathcal{V}, W)$ is invertible. We consider the integral

$$\int_{Z} \eta \operatorname{Eul}^{-1}(\mathcal{V}, W) \tag{2.27}$$

Let τ be a compactly supported *H*-equivariant Thom form for the vector bundle \mathcal{V} . Using the defining

property of the Thom form, we can re-write equation (2.27) as an integral over \mathcal{V} :

$$\int_{Z} \eta \operatorname{Eul}^{-1}(\mathcal{V}, W) = \int_{\mathcal{V}} \eta \tau(W) \pi^* \operatorname{Eul}^{-1}(\mathcal{V}, W).$$
(2.28)

 τ is equivariantly cohomologous to the pullback of the equivariant Euler form $\pi^* \text{Eul}(\mathcal{V})$, hence

$$\tau - \pi^* \mathrm{Eul}(\mathcal{V}) = d_H \lambda,$$

where $\lambda \in \Omega_H(\mathcal{V})$. Equation (2.28) becomes

$$\int_{Z} \eta \operatorname{Eul}^{-1}(\mathcal{V}, W) = \int_{\mathcal{V}} \eta(d_{W}\lambda(W) + \pi^{*}\operatorname{Eul}(\mathcal{V}, W))\pi^{*}\operatorname{Eul}^{-1}(\mathcal{V}, W).$$

One way to obtain a suitable form λ is by using the standard de Rham homotopy operator. This involves pulling τ back along the scalar multiplication map $[0,1] \times \mathcal{V} \to \mathcal{V}$, $(t,v) \mapsto tv$, and then integrating over [0,1]. For λ obtained in this way, the norms $|\lambda(W)|$ and $|d_W\lambda(W)|$ grow slowly along the fibres (in fact both can be bounded by a constant). Since η is rapidly decreasing along the fibres, its product with either $d_W\lambda(W)$ or $\lambda(W)$ is again rapidly decreasing along the fibres. It follows that the integral

$$\int_{\mathcal{V}} \eta \pi^* \operatorname{Eul}^{-1}(\mathcal{V}, W) d_W \lambda(W), \qquad (2.29)$$

exists, and vanishes by Stokes' theorem.

For the remaining term, $\pi^* \text{Eul}(\mathcal{V}, W)$ cancels, and we obtain:

$$\int_{\mathcal{V}} \eta = \int_{Z} \eta \mathrm{Eul}^{-1}(\mathcal{V}, W).$$

We now specialize somewhat. Let H, \mathcal{V}, Z be as above. Let α be a bounded H-equivariantly closed form on \mathcal{V} . Let $\omega - \phi$ be a H-equivariantly closed 2-form on \mathcal{V} . Suppose there exists a vector $\overline{\beta} \in \mathfrak{h}$ such that (1) $\mathcal{V}^{\overline{\beta}} = Z$, (2) there is an open cone \mathcal{C} of directions around $\overline{\beta} \in \mathfrak{h}$ such that for $\xi \in \mathcal{C}, \langle \phi, \xi \rangle$ has homogeneous growth along the fibres of π , for some positive exponent (ϕ is *polarized*). More precisely, assume that there are constants C > 0, a > 0, D such that for $\xi \in \mathcal{C}, |\xi| = 1$ we have

$$|\langle \phi, \xi \rangle|(v) \ge C|v|^a + D.$$

The polarization condition implies that ϕ is proper, and moreover, that if f is a Schwartz function on \mathfrak{h} then $\phi^* f$ is rapidly decreasing along the fibres of π .

Since $\mathcal{V}^{\overline{\beta}} = Z$, the weights of the *H*-action on the fibres of \mathcal{V} do not vanish on $\overline{\beta}$. Consequently, the form $\operatorname{Eul}(\mathcal{V}, X - is\overline{\beta})$ is invertible for all $X \in \mathfrak{h}, s \in \mathbb{R} \setminus \{0\}$. Following Paradan [48], we introduce the following *H*-equivariantly closed differential form with tempered generalized coefficients:

$$\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, X) := \lim_{s \to 0^+} \frac{1}{\operatorname{Eul}(\mathcal{V}, X - is\overline{\beta})}$$

(The intended meaning of this expression is that the fraction should be integrated against a smooth rapidly decreasing measure on \mathfrak{h} , before the limit is taken.) That this form has tempered generalized

coefficients follows from the fact that the generalized function

$$\phi(x) = \lim_{s \to 0^+} \frac{1}{x - is}$$

is tempered.

Theorem 2.4.2. Let $H, \mathcal{V}, Z, \omega - \phi, \overline{\beta}, \alpha$ be as above. We have the following equality of tempered generalized functions on \mathfrak{h} :

$$\int_{\mathcal{V}} \alpha(2\pi i X) e^{\omega - 2\pi i \langle \phi, X \rangle} = \int_{Z} \alpha(2\pi i X) e^{\omega - 2\pi i \langle \phi, X \rangle} \operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X).$$
(2.30)

(In this expression, the integrands should be paired with a rapidly-decreasing smooth measure on \mathfrak{h} before the integrals over \mathcal{V} , Z are performed.)

Proof. Let f dX be a rapidly decreasing smooth measure on \mathfrak{h} , with Fourier transform \hat{f} . Write $\alpha = \sum_k \alpha_k p_k$. Then

$$\int_{\mathfrak{h}} \alpha(2\pi i X) e^{\omega - 2\pi i \langle \phi, X \rangle} f(X) \, dX = \sum_{k} \alpha_{k} e^{\omega} \phi^{*}(p_{k}(-\partial)\hat{f})$$

Since $p_k(-\partial)\hat{f}$ is a Schwartz function, its pullback by ϕ is rapidly decreasing along the fibres of π . It follows that the left-hand-side of (2.30) defines a tempered generalized function. Moreover, we have:

$$\begin{split} \int_{\mathcal{V}} \int_{\mathfrak{h}} \alpha(2\pi i X) e^{\omega - 2\pi i \langle \phi, X \rangle} f(X) \, dX &= \int_{\mathcal{V}} \int_{\mathfrak{h}} \lim_{s \to 0^+} \alpha(2\pi i (X - i s \overline{\beta})) e^{\omega - 2\pi i \langle \phi, X - i s \overline{\beta} \rangle} f(X) \, dX, \\ &= \lim_{s \to 0^+} \int_{\mathcal{V}} \int_{\mathfrak{h}} \alpha(2\pi i (X - i s \overline{\beta})) e^{\omega - 2\pi i \langle \phi, X - i s \overline{\beta} \rangle} f(X) \, dX, \\ &= \lim_{s \to 0^+} \int_{\mathfrak{h}} f(X) \, dX \int_{\mathcal{V}} \alpha(2\pi i (X - i s \overline{\beta})) e^{\omega - 2\pi i \langle \phi, X - i s \overline{\beta} \rangle}. \end{split}$$

Here, the second line follows from the dominated convergence theorem, since the integrand is rapidly decreasing along the fibres of \mathcal{V} and on \mathfrak{h} . The third line follows from Fubini's theorem, using the fact that $e^{-2\pi i \langle \phi, X - is\overline{\beta} \rangle}$ is rapidly decreasing along the fibres (so that the decay of f(X) is no longer needed to guarantee convergence of the integral over \mathcal{V}). We can now apply the previous Proposition 2.4.1 to the integral over \mathcal{V} , since the integrand is rapidly decreasing along the fibres, and is closed for the differential d_W , $W = 2\pi i (X - is\overline{\beta})$. The result is:

$$\int_{\mathcal{V}} \int_{\mathfrak{h}} \alpha(2\pi i X) e^{\omega - 2\pi i \langle \phi, X \rangle} f(X) \, dX = \lim_{s \to 0^+} \int_{\mathfrak{h}} f(X) \, dX \int_{Z} \frac{\alpha(2\pi i (X - i s\overline{\beta})) e^{\omega - 2\pi i \langle \phi, X - i s\overline{\beta} \rangle}}{\operatorname{Eul}(\mathcal{V}, 2\pi i (X - i s\overline{\beta}))}.$$

Applying Fubini's theorem and the dominated convergence theorem now in the opposite direction gives the result. $\hfill \Box$

In order to directly apply Theorem 2.4.2 in the main part of the paper, we would need to have a version of 2.4.2 for the orbifold \mathcal{V}/T'_{Δ} . Alternatively, we can slightly modify 2.4.2 to work with basic forms on \mathcal{V} , which is what we do now.

Corollary 2.4.3. Let $H, \mathcal{V}, Z, \omega - \phi, \overline{\beta}, \alpha$ be as in Theorem 2.4.2. Let H' be a second torus acting on \mathcal{V} and commuting with H. Suppose the action of H' on the base Z is locally free, and that α , $(\omega - \phi)$

are H'-basic. Let $q: \mathcal{V} \to \mathcal{V}/H'$ be the quotient map, and let $\nu \in \Omega(Z)$ be a form such that $q_*\nu = c$, a constant. Then we have the following equality of tempered generalized functions on \mathfrak{h} :

$$\int_{\mathcal{V}} \alpha(2\pi i X) \nu e^{\omega - 2\pi i \langle \phi, X \rangle} = \int_{Z} \alpha(2\pi i X) \nu e^{\omega - 2\pi i \langle \phi, X \rangle} \operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X).$$
(2.31)

Proof. Since H' acts locally freely on the base Z, we can choose an H'-invariant connection σ on $\pi : \mathcal{V} \to Z$ such that the vector fields generated by the H' action are horizontal. The connection 1-form is then H'-basic, and thus the corresponding representative for the Euler form is H'-basic. We can also take the Thom form τ to be H'-basic (for example, using the Mathai-Quillen representative built using σ). The proof is now almost exactly the same: we repeat the proof of Proposition 2.4.1 with $\eta\nu$ replacing η , and then repeat the proof of Theorem 2.4.2 with $\alpha\nu$ replacing α . The only step requiring modification is the final step (2.29) in the proof of Proposition 2.4.1: we are left with the integral

$$\int_{\mathcal{V}} \eta \nu \pi^* \mathrm{Eul}^{-1}(\mathcal{V}, W) d_W \lambda(W) = \int_{\mathcal{V}} d_W \bigg(\eta \pi^* \mathrm{Eul}^{-1}(\mathcal{V}, W) \lambda(W) \bigg) \nu.$$

All forms in the integrand are H'-basic, except ν (λ is basic if we use the de Rham homotopy operator, for example). Integrating over the fibres of q and applying the orbifold version of Stokes' theorem on \mathcal{V}/H' shows that this integral vanishes.

In the main part of the paper, we apply 2.4.3 to the case $H = T_{\Delta}$, $H' = T'_{\Delta}$, $\omega - \phi = \omega_{\mathcal{V}} - \phi_{\mathcal{V}}$, and α is the T_{Δ} -equivariantly closed, T'_{Δ} -basic (in fact, *T*-basic) form $\overline{\alpha}(X)e^{\langle \xi, F \rangle}$ with $\xi \in (\mathfrak{t}_{\Delta}^{\perp})^*$ fixed.

Finally, we describe briefly how to compute the inverse Fourier transform of $\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V})(2\pi i X)$. Choose a complex structure on \mathcal{V} such that the complex weights $\alpha_1^-, ..., \alpha_n^-$ of the *H*-action have negative pairing with $\overline{\beta}$. The bundle \mathcal{V} splits into a direct sum of weight sub-bundles \mathcal{V}_k :

$$\mathcal{V} = \bigoplus_{k=1}^n \mathcal{V}_k,$$

where H acts on \mathcal{V}_k with weight $\alpha_k^- \in \mathfrak{h}^*$. As the Euler class is multiplicative, it suffices to give an expression for the inverse Fourier transform of $\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}_k)(2\pi i X)$ (the full result will then be a convolution). So assume \mathcal{V} has complex rank r and that H acts on \mathcal{V} by a single weight $\alpha^- \in \mathfrak{h}^*$.

Choose an H'-basic connection on \mathcal{V} with $\mathfrak{u}(\mathcal{V})$ -valued curvature R. One has

$$\operatorname{Eul}(\mathcal{V}, 2\pi i X) = \epsilon \, \det_{\mathbb{C}} \left(\frac{i}{2\pi} R - 2\pi i \langle \alpha^{-}, X \rangle \right),$$

where $\epsilon = \pm 1$ is a sign, equal to -1 iff the orientation on \mathcal{V} induced by the complex structure is opposite the original orientation on \mathcal{V} . For example, if \mathcal{V} is a (complex) line bundle, $\operatorname{Eul}(\mathcal{V}, 2\pi i X) = \epsilon(x - 2\pi i \langle \alpha^-, X \rangle)$ where $x = c_1(\mathcal{V})$. Set $z = -2\pi i \langle \alpha^-, X \rangle$ and expand the determinant

$$\operatorname{Eul}(\mathcal{V}, 2\pi i X) = \epsilon z^r \left(1 + \sum_{m=1}^r c_m(\mathcal{V}) z^{-m} \right),$$

where $c_m(\mathcal{V})$ is the m^{th} Chern class of \mathcal{V} . Inverting and expanding in power series:

$$\frac{1}{\operatorname{Eul}(\mathcal{V}, 2\pi i X)} = \frac{\epsilon}{z^r} \sum_{\ell \ge 0} \left(-\sum_{m=1}^r c_m(\mathcal{V}) z^{-m} \right)^{\ell}.$$

Thus

$$\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X) = \lim_{s \to 0^+} \frac{\epsilon}{(-2\pi i \langle \alpha^-, X - i s \overline{\beta} \rangle)^r} \sum_{\ell \ge 0} \left(-\sum_{m=1}^r \frac{c_m(\mathcal{V})}{(-2\pi i \langle \alpha^-, X - i s \overline{\beta} \rangle)^m} \right)^\ell$$

The sum over ℓ truncates since $c_m(\mathcal{V})^\ell = 0$ for ℓ larger than half the dimension of the base. Using the fact that $\langle \alpha^-, \overline{\beta} \rangle < 0$,

$$\mathcal{F}^{-1}\left(\lim_{s\to 0^+}\frac{1}{-2\pi i\langle\alpha^-, X-is\overline{\beta}\rangle}\right) = H_{-\alpha^-} = H_{\alpha^+}$$

where $\alpha^+ = -\alpha^-$ (note $\langle \alpha^+, \overline{\beta} \rangle > 0$) and H_{α^+} denotes the Heaviside distribution defined by

$$\langle H_{\alpha^+}, f \rangle = \int_0^\infty dt f(t\alpha^+).$$

Since the Fourier transform of a product is the convolution of the Fourier transforms, we obtain

$$\mathcal{F}^{-1}(\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi i X)) = \epsilon H_{\alpha^+}^r * \sum_{\ell \ge 0} \left(-\sum_{m=1}^r c_m(\mathcal{V}) H_{\alpha^+}^m \right)^{\ell}.$$

Hence in general we have

$$\mathcal{F}^{-1}(\operatorname{Eul}_{\overline{\beta}}^{-1}(\mathcal{V}, 2\pi iX)) = \epsilon \prod_{k=1}^{n} H_{\alpha_{k}^{+}}^{r_{k}} * \sum_{\ell \ge 0} \left(-\sum_{m=1}^{r_{k}} c_{m}(\mathcal{V}_{k}) H_{\alpha_{k}^{+}}^{m} \right)^{\ell}.$$

The exponents and product over k in this expression denote convolution, and r_k is the complex rank of \mathcal{V}_k .

2.5 Hamiltonian cobordisms

In this section, we recall the definition of Hamiltonian cobordisms ([21],[22],[26]), and then outline proofs of some key results referred to in Section 1.

Definition 2.5.1. Let $(N_i, \omega_i, \phi_i, \alpha_i)$, i = 0, 1 and $(\tilde{N}, \tilde{\omega}, \tilde{\phi}, \tilde{\alpha})$ be Hamiltonian *G*-spaces equipped with closed equivariant differential forms. We say that $(\tilde{N}, \tilde{\omega}, \tilde{\phi}, \tilde{\alpha})$ is a *proper Hamiltonian cobordism* between $(N_0, \omega_0, \phi_0, \alpha_0)$ and $(N_1, \omega_1, \phi_1, \alpha_1)$ if $\tilde{\phi}$ is proper on the support of $\tilde{\alpha}$ and

$$\partial \tilde{N} = N_0 \sqcup (-N_1), \quad \iota^*_{\partial \tilde{N}} (\tilde{\omega} - \tilde{\phi}) = (\omega_0 - \phi_0) \sqcup (\omega_1 - \phi_1), \quad \iota^*_{\partial \tilde{N}} \tilde{\alpha} = \alpha_0 \sqcup \alpha_1.$$

Remark 2.5.2. Note that the definition implies $\tilde{\phi}_i$ is proper on the support of α_i , i = 0, 1.

Theorem 2.5.3 ([26] Lemma 5.10). In the setting of Definition 2.5.1,

$$DH(N_0, \omega_0, \phi_0, \alpha_0) = DH(N_1, \omega_1, \phi_1, \alpha_1).$$

Proof. Using the definition, the difference is a distribution \mathfrak{m} whose pairing with $f \in C^{\infty}_{\text{comp}}(\mathfrak{g}^*)$ is

$$\begin{split} \langle \mathfrak{m}, f \rangle &= \int_{\partial \tilde{N}} \iota^*_{\partial \tilde{N}} (e^{\tilde{\omega}} \tilde{\alpha}(-\partial) f \circ \tilde{\phi}) \\ &= \int_{\tilde{N}} d(e^{\tilde{\omega}} \tilde{\alpha}(-\partial) f \circ \tilde{\phi}). \end{split}$$

Let $\{U_k\}$ be a locally finite open cover of \mathfrak{g}^* by relatively compact sets, and let $\{\rho_k\}$ be a partition of unity subordinate to this open cover. Define distributions \mathfrak{m}_k ,

$$\langle \mathfrak{m}_k, f \rangle = \int_{\tilde{N}} (\tilde{\phi}^* \rho_k) \, d(e^{\tilde{\omega}} \tilde{\alpha}(-\partial) f \circ \tilde{\phi})$$

Then $\operatorname{supp}(\mathfrak{m}_k) \subset \overline{U_k}$, and it follows that $\mathfrak{m} = \sum_k \mathfrak{m}_k$ (the sum is locally finite) and each \mathfrak{m}_k is compactly supported. To show $\mathfrak{m}_k = 0$ it suffices to show all of its Fourier coefficients vanish. For $X \in \mathfrak{g}$ fixed

$$\langle \mathfrak{m}_{k}, e^{-2\pi i \langle -, X \rangle} \rangle = \int_{\tilde{N}} (\tilde{\phi}^{*} \rho_{k}) d(e^{\tilde{\omega} - 2\pi i \langle \tilde{\phi}, X \rangle} \tilde{\alpha}(2\pi i X))$$
$$= 2\pi i \int_{\tilde{N}} (\tilde{\phi}^{*} \rho_{k}) \iota(X_{N}) (e^{\tilde{\omega} - 2\pi i \langle \tilde{\phi}, X \rangle} \tilde{\alpha}(2\pi i X)),$$

where we have used $(d - 2\pi i \iota(X_N))(e^{\tilde{\omega} - 2\pi i \langle \tilde{\phi}, X \rangle} \tilde{\alpha}(2\pi i X)) = 0$. The last line vanishes, because the integrand has no top-degree part.

The next result can be compared to Theorem 2.1.1.

Theorem 2.5.4 ([26]). Let N be a G-manifold, and α a closed equivariant differential form on N. Let $v: N \to \mathfrak{g}$ be a bounded taming map. Let $\omega_i - \phi_i$, i = 0, 1 be two equivariant 2-forms on N. Suppose that

- 1. $[\omega_0 \phi_0] = [\omega_1 \phi_1] \in H_G(N)$
- 2. $\langle \phi_i, v \rangle$ is proper and bounded below on supp (α) , i = 0, 1.

Then $DH(N, \omega_0, \phi_0, \alpha) = DH(N, \omega_1, \phi_1, \alpha).$

Proof. Write $(\omega_1 - \phi_1) - (\omega_0 - \phi_0) = d_G \eta$ for some $\eta \in \Omega^1_G(N)$. Let $W = N \times [0, 1]$, and pull back all equivariant forms as well as v by pr₁ (pullbacks omitted from notation). Let t be the coordinate on [0, 1]. Define $\tilde{\omega}_G = \tilde{\omega} - \tilde{\phi}$ by

$$\tilde{\omega}_G = (\omega_0)_G + d_G(t\eta)$$

We have $\tilde{\phi} = (1-t)\phi_0 + t\phi_1$, and thus

$$\langle \phi, v \rangle = (1-t) \langle \phi_0, v \rangle + t \langle \phi_1, v \rangle.$$

Since a convex linear combination of proper, bounded below maps is proper and bounded below, $\langle \phi, v \rangle$ is proper and bounded below on supp (α) . Since v is bounded, $\tilde{\phi}$ is proper (Prop. 2.1.5). $(W, \tilde{\omega}, \tilde{\phi}, \alpha)$ is the desired proper Hamiltonian cobordism.

Remark 2.5.5. Recall (see remark 2.1.8) that one situation in which the condition $[\omega_0 - \phi_0] = [\omega_1 - \phi_1] \in H_G(N)$ is satisfied, is if N can be smoothly collapsed to part of a G-invariant submanifold Z, and $\iota_Z^*(\omega_0 - \phi_0) = \iota_Z^*(\omega_1 - \phi_1)$.

The Harada-Karshon Theorem is proven by showing that N is cobordant to a small neighbourhood of the critical set Z. The main challenge in the cobordism approach ([21],[22],[26]) is to construct a proper moment map on the cobording manifold. As already pointed out in Lemma 2.1.5, when v is bounded, $\langle \phi, v \rangle$ proper and bounded below implies that ϕ is proper. The convenience of working with the condition that $\langle \phi, v \rangle$ be proper and bounded below comes from two point-set topology facts:

- (A) A finite collection of proper, bounded below functions f_i can be patched together with a partition of unity, and the result is again proper and bounded below ([26], Lemma 3.5).
- (B) For G compact, the G-average of a proper, bounded below function is again proper and bounded below ([26], Lemma 3.6).

We next outline the proof of Lemma 2.1.9, which we break into 'existence' and 'uniqueness' parts.

Lemma 2.5.6 ('existence', [26] Proposition 3.4). Let (N, ω, ϕ) be a Hamiltonian G-manifold, possibly with boundary, and α a closed equivariant differential form. Let $v : N \to \mathfrak{g}$ be a bounded taming map with localizing set Z.² Let $Y \supset Z$ be a G-invariant closed set. Suppose that $\langle \phi, v \rangle$ is proper and bounded below on $Y \cap \operatorname{supp}(\alpha)$. Then there exists a v-polarized completion $(N, \tilde{\omega}, \tilde{\phi})$ of $(N, \omega, \phi, \alpha)$ such that $\tilde{\omega} - \tilde{\phi}$ equals $\omega - \phi$ on a neighbourhood of Y.

Proof. Since $\langle \phi, v \rangle$ is proper and bounded below on $Y \cap \operatorname{supp}(\alpha)$, it is possible to find a *G*-invariant open neighbourhood $U \supset Y \cap \operatorname{supp}(\alpha)$ such that $\langle \phi, v \rangle$ is proper and bounded below on \overline{U} ([26], Lemma 3.8). Let $U_Y = (N \setminus \operatorname{supp}(\alpha)) \cup U$ (an open neighbourhood of *Y*). Let ρ_1, ρ_2 be a partition of unity subordinate to the open cover $U_Y, N \setminus Y$, and let $f : N \to \mathbb{R}$ be any proper and bounded below smooth function. Let

$$\psi' = \rho_1 \langle \phi, v \rangle + \rho_2 f.$$

It's clear that ψ' agrees with $\langle \phi, v \rangle$ on a neighbourhood of Y. Note that ψ' is proper and bounded below on \overline{U} , by point-set topology fact (A). Also, ψ' agrees with f on $N \setminus U_Y$, and so $\psi'|_{N \setminus U_Y}$ is proper and bounded below. Since $\operatorname{supp}(\alpha) \subset \overline{U} \cup (N \setminus U_Y)$ (a union of two closed sets), this proves that ψ' is proper and bounded below on $\operatorname{supp}(\alpha)$. Now set ψ to be the G-average of ψ' . By point-set topology fact (B), ψ is proper and bounded below on $\operatorname{supp}(\alpha)$. Moreover ψ equals $\langle \phi, v \rangle$ near Y.

Choose a G-invariant Riemannian metric g on N, and define a 1-form Θ on $N \setminus Z$,

$$\Theta = \frac{g(v_N, -)}{g(v_N, v_N)}$$

Finally, let

$$\tilde{\omega} - \tilde{\phi} = \tilde{\omega}_G := \omega_G + d_G \big((\psi - \langle \phi, v \rangle) \Theta \big).$$

Since ψ and $\langle \phi, v \rangle$ are equal near $Y, \tilde{\omega}_G$ is defined on all of N and equals ω_G on a neighbourhood of Y. And

$$\langle \phi, v \rangle = \psi$$

which is proper and bounded below on $\operatorname{supp}(\alpha)$ by construction. $(N, \tilde{\omega}, \tilde{\phi})$ is the desired *v*-polarized completion.

²We do not require that Z be smooth here.

Lemma 2.5.7 ('uniqueness', [26] Lemmas 4.12, 4.17). Let (N, ω, ϕ) be an oriented Hamiltonian Gmanifold, and α a closed equivariant differential form. Let $v_i, Z_i, (U_i, \tilde{\omega}_i, \tilde{\phi}_i), i = 0, 1$ be two sets of data satisfying the following conditions:

- 1. $v_i: N \to \mathfrak{g}$ is a bounded taming map with smooth localizing set Z_i ,
- 2. U_i is a G-invariant open set that can be smoothly collapsed to part of Z_i ,
- 3. $\langle \tilde{\phi}_i, v_i \rangle$ is proper and bounded below on $Z_i \cap \text{supp}(\alpha)$.

Suppose further that v_0 , v_1 agree on $supp(\alpha)$. Then

$$\mathrm{DH}(U_1, \tilde{\omega}_1, \tilde{\phi}_1, \alpha) = \mathrm{DH}(U_0, \tilde{\omega}_0, \tilde{\phi}_0, \alpha).$$

Proof. Put $U = U_0 \cup U_1$ and

$$W = (U \times [0,1]) \setminus ((U \setminus U_0) \times \{0\} \cup (U \setminus U_1) \times \{1\})$$

Pull back ω , ϕ , v_0, v_1 , α along the projection $U \times [0, 1] \to U$, and then restrict to W (pullbacks omitted from notation). Using t as a coordinate on [0, 1], let

$$v = (1 - t)v_0 + tv_1,$$

and let $Z = \{p|v_W(p) = 0\}$ be the corresponding localizing set (it need not be smooth). Since v_1, v_0 agree on supp (α) ,

$$v|_{\operatorname{supp}(\alpha)} = v_1|_{\operatorname{supp}(\alpha)} = v_0|_{\operatorname{supp}(\alpha)}$$

It follows that $\langle \phi, v \rangle$ is proper and bounded below on $\operatorname{supp}(\alpha) \cap Z$. Let $(W, \tilde{\omega}, \tilde{\phi})$ be a *v*-polarized completion of $(W, \omega, \phi, \alpha)$ (using Lemma 2.5.6). Since *v* is bounded, $\tilde{\phi}$ is proper on the support of α . Thus *W* is a proper Hamiltonian cobordism between the two boundary components U_0, U_1 . By Theorem 2.5.3

$$DH(U_0, \tilde{\omega}|_{U_0}, \phi|_{U_0}, \alpha) = DH(U_1, \tilde{\omega}|_{U_1}, \phi|_{U_1}, \alpha).$$
(2.32)

For $i = 0, 1, (U_i, \tilde{\omega}|_{U_i}, \tilde{\phi}|_{U_i})$ is a $v|_{U_i} = v_i$ -polarized completion of $(U_i, \omega, \phi, \alpha)$. However, recall that U_i is a *G*-invariant open set that can be smoothly collapsed to part of Z_i . Remark 2.5.5 shows that in this case, any v_i -polarized completion has the *same* DH distribution. Hence, the result actually follows from (2.32).

Finally, we outline the proof of Theorem 2.1.11.

Theorem 2.5.8 ([26] Theorem 5.20). Consider the setting of Lemma 2.1.9. Suppose further that $\langle \phi, v \rangle$ is proper and bounded below on the support of α . Then

$$\mathrm{DH}(N,\omega,\phi,\alpha) = \mathrm{DH}_Z^v(N,\omega,\phi,\alpha).$$

Proof. Let U be a G-invariant tubular neighbourhood of Z, and put

$$W = (N \times [0,1]) \setminus ((N \setminus U) \times \{1\}), \qquad Y = (Z \times [0,1]) \cup (N \times \{0\}).$$

Pull back the equivariant forms and v to $N \times [0, 1]$ and restrict to W (pullbacks omitted from notation). By assumption, $\langle \phi, v \rangle$ is proper and bounded below on $Y \cap \text{supp}(\alpha)$. Apply Lemma 2.5.6 to obtain a v-polarized completion $(W, \tilde{\omega}, \tilde{\phi})$ of $(W, \omega, \phi, \alpha)$, with $\tilde{\omega} - \tilde{\phi}$ being equal to $\omega - \phi$ in a neighbourhood of Y (in particular, near $N \times \{0\}$). Then W is a proper Hamiltonian cobordism between $(N, \omega, \phi, \alpha)$ and $(U, \tilde{\omega}|_{U \times \{1\}}, \tilde{\phi}|_{U \times \{1\}}, \alpha|_U)$. The result follows from Theorem 2.5.3 and Remark 2.5.5.

2.6 Smoothness of Z

Here we discuss the role of the assumption (Lemma 2.1.9) that the localizing set Z is a smooth submanifold. The smoothness assumption leads to a particularly simple description of the contribution of a component $Z_i \subset Z$ to (2.4): it is the twisted DH distribution of any v-polarized completion of a tubular neighbourhood of Z_i .

However, in general $Z = \{v_N = 0\}$ is not smooth. On the other hand, the cobordism used in the H-K Theorem (see Theorem 2.5.8, or [26] Theorem 5.20) does not use the smoothness assumption. Neither does the 'existence' part of Lemma 2.1.9, since, for example, for the 'polarized completion' $(U, \tilde{\omega}, \tilde{\phi})$ which is constructed (see Lemma 2.5.6, or [26] Proposition 3.4), $\tilde{\omega} - \tilde{\phi}$ agrees with $\omega - \phi$ on an open neighbourhood of Z, and so also makes sense when Z is singular. The assumption only plays a role in the 'uniqueness' part of Lemma 2.1.9 (see Lemma 2.5.7, or [26] Lemmas 4.12, 4.17). Thus, one obtains a localization formula for the Duistermaat-Heckman distribution quite generally, but the contribution DH $(U_i, \tilde{\omega}, \tilde{\phi}, \alpha)$ of a component Z_i is not as simple to describe.

In the general case, one can use the polarized completion appearing in the proof of Lemma 2.5.6. Alternatively, to get hold of the contribution $DH(U_i, \tilde{\omega}, \tilde{\phi}, \alpha)$, one can try to choose a new taming map v' on $N' := U_i$ now with a smooth localizing set, and then apply the H-K Theorem to N' obtaining a new Hamiltonian cobordism from N' to a collection of smaller open sets U'_j around the components Z'_j of the localizing set for v'. Then

$$\mathrm{DH}(N',\tilde{\omega},\tilde{\phi},\alpha) = \sum_{j} \mathrm{DH}_{Z'_{j}}^{v'}(N',\omega',\phi',\alpha)$$

and the uniqueness part of Lemma 2.1.9 applies to the terms on the right side of the equation. In this way one obtains a formula for $DH(N, \omega, \phi, \alpha)$ using a composition of Hamiltonian cobordisms.

In the case $v = \phi$ where $\phi : N \to \mathfrak{t}^* \simeq \mathfrak{t}$ is the moment map for the action of a torus T, a suitable v'on $N' = U_i$ can be obtained by perturbing v to $v' = v - \gamma$, where γ is a small 'generic' element of \mathfrak{t} (see Section ??). The perturbation γ can even be chosen independently for each component Z_i of Z. This is similar to the perturbation used by Paradan in [49].

To see that the Harada-Karshon Theorem applies to $(N' = U_i, \tilde{\omega}, \tilde{\phi}, \alpha)$ equipped with taming map $v' = v - \gamma$ (with γ sufficiently small), one needs to check that $\langle \tilde{\phi}, v' \rangle$ is proper and bounded below on $N' \cap \operatorname{supp}(\alpha)$. The moment map $\tilde{\phi}$ of the polarized completion $(N', \tilde{\omega}, \tilde{\phi})$ can be obtained from the proof of Lemma 2.5.6:

$$\tilde{\phi} = \phi - (\psi - \|\phi\|^2)\Theta_N, \quad \langle\Theta_N, X\rangle = \frac{g(v_N, X_N)}{g(v_N, v_N)}, \quad X \in \mathfrak{t}$$

where g is a T-invariant Riemannian metric on N and $\psi \ge \|\phi\|^2$ is a smooth T-invariant function on N' which is proper and bounded below on supp (α) , and agrees with $\langle \phi, v \rangle = \|\phi\|^2$ on a neighbourhood of Z_i . (Note that $\psi - \|\phi\|^2 = 0$ on a neighbourhood of $Z_i = \{v_N = 0\} \cap N'$, so it is not a problem that Θ_N is not defined there.) Then

$$\langle \tilde{\phi}, v' \rangle = \psi - \langle \phi, \gamma \rangle + (\psi - \|\phi\|^2) \frac{g(v_N, \gamma_N)}{g(v_N, v_N)}$$

In our case, the neighbourhood $N' = U_i$ is such that $\operatorname{supp}(\alpha) \cap \overline{N'}$ is compact and $\overline{N'} \cap \{v_N = 0\} = Z_i$. This implies that $|\langle \phi, \gamma \rangle| \leq C_1$ and $|g(v_N, \gamma_N)| \leq C_2$ are bounded on $\operatorname{supp}(\alpha) \cap \overline{N'}$, and that $g(v_N, v_N)$ can be bounded below by a positive constant $0 < C_3 \leq 1$ on $(\overline{N'} \cap \operatorname{supp}(\alpha)) \setminus \operatorname{supp}(\psi - \|\phi\|^2)$. Replacing γ with $C_3(C_2 + 1)^{-1}\gamma$, we have

$$\langle \tilde{\phi}, v' \rangle \ge \psi - C_1 - \frac{C_2}{C_2 + 1}\psi = \frac{1}{C_2 + 1}\psi - C_1$$

on $N' \cap \operatorname{supp}(\alpha)$. This implies that $\langle \tilde{\phi}, v' \rangle$ is proper and bounded below on $N' \cap \operatorname{supp}(\alpha)$.

Because of the 'uniqueness' part of Lemma 2.1.9, the formula obtained from the above composition of Hamiltonian cobordisms is the same as that obtained from perturbing the taming map $v' = v - \gamma$ from the beginning, as we did in the main part of the paper. From this one can see that the terms \mathfrak{m}_{β} of (2.14) are the same as the contributions obtained using the unperturbed taming map $v = \phi$.

Remark 2.6.1. Harada-Karshon work with a weaker condition: that a neighbourhood of the localizing set Z admit a smooth equivariant weak deformation retract to Z.³ This condition is appealing in that it guarantees the 'uniqueness' part of Lemma 2.1.9. An interesting question left open in [26] is whether the localizing set Z in the case $v = \phi$ always satisfies this weaker condition.

³This is the same as what we have shortened to 'smooth collapse' (Definition 2.1.7), except with Z' not required to be smooth.

Chapter 3

Duistermaat-Heckman measures and Bernoulli series

In this chapter we derive a 'norm-square localization' formula for a Duistermaat-Heckman measure \mathfrak{m} on t associated to a Hamiltonian *LG*-space (see Chapter 2 for background). The terms of the formula agree with those in the formula proved in Chapter 2, but they are expressed in terms of 'germs' of piece-wise polynomial functions convolved with multi-splines, rather than Duistermaat-Heckman distributions of polarized completions. The method of proof is different: it is based on the abelian localization formula for the distribution \mathfrak{m} (c.f. [5], [39], [1]) and a combinatorial 'decomposition formula' for multiple Bernoulli series due to A. Boysal and M. Vergne ([15]).

As we will see, there is a close connection between the abelian localization formula for quasi-Hamiltonian G-spaces and multiple Bernoulli series. Indeed, a collection of examples are related to moduli spaces of flat connections on a compact Riemann surface having at least 1 boundary component. In these examples, the twisted Duistermaat-Heckman distributions are essentially Bernoulli series, and the norm-square localization formula coincides with the Boysal-Vergne decomposition formula. (These examples are closely related to Witten's formulas for intersection pairings on moduli spaces of flat connections on Riemann surfaces.)

This chapter is fairly brief, as it is meant mainly as a warm-up for the analogous but more complicated 'quantum version' of the problem studied in the next chapter.

3.1 Multiple Bernoulli series

In this section we study a certain class of periodic distributions known as *multiple Bernoulli series*. They were studied in depth by several authors, including [58], [17], [15], [9]. Part of the interest in these series comes from their appearance in Witten's formulas [63] for intersection pairings on moduli spaces of flat connections on compact Riemann surfaces.

3.1.1 Lattices

We collect here some simple facts about lattices. Some of these facts will not be use until Chapter 4. Let V be a finite dimensional vector space, and Γ a lattice in V. Let V^{*} be the dual vector space and $\Gamma^* = \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ the dual lattice.

Proposition 3.1.1. Let W be a subspace of V such that $W \cap \Gamma$ is a full-rank lattice in W. Let $e_1, ..., e_k$ be a \mathbb{Z} -basis of $W \cap \Gamma$. Then $e_1, ..., e_k$ can be extended to a \mathbb{Z} -basis of Γ .

Proof. The proof is by induction on the codimension, so it suffices to consider the case when $W \subset V$ is codimension 1. Let $\bar{v} \in \Gamma/W$ be a \mathbb{Z} -basis for Γ/W viewed as a lattice in V/W. Let $v \in \Gamma$ be a lift of \bar{v} . We claim that $e_1, ..., e_k, v$ is a \mathbb{Z} -basis for Γ . Indeed, suppose $u \in \Gamma$ and let $\bar{u} \in \Gamma/W$ be its image in the quotient. We have $\bar{u} = n\bar{v}$ for some $n \in \mathbb{Z}$. Therefore $u - nv \in W \cap \Gamma$, and so can be expressed as an integer linear combination of $e_1, ..., e_k$ (as the latter forms a \mathbb{Z} -basis for $W \cap \Gamma$).

Definition 3.1.2. A subspace $W \subset V$ is *rational* if W is cut out by elements of Γ^* .

Definition 3.1.3. An integral inner product B on (V, Γ) is one that takes values in \mathbb{Z} when restricted to Γ . Equivalently, $B^{\flat}(\Gamma) \subset \Gamma^*$.

Let B be an integral inner product, and $W \subset V$ a rational subspace. Using Proposition 3.1.1 one can easily show:

- 1. $U \subset V$ is rational iff $\Gamma \cap U$ is a full rank lattice in U.
- 2. The dual lattice $(\Gamma \cap W)^*$ is $\Gamma^*/\operatorname{ann}(W)$.
- 3. The orthogonal complement W^{\perp} is rational.
- 4. If pr : $V \to W$ is the orthogonal projection, then the isomorphism $B^{\flat} : W \to W^*$ identifies $(W \cap \Gamma)^*$ with pr(Γ^*).

Let $H = V/\Gamma$ (a torus). Suppose there is a second lattice $\Xi \supset \Gamma$ and set

$$H_{\ell} = \ell^{-1} \Xi / \Gamma \subset H.$$

The following result is used in Chapter 4.

Proposition 3.1.4. Let $h \in H$ and suppose there exists some ℓ such that $h \in H_{\ell}$. Let ℓ_0 be minimal such that $h \in H_{\ell_0}$. Then the set of ℓ such that $h \in H_{\ell}$ is the set of positive integer multiples of ℓ_0 .

Proof. Note that $h \in H_{\ell}$ iff $h^{\ell} \in H_1$. The set of $\ell \in \mathbb{Z}$ such that $h^{\ell} \in H_1$ is clearly an ideal in \mathbb{Z} , and ℓ_0 is the unique positive generator.

The following version of the Poisson summation formula is used in this chapter as well as Chapter 4.

Lemma 3.1.5. Let $\Gamma^* \subset \Lambda^*$ be full rank lattices in V^* , and let $W \subset V$ be a rational subspace. Let $d\xi$ be Lebesgue measure on V, normalized such that the induced measure on $T = V/\Lambda$ is normalized Haar measure. Then we have an equality of distributions

$$\sum_{\nu\in\Gamma^*\cap\mathrm{ann}(W)}e^{2\pi i\langle\nu,\xi\rangle}d\xi=\sum_{U\in\Gamma/W}\delta_U(\xi).$$

In this sum, U is understood as a coset $\gamma + W$ for $\gamma \in \Gamma$, and the normalization for δ_U is fixed to be the pullback of normalized Haar measure on the (possibly disconnected) group $\exp(\Gamma + W) \subset T$.

Proof. View the left-hand-side as a periodic distribution on V, hence a distribution on $T = V/\Lambda$. The Fourier coefficients of this distribution are 1 exactly for $\lambda \in \Gamma^* \cap \operatorname{ann}(W) \subset \Lambda^*$ and 0 otherwise. The delta distribution on $\exp(\Gamma + W)$ (using its normalized Haar measure) has the same Fourier coefficients. Now pullback to periodic distributions on V.

3.1.2 Definitions and piece-wise polynomial behaviour.

Let \mathfrak{t} be a vector space and $\Lambda \subset \mathfrak{t}$ a lattice of full rank, and $T = \mathfrak{t}/\Lambda$. Let $B(-,-) = \langle -,-\rangle$ be an integral inner product on \mathfrak{t} . Throughout we use B to identify $\mathfrak{t} \simeq \mathfrak{t}^*$, and view both Λ and its dual lattice Λ^* as subsets of \mathfrak{t} . Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ denote a list (repeated elements allowed) of elements of $\Lambda^* \otimes \mathbb{Q}$.

Definition 3.1.6. The *multiple Bernoulli series* associated to the data $(\mathfrak{t}, \Lambda^*, \alpha)$ is the distribution on \mathfrak{t} given by

$$B_{\alpha}(\mu) = \sum_{\lambda \in \Lambda^*} \frac{e^{-2\pi i \langle \mu, \lambda \rangle}}{\prod_{\alpha \in \alpha} 2\pi i \langle \alpha, \lambda \rangle} dm_{\Lambda},$$

where the ' next to the summation sign indicates that the sum is over those $\lambda \in \Lambda^*$ such that the denominator is not zero, and m_{Λ} denotes the translation-invariant measure on \mathfrak{t} such that the induced volume of the torus $T = \mathfrak{t}/\Lambda$ is 1.

The distribution B_{α} is Λ -periodic. The inverse Fourier transform of B_{α} is the function $\dot{B}_{\alpha} : \Lambda^* \to \mathbb{C}$ given by

$$\check{B}_{\alpha}(\lambda) = \begin{cases} \frac{1}{\prod_{\alpha \in \alpha} 2\pi i \langle \alpha, \lambda \rangle} & \text{if } \lambda \in \Lambda^* \setminus \cup \{\alpha = 0\} \\ 0 & \text{otherwise.} \end{cases}$$

Later on, it is sometimes convenient to view this as a delta-type distribution supported on $\Lambda^* \subset \mathfrak{t}$.

By taking spans of subsets of α together with their translates by elements of Λ , we obtain a periodic affine subspace arrangement in t. Following [15], we refer to these as *admissible subspaces*. Let $S = S(\alpha, \Lambda)$ denote the collection of admissible subspaces.

Suppose that α spans t^{*}. Then the codimension-1 admissible subspaces form an infinite periodic hyperplane arrangement, and we call the connected components of the complement the *chambers*.

Theorem 3.1.7 ([58], [17]). Suppose α spans \mathfrak{t}^* . Then B_{α} is locally L^1 and the restriction of B_{α} to each chamber is polynomial.

Let $S \subset \mathfrak{t}$ be a rational subspace (S is cut out by elements of Λ^*). For any translate Δ of S define

$$\boldsymbol{\alpha}_{\Delta} = \boldsymbol{\alpha}_{S} = \{ \alpha \in \boldsymbol{\alpha} | \alpha \in S \}, \qquad \boldsymbol{\alpha}_{\Delta}^{\perp} = \boldsymbol{\alpha}_{S}^{\perp} = \boldsymbol{\alpha} \setminus \boldsymbol{\alpha}_{S}.$$

Let $\mathfrak{t}_{\Delta} = \mathfrak{t}_{S} = S^{\perp}$ be the orthogonal complement to S relative to the integral inner product B, and let $T_{\Delta} = T_{S} = \exp(\mathfrak{t}_{S}) \subset T$ be the corresponding subtorus. The translates $\Delta = \lambda_{\Delta} + S$ of S by elements of Λ are parametrized by the quotient Λ/S .

Proposition 3.1.8. The Bernoulli series $B(\alpha_S)$ is a sum of distributions $B(\alpha_{\Delta}; \Delta)$ supported on translates $\Delta = \lambda_{\Delta} + S$ of S, where $\lambda_{\Delta} \in \Delta \cap \Lambda$. Thus

$$B(\boldsymbol{\alpha}_S) = \sum_{\Delta \in \Lambda/S} B(\boldsymbol{\alpha}_{\Delta}; \Delta).$$
(3.1)

Let $\iota^{\Delta}: S \to \mathfrak{t}$ be the map $\iota^{\Delta}(\lambda) = \lambda + \lambda_{\Delta}$. Let $B_{\alpha,S}$ be the (lower-dimensional) Bernoulli series defined by the data

$$(\mathfrak{t}/\mathfrak{t}_S, \Lambda^*/\mathfrak{t}_S, \boldsymbol{\alpha}_S).$$

Note that $B_{\alpha,S}$ can be viewed as a distribution on S using the isomorphism $\mathfrak{t}/\mathfrak{t}_S \simeq S$ determined by the inner product. Then

$$B(\boldsymbol{\alpha}_{\Delta};\Delta) = \iota^{\Delta}_* B_{\boldsymbol{\alpha},S}.$$

Proof. We have a short exact sequence

$$0 \to \Lambda^* \cap \mathfrak{t}_S \to \Lambda^* \to \Lambda^*/\mathfrak{t}_S \to 0.$$

Choose representatives $\lambda' \in \Lambda^*$ for the elements of Λ^*/\mathfrak{t}_S . The linear functions α_S are constant on translates of \mathfrak{t}_S . Hence $\alpha(\lambda' + \lambda) = \alpha(\lambda')$ for $\lambda \in \Lambda^* \cap \mathfrak{t}_S$, $\alpha \in \alpha_S$. This means in particular that whether $\alpha(\lambda' + \lambda)$ is zero (i.e. whether $\lambda' + \lambda$ appears in the sum) depends only on λ' , not on $\lambda \in \Lambda^* \cap \mathfrak{t}_S$. Thus we can split up the sum over Λ^* into an iterated sum

$$B(\boldsymbol{\alpha}_S)(\mu) = \sum_{\lambda'}' \frac{e^{-2\pi i \langle \mu, \lambda' \rangle}}{\prod_{\boldsymbol{\alpha}_S} 2\pi i \langle \alpha, \lambda' \rangle} \sum_{\lambda \in \Lambda^* \cap \mathfrak{t}_S} e^{-2\pi i \langle \mu, \lambda \rangle} dm_{\Lambda}.$$

By the Poisson summation formula

$$\sum_{\lambda \in \Lambda^* \cap \mathfrak{t}_S} e^{-2\pi i \langle \mu, \lambda \rangle} dm_{\Lambda} = \sum_{\Delta \in \Lambda/S} \delta_{\Delta}(\mu),$$

where $\delta_{\Delta} := \iota_*^{\Delta} dm_{\Lambda \cap S}$. This shows that $B(\boldsymbol{\alpha}_S)$ is supported on $\Lambda + S$, the set of translates of S by elements of Λ . If $\mu = \lambda_{\Delta} + \mu'$ is in the support, with $\lambda_{\Delta} \in \Lambda \cap \Delta$ and $\mu' \in S$, then $e^{-2\pi i \langle \lambda_{\Delta}, \lambda' \rangle} = 1$ since elements of Λ , Λ^* pair to give an element of \mathbb{Z} . Thus on the support of the inner sum, the outer sum is $B_{\boldsymbol{\alpha},S}(\mu')$.

Remark 3.1.9. A corollary of this result is that if $S = \operatorname{span}(\overline{\alpha})$ then B_{α} is supported on the Λ -translates of S.

If $\Delta = \lambda_{\Delta} + S$ is admissible, then by definition α_S spans S. Thus Theorem 3.1.7 implies that $B_{\alpha,S}$ is locally L^1 and piecewise polynomial on S, more precisely, for each chamber \mathfrak{c} of a hyperplane arrangement in S, there is a polynomial p such that $p \cdot dm_{\Lambda \cap S}$ agrees with $B_{\alpha,S}$ on \mathfrak{c} . It follows that $B(\alpha_{\Delta}; \Delta)$ agrees with a polynomial distribution on certain chambers in Δ .

Definition 3.1.10. If $\gamma \in \mathfrak{t}$ is generic, then the orthogonal projection of γ onto Δ , denoted $\operatorname{pr}_{\Delta}(\gamma)$, lies in the interior of one of these chambers and thus selects a polynomial distribution

$$\operatorname{Ber}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) : \Delta \to \mathbb{C},$$

which agrees with $B(\boldsymbol{\alpha}_{\Delta}; \Delta)$ on the chamber containing $\mathrm{pr}_{\Delta}(\gamma)$. We will refer to $\mathrm{Ber}(\boldsymbol{\alpha}_{\Delta}; \mathrm{pr}_{\Delta}(\gamma))$ as the *polynomial germ* of $B(\boldsymbol{\alpha}_{\Delta}; \Delta)$ at $\mathrm{pr}_{\Delta}(\gamma)$.

For later reference it is convenient to record the inverse Fourier transform of $B(\alpha_{\Delta}; \Delta)$: this is a

distribution on $\mathfrak{t}^* \simeq \mathfrak{t}$ defined by the equation

$$\langle \mathcal{F}^{-1}B(\boldsymbol{\alpha}_{\Delta};\Delta), f \rangle := \langle B(\boldsymbol{\alpha}_{\Delta};\Delta), \mathcal{F}^{-1}f \rangle,$$

where $\mathcal{F}^{-1}(f)$ is computed by the usual formula, integrating against the measure dm_{Λ} . Under the orthogonal direct sum decomposition $\mathfrak{t} = S \oplus \mathfrak{t}_S$, the measure dm_{Λ} decomposes into a product

$$dm_{\Lambda} = dm_{\Lambda \cap S} \times dm_{\Lambda/S}.$$

where $dm_{\Lambda/S}$ is the normalized measure for Λ/S , viewed as a lattice in \mathfrak{t}_S . The sub-lattice $\Lambda \cap \mathfrak{t}_S \subset \Lambda/S$ has finite index equal to $|T_S \cap T_S^{\perp}|$ where $T_S = \exp(\mathfrak{t}_S)$, $T_S^{\perp} = \exp(S)$. This also equals the volume of T_S relative to the normalized measure determined by the lattice Λ/S . Thus

$$dm_{\Lambda/S} = \operatorname{vol}(T_S) dm_{\Lambda \cap \mathfrak{t}_S}.$$
(3.2)

The measure $dm_{\Lambda/S}$ defines a pull-back operation for distributions φ on S:

$$\langle \pi_S^* \varphi, f \rangle := \langle \varphi, (\pi_S)_* f \rangle$$

where $(\pi_S)_*$ denotes integration along the fibres of the orthogonal projection $\pi : \mathfrak{t} \to S$, using the measure $dm_{\Lambda/S}$.

Taking the push-forward by ι^{Δ} , the inverse Fourier transform $\mathcal{F}^{-1}B(\boldsymbol{\alpha}_{\Delta}; \Delta)$ is obtained by pulling back $\mathcal{F}^{-1}B_{\boldsymbol{\alpha},S}$ using π_{S}^{*} , and multiplying by a phase $e^{2\pi i \langle \lambda_{\Delta}, - \rangle}$, where $\lambda_{\Delta} \in \Lambda \cap \Delta$. Thus $\mathcal{F}^{-1}B(\boldsymbol{\alpha}_{\Delta}; \Delta)$ is supported on $\Lambda^{*} + \mathfrak{t}_{S} \subset \mathfrak{t}$, and can be expressed as

$$\mathcal{F}^{-1}B(\boldsymbol{\alpha}_{\Delta};\Delta) = b \cdot \delta(\Lambda^* + \mathfrak{t}_S)$$

where $\delta(\Lambda^* + \mathfrak{t}_S)$ is a delta-type distribution supported on $\Lambda^* + \mathfrak{t}_S \subset \mathfrak{t}$ defined by integration against the measure $dm_{\Lambda \cap \mathfrak{t}_S}$ on \mathfrak{t}_S , and b is the function of $\lambda \in \Lambda^*$, $\xi \in \mathfrak{t}_S$ given by

$$b(\lambda + \xi) = \begin{cases} |T_S \cap T_S^{\perp}| \frac{e^{2\pi i \langle \lambda_{\Delta}, \xi \rangle}}{\prod_{\alpha_{\Delta}} 2\pi i \langle \alpha, \lambda \rangle}, & \langle \alpha, \lambda \rangle \neq 0, \alpha \in \boldsymbol{\alpha}_{\Delta} \\ 0, & \text{otherwise.} \end{cases}$$
(3.3)

Remark 3.1.11. If $\mu \in \Lambda^* + \mathfrak{t}_S$, the decomposition $\mu = \lambda + \xi$ with $\lambda \in \Lambda^*$, $\xi \in \mathfrak{t}_S$ is clearly not unique. But elements of α_{Δ} pair with elements of \mathfrak{t}_S to give 0, while elements of Λ^* pair with $\lambda_{\Delta} \in \Lambda$ to give an integer, by definition. Thus equation (3.3) is independent of the choice. Also we can replace $\lambda_{\Delta} \in \Lambda \cap \Delta$ with its orthogonal projection μ_{Δ} to $\mathfrak{t}_S = \mathfrak{t}_{\Delta}$.

3.1.3 Boysal-Vergne decomposition formula.

Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ be a list of elements of \mathfrak{t}^* and suppose that $\gamma^+ \in \mathfrak{t}$ is *polarizing* for $\boldsymbol{\alpha}$, i.e. γ^+ satisfies $\langle \alpha_i, \gamma^+ \rangle \neq 0, i = 1, ..., n$. For each $\alpha_k \in \boldsymbol{\alpha}$, let $\alpha_k^+ = \epsilon_k \alpha_k$, where $\epsilon_k = \pm 1$ is such that $\langle \alpha_k^+, \gamma^+ \rangle > 0$. The *multi-spline* distribution $H(\boldsymbol{\alpha}, \gamma^+)$ is defined by

$$\langle H(\boldsymbol{\alpha},\gamma^+),f\rangle = \epsilon \int_{\mathbb{R}^n_+} f(t_1\alpha_1^+ + \dots + t_n\alpha_n^+)d^n t, \qquad \epsilon = \prod_k \epsilon_k$$

Apart from the sign ϵ , it is a convolution of Heaviside distributions $H_{\alpha_k^+}$, supported in the half-space $\gamma^+ \geq 0$. The inverse Fourier transform of $H_{\alpha_k^+}$ is the distribution

$$\check{H}_{\alpha_k^+}(\xi) = \lim_{s \to 0^+} \frac{1}{2\pi i \langle \alpha_k^+, \xi - is\gamma^+ \rangle} dm_{\Lambda}.$$

(The meaning of this expression is that the fraction should be integrated against a smooth compactly supported function before the limit is taken.) The inverse Fourier transform of $H(\boldsymbol{\alpha}; \gamma^+)$ is therefore

$$\check{H}(\boldsymbol{\alpha};\gamma^{+})(\xi) = \epsilon \lim_{s \to 0^{+}} \frac{1}{\prod_{k} 2\pi i \langle \alpha_{k}^{+}, \xi - is\gamma^{+} \rangle} dm_{\Lambda}.$$

We are now ready to state the decomposition formula of [15] for B_{α} . For this we need to choose a sufficiently generic element $\gamma \in \mathfrak{t}$. To be precise, we require two conditions. First, that for each admissible subspace $\Delta \in \mathcal{S}$, the orthogonal projection $\operatorname{pr}_{\Delta}(\gamma)$ of γ to Δ lies in a chamber for $B(\alpha_{\Delta}; \Delta)$ (this implies that the distribution $\operatorname{Ber}(\alpha_{\Delta}; \operatorname{pr}_{\Delta}(\gamma))$ is defined). Second, we require that $\gamma_{\Delta}^{+} := \operatorname{pr}_{\Delta}(\gamma) - \gamma$ be polarizing for the list α_{Δ}^{\perp} (this ensures that $H(\alpha_{\Delta}^{\perp}; \gamma_{\Delta}^{+})$ is defined).

Theorem 3.1.12 ([15]). Let $(\mathfrak{t}, \Lambda^*, \alpha)$ be data defining a multiple Bernoulli series $B(\alpha)$, and let $\gamma \in \mathfrak{t}$ be generic. Then

$$B(\boldsymbol{\alpha}) = \sum_{\Delta \in \mathcal{S}(\boldsymbol{\alpha},\Lambda)} \operatorname{Ber}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * H(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}).$$
(3.4)

Remark 3.1.13. This can be proved inductively using properties of Bernoulli series. If $\boldsymbol{\alpha}$ spans \mathfrak{t}^* , then $\Delta = \mathfrak{t}$ is an admissible subspace, and its contribution is $Ber(\boldsymbol{\alpha}; \gamma)$, the polynomial on \mathfrak{t} which agrees with $B_{\boldsymbol{\alpha}}$ on the chamber containing γ . All other contributions are contained in half-spaces which do not intersect the chamber containing γ .

In the simplest case when $\boldsymbol{\alpha} = \emptyset$, this reduces to the Poisson summation formula. Indeed, in that the set of admissible subspaces is $\mathcal{S} = \Lambda$, $H = \delta_0$, $\text{Ber}(\boldsymbol{\alpha}_{\Delta}; \text{pr}_{\Delta}(\gamma)) = \delta_{\lambda}$ where $\lambda = \Delta \in \Lambda = \mathcal{S}$.

3.2 Norm-square localization formula

In this section we derive a 'norm-square localization' formula for a Duistermaat-Heckman distribution \mathfrak{m} on \mathfrak{t} associated to a quasi-Hamiltonian *G*-space (see Chapter 2 for more background).

3.2.1 Fixed-point expressions.

Let G be a compact connected Lie group. We identify the Lie algebra \mathfrak{g} with its dual using an invariant inner product $B(-,-) = \langle -,-\rangle$. Let $T \subset G$ be a maximal torus with Lie algebra \mathfrak{t} . Let Λ denote the integral lattice, and Λ^* its dual, the weight lattice. We assume the inner product is *integral* in the sense that $B(\Lambda, \Lambda) \subset \mathbb{Z}$. Let \mathcal{R} denote the roots, and \mathcal{R}_+ a fixed set of positive roots. Let Λ^*_{reg} denote the set of regular elements, that is, the subset of $\lambda \in \Lambda^*$ such that $\langle \lambda, \alpha \rangle \neq 0$ for each root α . We equip $\mathfrak{g}/\mathfrak{t}$ with the complex structure such that the complex weights of the adjoint T-action are \mathcal{R}_+ .

Let

$$\operatorname{Eul}(\mathfrak{g}/\mathfrak{t},\xi) = \prod_{\alpha>0} -\langle \alpha,\xi\rangle,$$

be the *T*-equivariant Euler class of $\mathfrak{g}/\mathfrak{t}$ viewed as a *T*-equivariant vector space over a point, using the orientation induced from the complex structure on $\mathfrak{g}/\mathfrak{t}$ determined by the positive roots.

Let (M, ω, Φ) be a quasi-Hamiltonian *G*-space. We consider the Duistermaat-Heckman distribution on \mathfrak{t} given by the formula

$$\mathfrak{m}(\mu) = \sum_{\lambda \in \Lambda_{\mathrm{reg}}^*} \mathrm{Eul}(\mathfrak{g}/\mathfrak{t}, -2\pi i\lambda) \sum_{F \subset M^{\lambda}} \int_F \frac{e^{\omega} \Phi_F^{\lambda}}{\mathrm{Eul}(\nu_F, -2\pi i\lambda)} e^{-2\pi i \langle \mu, \lambda \rangle} dm_{\Lambda}.$$
(3.5)

Some comments on this expression:

- 1. The sum always converges in the sense of distributions. It defines a Λ -periodic distribution on t (equivalently, a distribution on T), which is alternating under the action of the Weyl group.
- 2. For $\lambda \in \Lambda^*_{\text{reg}}$ the fixed-point subset $M^{\lambda} \subset \Phi^{-1}(T)$. The phase $\Phi(m)^{\lambda}$ is constant for $m \in F \subset M^{\lambda}$ and is denoted Φ_F^{λ} .
- 3. We could replace the sum over Λ^*_{reg} with a sum over all of Λ^* since the pre-factor $\text{Eul}(\mathfrak{g}/\mathfrak{t}, -2\pi i\lambda)$ vanishes at all non-regular elements. The integrand makes sense for non-regular weights as well, since $F \subset M^{\lambda} \Rightarrow \Phi(F) \subset G^{\lambda}$, and $\lambda \in \Lambda^*$ defines a homomorphism $G^{\lambda} \to U(1)$.

Remark 3.2.1. Equation (3.5) is a different expression for the same distribution $\mathfrak{m}' := \mathrm{DH}(\mathcal{N}, \pi^* \omega_a, \phi, \pi^* \tau)$ (with $\alpha = 1$, and π the quotient map $\mathcal{N} \to \mathcal{N} = \mathcal{N}/\Lambda$) considered in Chapter 2. To see this recall that the latter distribution was periodic, hence could be viewed as a distribution on the maximal torus T; viewed in this way, its Fourier coefficients are given by

$$\langle \mathfrak{m}', e_{\lambda} \rangle = \int_{N} e^{\omega_{a}} \Phi_{a}^{\lambda} \tau(-2\pi i \lambda), \qquad \lambda \in \Lambda^{*}.$$

Using the equivariance properties under the action of $N_G(T)$ (and the anti-symmetry of τ), this expression vanishes for $\lambda \notin \Lambda^*_{\text{reg}}$, so we can assume $\lambda \in \Lambda^*_{\text{reg}}$. On the other hand, the integrand is closed with respect to the differential $d_{-2\pi i\lambda}$, and so localizes to the fixed-point set of λ . Since $\lambda \in \Lambda^*_{\text{reg}}$, $G^{\lambda} = T$ hence $N^{\lambda} \subset \Phi^{-1}(T)$. It follows $M^{\lambda} = N^{\lambda}$, and also that after pulling back to M^{λ} , the Thom form can be replaced with the Euler class of $\mathfrak{g}/\mathfrak{t}$. Thus

$$\langle \mathfrak{m}', e_{\lambda} \rangle = \operatorname{Eul}(\mathfrak{g}/\mathfrak{t}, -2\pi i\lambda) \sum_{F \subset M^{\lambda}} \int_{F} \frac{e^{\omega} \Phi_{F}^{\lambda}}{\operatorname{Eul}(\nu_{F}, -2\pi i\lambda)}.$$
(3.6)

Using the Fourier inversion formula

$$\mathfrak{m}' = \sum_{\lambda \in \Lambda^*} \langle \mathfrak{m}', e_\lambda \rangle e_{-\lambda} dm_\Lambda$$

we obtain equation (3.5). Equation (3.6) was described in greater detail in Theorem 5.3 of [39] (it is related to the abelian localization formula for q-Hamiltonian spaces in [5]). As explained already in Chapter 2, morally this is a formula for the DH measure of $\Phi^{-1}(T)$, viewed as a periodic distribution on t, and it in fact agrees with this in the case that Φ is transverse to T. See Chapter 2 as well as [39] for more details.

For each component F arising in the expression (3.5) above, let \mathfrak{t}_F denote the Lie algebra of its

stabilizer in \mathfrak{t} , and $T_F = \exp(\mathfrak{t}_F) \subset T$. Choose a complex structure on the normal bundle ν_F ; this can be done for example by choosing a generic $\delta \in \mathfrak{t}$, and using the almost complex structure on ν_F such that the complex weights of the action of T_F pair positively with δ . Let $\alpha_F \subset \mathfrak{t}_F^* = \Lambda^*/\mathfrak{t}_F^{\perp}$ denote the list of weights on the normal bundle ν_F . We exchange the order of the summations in (3.5), to first sum over orbit-types and then over weights:

$$\mathfrak{m} = \sum_{F} \mathfrak{m}_{F}, \quad \mathfrak{m}_{F}(\mu) = \operatorname{Eul}(\mathfrak{g}/\mathfrak{t}, \partial_{\mu}) \int_{F} \sum_{\lambda \in \Lambda^{*} \cap \mathfrak{t}_{F}} \frac{e^{\omega} \Phi_{F}^{\lambda}}{\operatorname{Eul}(\nu_{F}, -2\pi i \lambda)} e^{-2\pi i \langle \mu, \lambda \rangle} dm_{\Lambda}, \quad (3.7)$$

where the sum runs over $\lambda \in \Lambda^* \cap \mathfrak{t}_F$ such that the weights of the T_F action on ν_F do not vanish on λ .

Equation (3.7) suggests that we define slightly more general multiple Bernoulli series, given by similar sums but now involving also differential forms. We will be somewhat brief, as a similar development is carried out in the next chapter. Define

$$B(\nu_F) = \sum_{\lambda \in \Lambda^* \cap \mathfrak{t}_F}' \frac{e^{-2\pi i \langle \mu, \lambda \rangle}}{\operatorname{Eul}(\nu_F, -2\pi i \lambda)} dm_{\Lambda/\mathfrak{t}_F^{\perp}}.$$
(3.8)

Taylor expanding Eul⁻¹(ν_F , $-2\pi i\lambda$) in the curvatures, one sees that $B(\nu_F)$ is a finite linear combination of characteristic forms on F with coefficients that are multiple Bernoulli series for the data of the form ($\mathfrak{t}_F, \Lambda^* \cap \mathfrak{t}_F, \boldsymbol{\alpha}_F^L$), where $L = (\ell_1, ..., \ell_n)$ is a multi-index, and by $\boldsymbol{\alpha}^L$ we mean the list $\boldsymbol{\alpha}$ with elements repeated according to the multi-index L. The Boysal-Vergne decomposition formula (equation (3.4)) can be applied to each of these ordinary multiple Bernoulli series, resulting in a expression for $B(\nu_F)$.

In order to write the resulting formula in a relatively compact form, we generalize the definitions of the distributions $Ber(\boldsymbol{\alpha}_{\Delta}; pr_{\Delta}(\gamma))$ and $H(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+})$ similar to (3.8).

Fix $\Delta \subset \mathfrak{t}_F$, an admissible subspace for α_F . Then

$$\boldsymbol{lpha}_F = \boldsymbol{lpha}_{F,\Delta} \cup \boldsymbol{lpha}_{F,\Delta}^{\perp}, \qquad \boldsymbol{lpha}_{F,\Delta} = \{ lpha \in \Delta \}$$

Let $\nu_{F,\Delta}$ be the sum of all weight sub-bundles whose weight is contained in $\alpha_{F,\Delta}$. It is the sub-bundle which is stabilized by \mathfrak{t}_{Δ} . We have a direct sum decomposition

$$\nu_F = \nu_{F,\Delta} \oplus \nu_{F,\Delta}^{\perp}.$$

Then $\alpha_{F,\Delta}$ and $\alpha_{F,\Delta}^{\perp}$ are exactly the lists of weights on $\nu_{F,\Delta}$ and $\nu_{F,\Delta}^{\perp}$ respectively.

Define $B(\nu_{F,\Delta})$ similar to (3.8). Using a Taylor expansion in the curvatures, one defines $B(\nu_{F,\Delta}; \Delta)$ and $Ber(\nu_{F,\Delta}; pr_{\Delta}(\gamma))$ by applying the definitions from previous sections to the coefficients appearing in the Taylor expansion (which are multiple Bernoulli series).

The definition of $H(\nu_{F,\Delta}^{\perp}; \gamma_{\Delta}^{+})$ can be given in the same way. Alternatively, $H(\nu_{F,\Delta}^{\perp}; \gamma_{\Delta}^{+})$ can be defined as the Fourier transform of the form with generalized coefficients (c.f. [48], or Chapter 2):

$$\lim_{s \to 0^+} \frac{\epsilon}{\operatorname{Eul}(\nu_{F,\Delta}^{\perp}, -2\pi i(\lambda - is\gamma_{\Delta}^{+}))},\tag{3.9}$$

where $\epsilon = \pm 1$ is a sign relating the orientations of $\nu_{F,\Delta}^{\pm}$ induced by the complex structures for which the list of weights are α_F , α_F^{\pm} respectively (in other words, $\epsilon = (-1)^k$ where k is the number of sign flips, when the weights are counted with multiplicity).

Proposition 3.2.2.

$$B(\nu_F) = \sum_{\Delta \in \mathcal{S}(\alpha_F, \Lambda/\mathfrak{t}_F^{\perp})} \operatorname{Ber}(\nu_{F,\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * H(\nu_{F,\Delta}^{\perp}, \gamma_{\Delta}^{+}).$$
(3.10)

We omit the proof, since a similar proof is given in the next chapter.

Note that $B(\nu_F)$ was defined with coefficients which are distributions on \mathfrak{t}_F . Under the orthogonal direct sum decomposition $\mathfrak{t} = \mathfrak{t}_F \oplus \mathfrak{t}_F^{\perp}$, the measures dm_{Λ} and $dm_{\Lambda/\mathfrak{t}_T^{\perp}}$ satisfy

$$dm_{\Lambda} = dm_{\Lambda/\mathfrak{t}_{E}^{\perp}} \times dm_{\Lambda\cap\mathfrak{t}_{E}^{\perp}}$$

The measure $dm_{\Lambda\cap\mathfrak{t}_F^{\perp}}$ on \mathfrak{t}_F^{\perp} defines an integration along the fibres map

$$(\pi_F)_*: C^{\infty}_{\text{comp}}(\mathfrak{t}) \to C^{\infty}_{\text{comp}}(\mathfrak{t}_F)$$

and by duality, a pullback map π_F^* on distributions. The pullback $\pi_F^*B(\nu_F)$ appears in (3.7).

3.2.2 Applying the decomposition formula.

We next apply (3.10) to each contribution \mathfrak{m}_F in (3.7). We are free to choose the center γ of the decomposition independently for each F. The factor Φ_F^{λ} means that the result will be translated compared to (3.10).

For each F, choose a point $x \in F$, then $\Phi(x) \in T$ and $\Phi_F^{\lambda} = \Phi(x)^{\lambda}$. Choose a logarithm in $\mathfrak{t} = \mathfrak{t}^*$ of $\Phi(x) \in T$, and let ϕ_F be the image of this logarithm under the quotient map

$$\mathfrak{t}^* \to \mathfrak{t}^* / \operatorname{ann}(\mathfrak{t}_F) \simeq \mathfrak{t} / \mathfrak{t}_F^{\perp}.$$

The result is independent of the choice of $x \in F$ (by the moment map condition), but changes by an element of $\Lambda/\mathfrak{t}_F^{\perp}$ when the branch of the logarithm is changed.

Thus

$$\sum_{\lambda \in \Lambda^* \cap \mathfrak{t}_F}^{\prime} \frac{\Phi_F^{\lambda} e^{-2\pi i \langle \mu, \lambda \rangle}}{\operatorname{Eul}(\nu_F, -2\pi i \lambda)} dm_{\Lambda} = \pi_F^*(T_{\phi_F} B(\nu_F)),$$
(3.11)

where T_{ϕ_F} denotes translation by ϕ_F . To compensate for the translation by ϕ_F , we apply (3.10) to \mathfrak{m}_F choosing the center of the expansion (which we now denote γ'), to be

$$\gamma' = -\phi_F + \gamma,$$

where γ is a small perturbation to ensure that center γ' is generic. This ensures that the terms in (3.10) for $T_{\phi_F}B(\nu_F)$ will be centred around $\gamma \approx 0$. Applying (3.10), equation (3.11) becomes

$$\pi_F^*\left(T_{\phi_F}\sum_{\Delta'\in\mathcal{S}(\boldsymbol{\alpha}_F,\Lambda/\mathfrak{t}_F)}\operatorname{Ber}(\nu_{F,\Delta'};\operatorname{pr}_{\Delta'}(\gamma'))*H(\nu_{F,\Delta'}^{\perp};(\gamma')_{\Delta'}^{+})\right).$$

We make a change of variables to incorporate the translation T_{ϕ_F} : translating the subspaces Δ' by ϕ_F

we obtain a new set $S_F = S(\alpha_F, \Lambda/\mathfrak{t}_F, \phi_F)$ of affine subspaces $\Delta = \Delta' + \phi_F$. The translate

$$T_{\phi_F} \operatorname{Ber}(\nu_{F,\Delta'}; \operatorname{pr}_{\Delta'}(\gamma'))$$

is polynomial and supported on $\Delta = \Delta' + \phi_F$, and agrees with the translate

$$T_{\phi_F}B(\nu_{F,\Delta'};\Delta')$$

near the point

$$\operatorname{pr}_{\Delta'}(\gamma') + \phi_F = \operatorname{pr}_{\Delta}(\gamma).$$

This motivates the notation

$$B(\nu_{F,\Delta};\Delta) := T_{\phi_F} B(\nu_{F,\Delta'};\Delta')$$

and

$$\operatorname{Ber}(\nu_{F,\Delta};\operatorname{pr}_{\Delta}(\gamma)) := T_{\phi_F}\operatorname{Ber}(\nu_{F,\Delta'};\operatorname{pr}_{\Delta'}(\gamma')).$$

Note also that $\gamma_{\Delta}^{+} = \operatorname{pr}_{\Delta}(\gamma) - \gamma = \gamma_{\Delta'}^{+}$.

With this notation, equation (3.7) becomes

$$\mathfrak{m} = \sum_{F} \operatorname{Eul}(\mathfrak{g}/\mathfrak{t}, \partial_{\mu}) \int_{F} e^{\omega} \sum_{\Delta \in \mathcal{S}_{F}} \operatorname{Ber}(\nu_{F,\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * H(\nu_{F,\Delta}^{\perp}; \gamma_{\Delta}^{\perp}).$$
(3.12)

(Here and below we omit π_F^* to simplify notation.)

3.2.3 Grouping the terms.

The next step is to exchange the order of the two summations \sum_F, \sum_{Δ} . From now on view the affine subspaces Δ as subspaces of \mathfrak{t} , by taking inverse image under the orthogonal projection $\pi_F : \mathfrak{t} \to \mathfrak{t}_F$. Terms in the sum are indexed by pairs (F, Δ) consisting of a fixed-point submanifold $F \subset M$, and an affine subspace $\Delta \in S_F$. Let S be the union of all the collections S_F . Group the terms corresponding to $(F, \Delta), (F', \Delta')$ together if $\Delta = \Delta' (\Rightarrow \mathfrak{t}_{\Delta} = \mathfrak{t}_{\Delta'})$, and F, F' belong to the same connected component $C \subset M^{\mathfrak{t}_{\Delta}}$. Note that for such pairs, the projection $\operatorname{pr}_{\Delta}(\gamma)$ is the same (meaning that the 'polynomial germs' are taken about the same point), as is the polarization direction γ_{Δ}^+ .

For $F \subset C \subset M^{\mathfrak{t}_{\Delta}}$, let ν_C denote the normal bundle to C in M, and $\nu_{F,C}$ the normal bundle to F in C. Clearly

$$\nu_{F,\Delta} = \nu_{F,C}, \qquad \nu_{F,\Delta}^{\perp} = \nu_C|_F.$$

Exchanging the order of summation, equation (3.13) becomes

$$\mathfrak{m} = \sum_{\Delta,C} \mathfrak{m}_{\Delta,C}^{\mathrm{pol}},\tag{3.13}$$

where

$$\mathfrak{m}_{\Delta,C}^{\mathrm{pol}} = \mathrm{Eul}(\mathfrak{g}/\mathfrak{t},\partial_{\mu}) \sum_{F \subset C} \int_{F} e^{\omega} \mathrm{Ber}(\nu_{F,C};\mathrm{pr}_{\Delta}(\gamma)) * H(\nu_{C};\gamma_{\Delta}^{+}).$$
(3.14)

We make a few comments on equations (3.13), (3.14). The subspaces $\Delta \in \mathcal{S}$ correspond to \mathfrak{t}_{Δ} -fixedpoint submanifolds of the associated Hamiltonian *LG*-space. The generic point γ (near 0) is the center of the decomposition. There is a central contribution from $\Delta = \mathfrak{t}$, which is polynomial. More generally, the term $\operatorname{Ber}(\nu_{F,C}; \operatorname{pr}_{\Delta}(\gamma))$ has coefficients which are polynomial distributions supported on Δ . $H(\nu_C; \gamma_{\Delta}^+)$ has coefficients which are multispline functions polarized by $\gamma_{\Delta}^+ = \gamma_{\Delta} - \gamma$. In particular, the coefficients of the terms $\Delta \neq \mathfrak{t}$ are supported in half-spaces not containing γ .

3.2.4 A related distribution.

It is useful to consider a closely related distribution, defined by replacing Ber(-) with B(-) in equation (3.14):

$$\mathfrak{m}_{\Delta,C} = \operatorname{Eul}(\mathfrak{g}/\mathfrak{t},\partial_{\mu}) \sum_{F \subset C} \int_{F} e^{\omega} B(\nu_{F,C};\Delta) * H(\nu_{C};\gamma_{\Delta}^{+}).$$
(3.15)

Thus $\mathfrak{m}_{\Delta,C}^{\text{pol}}$ is obtained from $\mathfrak{m}_{\Delta,C}$ by taking 'polynomial germs' along Δ , around $\operatorname{pr}_{\Delta}(\gamma)$ —more precisely, replacing the piecewise polynomial distributions $B(\nu_{F,C}; \Delta)$ supported on Δ , with the unique polynomial distributions $\operatorname{Ber}(\nu_{F,C}; \operatorname{pr}_{\Delta}(\gamma))$ agreeing with them on a neighbourhood of $\operatorname{pr}_{\Delta}(\gamma)$ in Δ .

We next work out the inverse Fourier transform of $\mathfrak{m}_{\Delta,C}$. Let $\lambda \in \Lambda^* \cap \mathfrak{t}_F$ and $\xi \in \mathfrak{t}_\Delta$. Using equation (3.3) we have

$$\mathcal{F}^{-1}B(\nu_{F,C};\Delta)(\lambda+\xi) = |T_{\Delta} \cap T_{\Delta}^{\perp}| \frac{e^{2\pi i \langle \mu_{\Delta},\xi \rangle} \Phi_{F}^{\lambda}}{\operatorname{Eul}(\nu_{F,C}, -2\pi i\lambda)} \delta(\Lambda^{*} \cap \mathfrak{t}_{F} + \mathfrak{t}_{\Delta})$$
(3.16)

for λ such that $\langle \alpha, \lambda \rangle \neq 0$, $\alpha \in \alpha_{\Delta}$, and is zero otherwise. Here $\delta(\Lambda^* \cap \mathfrak{t}_F + \mathfrak{t}_{\Delta})$ is a delta-type distribution supported on $\Lambda^* \cap \mathfrak{t}_F + \mathfrak{t}_{\Delta}$, defined using the measure $dm_{\Lambda \cap \mathfrak{t}_{\Delta}}$ on \mathfrak{t}_{Δ} (see equation (3.3) for example), $T_{\Delta}^{\perp} = \exp(\mathfrak{t}_{\Delta}^{\perp})$, and $\mu_{\Delta} = \operatorname{pr}_{\Delta}(0)$.

This expression defines a distribution on \mathfrak{t}_F , supported on $\Lambda^* \cap \mathfrak{t}_F + \mathfrak{t}_\Delta$. To obtain a distribution on \mathfrak{t} , one must then push-forward under the inclusion $\mathfrak{t}_F \hookrightarrow \mathfrak{t}$ (the pull-back π_F^* of $B(\nu_F)$ corresponds to push-forward for the Fourier transform).

Remark 3.2.3. In general a point $\mu \in (\Lambda^* \cap \mathfrak{t}_F + \mathfrak{t}_\Delta)$ can be expressed as a sum in more than one way. If $\mu = \lambda + \xi = \lambda' + \xi'$, then $\lambda - \lambda' \in \Lambda^* \cap \mathfrak{t}_\Delta$. But for $F \subset C$, $\Phi_F^{\nu} = e^{2\pi i \langle \mu_\Delta, \nu \rangle}$ for $\nu \in \Lambda^* \cap \mathfrak{t}_\Delta$. Thus the expression above is independent of this decomposition.

Combining this with the inverse Fourier transform of H(-) gives the following.

Theorem 3.2.4. Let $\lambda + \xi \in \Lambda^* + \mathfrak{t}_{\Delta}$. The inverse Fourier transform of $\mathfrak{m}_{\Delta,C}$ is supported on $\Lambda^* + \mathfrak{t}_{\Delta}$, and is given by

$$\operatorname{Eul}(\mathfrak{g}/\mathfrak{t}, -2\pi i(\lambda + \xi)) \sum_{F \subset C^{\lambda}} \int_{F} \lim_{s \to 0^{+}} \frac{\epsilon_{F,C} e^{\omega} \Phi_{F}^{\lambda} e^{2\pi i \langle \mu_{\Delta}, \xi \rangle}}{\operatorname{Eul}(\nu_{F}, -2\pi i(\lambda + \xi - is\gamma_{\Delta}^{+}))} |T_{\Delta} \cap T_{\Delta}^{\perp}| \delta(\Lambda^{*} \cap \mathfrak{t}_{F} + \mathfrak{t}_{\Delta}).$$
(3.17)

Here $\epsilon_{F,C}$ is the sign relating two orientations on ν_C (see equation (3.9)).

Proof. Since the inverse Fourier transform of a convolution is the product of the inverse Fourier transforms, using the multiplicativity of the Euler class and equations (3.16), (3.9), $\mathcal{F}^{-1}\mathfrak{m}_{\Delta,C}(\lambda+\xi)$ is given by

$$\operatorname{Eul}(\mathfrak{g}/\mathfrak{t}, -2\pi i(\lambda+\xi)) \sum_{F\subset C} \int_{F} \lim_{s\to 0^+} \frac{\epsilon_{F,C} e^{\omega} \Phi_F^{\lambda} e^{2\pi i \langle \mu_{\Delta}, \xi \rangle}}{\operatorname{Eul}(\nu_F, -2\pi i(\lambda+\xi-is\gamma_{\Delta}^+))} |T_{\Delta} \cap T_{\Delta}^{\perp}| \delta(\Lambda^* \cap \mathfrak{t}_F + \mathfrak{t}_{\Delta}),$$

where the sum over $F \subset C$ is only over those F such that $\lambda \in (\Lambda^* \cap \mathfrak{t}_F) \setminus \bigcup_{\alpha_{F,\Delta}} \{\alpha = 0\}$. Since $\alpha_{F,\Delta}$ is the list of weights on the normal bundle to F in C, this condition means that F is a connected component of C^{λ} . (This is similar to the derivation of equation (3.7), but in reverse.)

Our reason for working out the inverse Fourier transform (Theorem 3.2.4) is that one can now see that $\mathfrak{m}_{\Delta,C}$ takes the form of fixed-point contributions for an integral over C—or even better perhaps, an integral over the total space of the normal bundle ν_C . Following this idea, one finds that $\mathfrak{m}_{\Delta,C}$ is essentially a Duistermaat-Heckman distribution for the total space ν_C , twisted by the pullback of a Thom form for the normal of the maximal torus T in G. In particular, by taking the support of this Thom form sufficiently small, it follows that the non-zero contributions $\mathfrak{m}_{\Delta,C}$ are labelled by critical points of the norm-square of the moment map for the LG-space—this part of the argument is identical to that in Chapter 1. This implies the corresponding vanishing results for the closely related distributions $\mathfrak{m}_{\Delta,C}^{\text{pol}}$, and justifies referring to the formula

$$\mathfrak{m} = \sum_{\Delta,C} \mathfrak{m}^{\mathrm{pol}}_{\Delta,C}$$

as a 'norm-square localization' formula. We do not pursue this in any more detail—the result is similar to the results in Chapter 1—since the developments in this chapter are intended mainly as a warm-up for the analogous quantum version of this problem in the next chapter.

Chapter 4

[Q, R] = 0 and Verlinde Series

Let G be a compact Lie group with Lie algebra \mathfrak{g} , and (M, ω) a compact Hamiltonian G-space with moment map $\phi : M \to \mathfrak{g}^*$. Suppose M has a G-equivariant prequantum line bundle L. Choose a G-invariant almost complex structure on M which is compatible with the symplectic form. The index of the Dolbeault-Dirac operator twisted by L defines an element

$$Q(M,L) := \operatorname{index}(D_L) \in R(G).$$

The [Q, R] = 0 Theorem (conjectured in [24], and proved in this degree of generality in [38], [45]) states that

$$Q(M,L)^G = Q(M//G,L//G),$$
(4.1)

the multiplicity of the trivial representation in Q(M, L) equals the corresponding index for the symplectic quotient $\phi^{-1}(0)/G$ (suitably defined when the quotient is singular).

An early approach of Meinrenken [37] to the [Q, R] = 0 Theorem involved replacing L with L^k and studying the limit $k \to \infty$ using the stationary phase approximation. Using Ehrhart's Theorem [19] and the fixed-point formula, one shows that, at least for sufficiently large k, the multiplicity of the trivial representation in $Q(M, L^k)$ is a quasi-polynomial function of k (c.f. [37]). This allows the $o(k^{-\infty})$ errors in the stationary phase approximation to be eliminated, leading to a proof of (4.1) for L^k , assuming kis sufficiently large. To make this approach work for k = 1, one would need to prove that $Q(M, L^k)^G$ is quasi-polynomial in k for all $k \ge 1$.

A powerful approach to the [Q, R] = 0 Theorem was developed by Paradan [50]. The geometry underlying this approach is strongly related to the norm-square of the moment map $\|\phi\|^2$. Paradan used the Hamiltonian vector field for $\|\phi\|^2$ to deform the symbol of the operator D_L to a transversally elliptic symbol with support contained in a small neighbourhood of the set of critical points of $\|\phi\|^2$. In this way he obtains a formula for the multiplicities in Q(M, L), with terms labelled by the components of the critical set of $\|\phi\|^2$. If 0 is a regular value of ϕ , then the term corresponding to $\phi^{-1}(0) \subset \operatorname{Crit}(\|\phi\|^2)$ is quasi-polynomial, and the 'correction' terms labelled by other components of $\operatorname{Crit}(\|\phi\|^2)$ are supported away from $0 \in \mathfrak{g}^*$.

Szenes-Vergne [60] noticed that Paradan's norm-square localization formula for $Q(M, L^k)$ could be derived from the Atiyah-Bott fixed-point formula using a combinatorial argument involving partition functions. They explained how the resulting formula leads to a proof that $Q(M, L^k)^G$ is a quasipolynomial function of k, for all $k \ge 1$. Combined with the earlier approach of Meinrenken [37], this provides possibly the most elementary known proof of the [Q, R] = 0 Theorem for compact Hamiltonian G-spaces. The goal of this paper is to carry out the first part of this strategy in the case of Hamiltonian loop-group spaces.

Many well-known results for compact Hamiltonian G-spaces have analogs for proper Hamiltonian LG-spaces (LG=the infinite dimensional loop group). Examples include the cross-section theorem, the convexity theorem, Duistermaat-Heckman formulas, c.f. [46], [2], [5]. It is thus natural to ask whether there is an analog of Q(M) and of the [Q, R] = 0 Theorem for a Hamiltonian LG-space \mathcal{M} . A definition of $Q(\mathcal{M})$ was proposed in [44], in terms of a push-forward in twisted K-homology from a certain finitedimensional quotient $M = \mathcal{M}/\Omega G$ to G; the result can be viewed as a positive-energy representation of LG via the Freed-Hopkins-Teleman Theorem. There is a [Q, R] = 0 Theorem in this context, proved in [4] by quite complicated means.¹

Let G be simple and simply connected, $T \subset G$ a maximal torus, and $\Lambda^* \subset \mathfrak{t}^*$ the weight lattice. Let $\Psi: \mathcal{M} \to L\mathfrak{g}^*$ be a proper Hamiltonian LG-space, with level $k_1 \geq 1$ prequantum line bundle L. Then the level k quantization of \mathcal{M} , denoted $Q(\mathcal{M}, k)$, is defined for each $k = nk_1$, with n a positive integer.² For convenience we define $Q(\mathcal{M}, k)$ to be zero when k is not a positive integer multiple of k_1 .

The level k irreducible projective positive energy representations of LG are labelled by the set Λ_k^* of dominant weights λ satisfying $B(\lambda, \theta) \leq k$, where θ is the highest root and B is the basic inner product (normalized such that $B(\alpha, \alpha) = 2$ for each long root α). Thus the multiplicities of the irreducible representations in $Q(\mathcal{M}, k)$ are given by a function $\Lambda_k^* \to \mathbb{Z}$. We define the multiplicity function

$$m: \Lambda^* \times \mathbb{N} \to \mathbb{Z}$$

to be the unique function such that:

- 1. For $\lambda \in \Lambda_k^*$, $m(\lambda, k)$ is the multiplicity of the irreducible representation labelled by λ in $Q(\mathcal{M}, k)$.
- 2. The function m(-,k) is alternating under the shifted level $k + h^{\vee}$ -action of the affine Weyl group W_{aff} on Λ^* , given by

$$(w,\xi) \cdot \lambda = w(\lambda + \rho) - \rho + (k + \mathbf{h}^{\vee})B^{\flat}(\xi),$$

where $(w,\xi) \in W \ltimes \Lambda = W_{\text{aff}}$, \mathbf{h}^{\vee} is the dual Coxeter number of \mathfrak{g} , and ρ is the half-sum of the positive roots of \mathfrak{g} .

The critical set of the norm-square of the moment map is a disjoint union (c.f. [31], [14]):

$$\operatorname{Crit}(\|\Psi\|^2) = G \cdot \bigcup_{\beta \in \mathcal{B}} \mathcal{M}^{\beta} \cap \Psi^{-1}(\beta),$$

where $\mathcal{B} \subset \mathfrak{t}_{+}^{*}$ is a discrete subset. Our main result is a 'norm-square localization' formula for the multiplicity function $m(\lambda, k), m : \Lambda^* \times \mathbb{N} \to \mathbb{Z}$ of the quantization $Q(\mathcal{M}, k)$.

Theorem 4.0.1. The multiplicity function $m : \Lambda^* \times \mathbb{N} \to \mathbb{Z}$ for $Q(\mathcal{M}, k)$ has a locally finite decomposition

$$m = \sum_{\Delta} m_{\Delta}^{\text{qpol}},$$

¹Note that [4] appeared much earlier than [44]. In [4], $Q(\mathcal{M})$ was 'defined' in terms of a fixed-point formula on M, and the [Q, R] = 0 Theorem was proved in that context. ²Note that $Q(\mathcal{M}, k)$ depends on L as well, although we suppress it from the notation.

where Δ ranges over a certain infinite collection of affine subspaces of \mathfrak{t}^* . The contribution m_{Δ}^{qpol} vanishes unless $\mathcal{M}^{\beta} \cap \Psi^{-1}(\beta) \neq \emptyset$, where β is the nearest point in Δ to the origin. Furthermore m_{Δ}^{qpol} is obtained by taking 'quasi-polynomial germs along Δ ' of a multiplicity function $m_{\Delta} = \mathcal{F}(Q_{\Delta})$, where

$$Q_{\Delta}(t,k) = \sum_{C \subset N^{T_{\Delta}}} \sum_{F \subset C^{t}} \int_{F} \frac{\hat{A}(F) \mathrm{Ch}(\mathcal{L}_{k,\Delta,C},t)^{1/2} \mathrm{Ch}(\wedge_{\mathbb{C}} \mathfrak{n}_{-} \otimes S\nu_{C}(\gamma_{\Delta}^{+}),t)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} \delta_{T_{\ell}T_{\Delta}}(t).$$

 $Q_{\Delta}(-,k) = q_{\Delta}(-,k)\delta_{T_{\ell}T_{\Delta}}$ is a distribution supported on $T_{\ell}T_{\Delta} \subset T$. The generalized function $q_{\Delta}(t,k)$ on $T_{\ell}T_{\Delta}$ takes the form of fixed-point contributions for $T_{\ell}T_{\Delta}$ -equivariant spin-c structures $S_{k,\Delta,C}$ on submanifolds $C \subset N^{T_{\Delta}}$, tensored with the \mathbb{Z}_2 -graded bundle $\wedge_{\mathbb{C}} \mathfrak{n}_- \otimes S\nu_C(\gamma_{\Delta}^+)$.

In this theorem, T_{ℓ} is a certain finite subgroup of the maximal torus T depending on the level k, and $T_{\Delta} = \exp(\mathfrak{t}_{\Delta}) \subset T$ where \mathfrak{t}_{Δ} is the subspace of \mathfrak{t} orthogonal to Δ . N is an open set in M of the form $N = \Phi^{-1}(U)$ where U is a tubular neighbourhood of the maximal torus (the 'abelianization' from Chapter 1). $\gamma \in \mathfrak{t}^*$ is a generic point near 0, and $\gamma_{\Delta}^+ = \operatorname{pr}_{\Delta}(\gamma) - \gamma$. $S\nu_C(\gamma_{\Delta}^+)$ denotes the symmetric algebra of the normal bundle ν_C , where the latter is equipped with a complex structure such that the weights of the T_{Δ} action are γ_{Δ}^+ -polarized. The notation, the properties of the spin-c structure $S_{k,\Delta,C}$, and the meaning of taking 'quasi-polynomial germs' along Δ will be explained in detail.

Our proof is modelled on the approaches of Paradan and Szenes-Vergne described above, and the formula is analogous to Paradan's formula for a compact Hamiltonian G-space. The result can be viewed as a 'quantum version' of the norm-square localization formula for Duistermaat-Heckman distributions studied in [35].

Recall that the level k irreducible positive energy representations of LG are labelled by dominant weights λ satisfying $B(\lambda, \theta) \leq k$, where θ is the highest root, and B is the basic inner product (normalized such that $B(\alpha, \alpha) = 2$ for each long root α). We refer to the level k representation labelled by $\lambda = 0$ as the *minimal* level k representation. An important application of Theorem 4.0.1 is to prove that the multiplicity m(0, k) of the minimal level k representation in $Q(\mathcal{M}, k)$ is a quasi-polynomial function of k. This follows from Theorem 4.0.1 together with an inequality for the Lie algebra of G, proved in Section 4.5:

Theorem 4.0.2. Let G be simple and simply connected. Let $\kappa \in \mathfrak{t}_+$ be a vertex of the Stiefel diagram. Let ρ be the half-sum of the positive roots of \mathfrak{g} and ρ_{κ} the half-sum of the positive roots of $G_{\exp(\kappa)}$. Then

$$\mathbf{h}^{\vee} \|\kappa\|^2 - \langle \rho + \rho_{\kappa}, \kappa \rangle \ge 0, \tag{4.2}$$

where $\|\cdot\|^2$ is the norm defined by the basic inner product, and h^{\vee} is the dual Coxeter number.

This inequality (and our use for it) is analogous to Paradan-Vergne's 'magic inequality' ([51]), which they used to prove a spin-c [Q, R] = 0 theorem ([52]).

We briefly summarize the contents of the sections. In Section 1 we study Verlinde series. These are complicated discrete sums which appear in the fixed-point formula for $Q(\mathcal{M})$. We provide a proof of an unpublished result of Boysal-Vergne (c.f. [15] for the analogous result for Bernoulli series) giving a kind of combinatorial 'norm-square localization' formula for Verlinde series (Theorem 4.1.20). Section 2 describes a mild extension of the result in Section 1 to Verlinde series that are differential form-valued. Section 3 provides a short introduction to the quantization $Q(\mathcal{M})$ of Hamiltonian LG-spaces \mathcal{M} (or equivalently, the corresponding finite-dimensional quasi-Hamiltonian G-spaces $M = \mathcal{M}/\Omega G$. The heart
of the proof is contained in Section 4: the idea is to break the fixed-point expressions for $Q(\mathcal{M}, k)$ down into their basic constituents—Verlinde series—apply the Boysal-Vergne formula, and then re-organize the terms into the norm-square localization formula (Theorem 4.0.1). In Section 5 we deduce the quasipolynomial behaviour of $Q(\mathcal{M}, k)$ from the norm-square localization formula, combined with Theorem 4.0.2 for the Lie algebra of G.

Finally, we should mention that in forthcoming work with E. Meinrenken and Y. Song [36], we construct spinor modules for Hamiltonian LG-spaces, and then study norm-square localization for LG-spaces from a more analytic perspective.

Additional notation for Chapter 4. For α an affine linear function on a vector space \mathfrak{t} , we write e_{α} or $e(\alpha)$ for the exponential function $e(\alpha)(X) = e_{\alpha}(X) = e^{2\pi i \alpha(X)}$. We write $\mathbf{1}_{p\mathbb{N}} : \mathbb{N} \to \mathbb{C}$ for the function $\mathbf{1}_{p\mathbb{N}}(\ell)$ which is 1 if ℓ is congruent to 0 modulo p and is 0 otherwise.

If A_1, A_2 are subgroups of an abelian group A, then we write A_2/A_1 for the image $q(A_2)$ under the quotient map $q: A \to A/A_1$. A_2/A_1 is identified with the quotient group $A_2/(A_1 \cap A_2)$.

For K a compact (possibly disconnected) Lie group, by *normalized Haar measure* we mean the unique translation-invariant smooth measure on K whose total integral is 1. Such a measure determines an isomorphism between generalized functions and distributions on K.

Throughout this chapter we take G to be a compact, connected, simply connected, simple Lie group with Lie algebra \mathfrak{g} . Fix a choice of maximal torus $T \subset G$ with Lie algebra \mathfrak{t} , and let $W = N_G(T)/T$ be the Weyl group. We have a triangular decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{-} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}_{+},$$

where \mathfrak{n}_+ (\mathfrak{n}_-) is the sum of the positive (resp. negative) root spaces. Let $\Lambda = \ker(\exp : \mathfrak{t} \to T)$ be the integral lattice. Since G is simply connected, Λ coincides with the coroot lattice. Let $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z})$ be the (real) weight lattice. Let $\mathcal{R} \subset \Lambda^*$ denote the roots, and fix a positive Weyl chamber \mathfrak{t}_+ and corresponding set of positive roots \mathcal{R}_+ . The fundamental alcove is

$$\mathfrak{a} = \{ X \in \mathfrak{t}_+ | \langle \theta, X \rangle \le 1 \},\$$

where θ denotes the highest root. The interiors of \mathfrak{a} , \mathfrak{t}_+ will be denoted \mathfrak{a} , \mathfrak{t}_+ . The half-sum of the positive roots is

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha$$

Let *B* be the basic invariant inner product on $\mathfrak{g} \simeq \mathfrak{g}^*$, normalized such that $B(\alpha, \alpha) = 2$ for each long root. The corresponding isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ is denoted B^{\flat} , and $B^{\sharp} = (B^{\flat})^{-1}$. It has the key property that $\Lambda \subset B^{\sharp}(\Lambda^*)$. For $\ell \in \mathbb{N}$ the quotient $T_{\ell} := \ell^{-1}B^{\sharp}(\Lambda^*)/\Lambda \subset \mathfrak{t}/\Lambda = T$ is a finite subgroup. If *G* is simply laced then $B^{\sharp}(\Lambda^*)$ coincides with the coweight lattice, which is the set of elements of \mathfrak{t} which exponentiate to an element of the center of *G*. The dual Coxeter number is given by

$$\mathbf{h}^{\vee} = 1 + B(\rho, \theta).$$

In Sections 3 and 4, formulas appear involving positive integers k and ℓ ; k always refers to the level of the prequantization (defined in Section 3), and $\ell = k + h^{\vee}$. Moreover $k = nk_1$ where $k_1 \in \mathbb{N}$ is the smallest level for which M is prequantized, and $n \in \mathbb{N}$ is the analog of the power of the prequantum line bundle.

A partial list of symbols used in this chapter is included below, together with a brief description and (in brackets) the subsection where the notation was first introduced (when applicable).

\mathbb{N}	positive integers $\{1, 2, 3,\}$
$1_{p\mathbb{N}}$	indicator function for the subset $\{p,2p,3p,4p,\ldots\}\subset\mathbb{N}$
S	number of elements in the finite set S
λ	element of the weight lattice Λ^*
α	list of affine linear functions $(4.1.1)$
\overline{lpha}	linear part of the affine linear function α (4.1.1)
b_{lpha}	phase $e^{2\pi i \alpha(0)}$ (4.1.1)
$e_{lpha}, e(lpha)$	exponential function $e^{2\pi i \alpha(X)}$ (4.1.1)
$ abla_{lpha}$	finite difference operator $(4.1.3)$
$\operatorname{ann}(W)$	W is a subspace of a vector space $U,\mathrm{ann}(W)\subset U^*$ is the annihi-
	lator of W
$V(oldsymbol{lpha}), V_{oldsymbol{lpha}}$	Verlinde series (4.1.1)
$P(\boldsymbol{\alpha};\gamma^+)$	generalized partition function $(4.1.4)$
$lpha_{\Delta}$	sub-list of affine linear functions with linear part parallel to Δ
	(4.1.5)
$oldsymbol{lpha}_{\Delta}^{\perp}$	complement of α_{Δ} in α (4.1.5)
$V(\boldsymbol{\alpha}_{\Delta};\Delta)$	Verlinde series for α_{Δ} , restricted to $\cup_{\ell} \ell \Delta \times \{\ell\}$ (4.1.5)
$\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma))$	quasi-polynomial on $\cup_{\ell} \ell \Delta \times \{\ell\}$ agreeing with $V(\boldsymbol{\alpha}_{\Delta}; \Delta)$ on cone
	containing $(pr_{\Delta}(\gamma), 1)$ (4.1.5)
$\mathcal{S}, \mathcal{S}(oldsymbollpha, \Xi^*)$	affine 'admissible subspaces' of the form $\Delta = \xi + \Delta_0$ where $\xi \in \Xi^*$,
	Δ_0 is a subspace through the origin in \mathfrak{t}^* spanned by some subset
	of the linear parts of the affine linear functions $\boldsymbol{\alpha}$ (4.1.5)
V(u, b)	differential form-valued Verlinde series $(4.2.3)$
$P(\nu, b; \gamma^+)$	differential form-valued partition function $(4.2.2)$
$V(\nu_{\Delta}, b; \Delta)$	differential form version of $V(\boldsymbol{\alpha}_{\Delta}; \Delta)$ (4.2.3)
$\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma))$	differential form version of $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma))$ (4.2.3)
$\Psi:\mathcal{M} ightarrow L\mathfrak{g}^*$	Hamiltonian LG -space (1.3)
\mathcal{B}	discrete subset of $\beta \in \mathfrak{t}_+$ such that $\mathcal{M}^{\beta} \cap \Psi^{-1}(\beta) \neq \emptyset$, indexes
	parts of $\operatorname{Crit}(\ \Psi\ ^2)$ (1.2.1)
$\Phi:M\to G$	quasi-Hamiltonian G -space (1.3.2)
U	tubular neighbourhood of the maximal torus $T \subset G$ (1.3.4)
π_T	projection map $U \to T$ (1.3.4)
N	inverse image $\Phi^{-1}(U)$, 'abelianization' of M (1.3.4)
ω_a	presymplectic form on N (1.3.4)
Φ_a	composition $\pi_T \circ \Phi _N$ (1.3.4)
\mathcal{N}	fibre product $\mathfrak{t} \times_T N$, Λ -covering space of N (1.3.4)
ϕ	induced map $\mathcal{N} \to \mathfrak{t}$ (1.3.4)
π	quotient map $\mathcal{N} \to \mathcal{N} = \mathcal{N}/\Lambda$ (1.3.4)

k_1	smallest positive integer such that M is prequantizable at level k_1
	(4.3)
n	power of the prequantum line bundle $(4.3.3)$
k_{\perp}	level of the prequantization $k = nk_1$ (4.3, 4.3.3)
h^{\vee}	dual Coxeter number of the simple Lie algebra ${\mathfrak g}$
ℓ	the sum $k + \mathbf{h}^{\vee}$ (4.3.3)
B	the basic inner product of the simple Lie algebra ${\mathfrak g}$
B^{\sharp}	isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ induced by B
B^{\flat}	inverse of B^{\sharp}
T_{ℓ}	finite subgroup $\ell^{-1}B^{\sharp}(\Lambda^*)/\Lambda \subset T$
T_{ℓ}^{reg}	regular elements of T_{ℓ} , i.e. having stabilizer T under conjugation
	by G
Q(M,k)	level k quantization of M, function defined on $T_{\ell}^{\rm reg}$ (4.3.4)
$m(\lambda,k)$	multiplicity function for the quantization $Q(M,k)$, related to
	Q(M,k) by finite Fourier transform (4.3.4)
L	T_{k_1} -equivariant prequantum line bundle on N (sometimes on \mathcal{M})
	(4.3.3)
π^*L	T-equivariant prequantum line bundle on \mathcal{N} , $L = \pi^* L / \Lambda$, moment
	map $k_1 \phi$ (4.3.3)
S	$T_{\rm hv}$ -equivariant spinor bundle on N, also denoted S_0 (4.3.3)
$\approx \pi^*S$	<i>T</i> -equivariant spinor bundle on \mathcal{N} $S = \pi^* S / \Lambda$ (4.3.3)
$\int \det(S)$	'determinant line bundle' Hom $_{CUT}(S^*, S)$ for the spinor bundle S
$\boldsymbol{\mathcal{L}}, \operatorname{det}(\boldsymbol{B})$	(4.3.4)
arphi	spin-c moment map $\mathcal{N} \to \mathfrak{t}^*$ for $\pi^* S$ (1/2 times the moment map
	for the determinant line bundle $det(\pi^*S)$) (4.3.3)
S_k	T_{ℓ} -equivariant spinor bundle on $N, S_k = S \otimes L^n$ (4.3.3)
π^*S_k	T-equivariant spinor bundle on \mathcal{N} , $\pi^* S_k / \Lambda = S_k$ (4.3.3)
$\mathcal{L}_k, \det(S_k)$	'determinant line bundle' $\operatorname{Hom}_{\operatorname{Cliff}}(S_k^*, S_k)$ for S_k (4.3.4)
$k\phi + \varphi$	spin-c moment map for $\pi^* S_k$ (4.3.3)
F	connected component of M^t for some $t \in T_{\ell}^{\text{reg}}$ (4.3.4)
$ ilde{T}_F$	stabilizer in T of F (4.4.1)
T_F	identity component of \tilde{T}_F , also denoted $T_{F,0}$ (4.4.1)
$T_{F,a}$	components of \tilde{T}_F , $a = 0, 1, 2, (4.4.1)$
$\ell_{F,a}$	minimal positive integer such that $T_{F,a} \cap T_{\ell_{F,a}} \neq \emptyset$ (4.4.1)
$t_{F,a}$	element of $T_{\ell_{F_a}} \cap T_{\ell}$ (4.4.1)
\mathfrak{t}_F	Lie algebra of T_F (4.4.1)
pr _{t*} _	quotient map $\mathfrak{t}^* \to \mathfrak{t}^*/\operatorname{ann}(\mathfrak{t}_F) = \mathfrak{t}_F^*$ (4.4.1)
$\kappa_{Fk}(t)^{1/2}$	phase factor for $F \subset M^t$ in the Atiyah-Segal-Singer formula, at
7 X /	level k (4.3.4)
$\rho_{F,k}(t)$	the phase $\kappa_{F,k}(t)^{1/2} = \rho_{F,k}(t) \det_{\mathbb{C}}(A(t)^{1/2})$ (4.3.4)
$\tilde{\nu}_F$	normal bundle to F in M (4.3.4)
ν_F	sub-bundle of $\tilde{\nu}_F$ on which \mathfrak{t}_F acts with non-zero weight (4.4.1)
	0 ()

$ u'_F$	T-invariant complement to ν_F , $\nu_F \oplus \nu'_F = \tilde{\nu}_F$ (4.4.1)
ϕ_F	element $\operatorname{pr}_{\mathfrak{t}_F^*}(\phi(\hat{x})) \in (\Lambda \cap \mathfrak{t}_F)^*$ where $x \in F$, and \hat{x} is a lift of x
	to \mathcal{N} (4.4.1)
φ_F	element $\operatorname{pr}_{\mathfrak{t}_F^*}(\varphi(\hat{x})) \in (\Lambda \cap \mathfrak{t}_F)^*$ where $x \in F$ and \hat{x} is a lift of x
	to \mathcal{N} (4.4.1)
s_F	element of $(\Lambda \cap \mathfrak{t}_F)^*$, weight for T_F action on $\det_{\mathbb{C}}(T_x N)$ where
	$x \in F (4.4.1)$
σ_F	the sum $\varphi_F - \frac{1}{2}s_F - \mathbf{h}^{\vee}\phi_F \in \mathfrak{t}_F^*$ (4.4.1)
$f_{F,a}(k)$	differential form with quasi-polynomial dependence on k (4.4.1)
Δ	affine subspace in t of the form $\phi + \xi + \Delta_0$ where $\xi \in \Lambda$, Δ_0 is a
	subspace containing 0, and ϕ is a shift (4.4.2, and end of 4.2.3)
\mathfrak{t}_Δ	subspace orthogonal to Δ_0
$ u_{F,\Delta}, u_{F,C}$	sub-bundle of ν_F fixed by T_{Δ} (4.2.3, 4.4.2)
$\nu_{F,\Delta}^{\perp}$	T-invariant complement to $\nu_{F,\Delta}$ in ν_F (4.2.3)
γ	sufficiently 'generic' point near $0 \in \mathfrak{t}$ (4.2.3)
$\operatorname{pr}_{\Delta}(\gamma)$	orthogonal projection of γ onto Δ
γ_{Δ}^+	'polarizing vector' $\operatorname{pr}_{\Delta}(\gamma) - \gamma$
C	connected component of $N^{T_{\Delta}}$ with $\Phi(C) \subset \exp(\Delta)$ (4.4.2)
$ u_C$	normal bundle to C, restriction $\nu_C _F = \nu_{F,\Delta}^{\perp}$ (4.4.2)
$m^{ m qpol}_{\Delta,C}$	contribution of the pair (Δ, C) to multiplicity function m (4.4.2)
$m_{\Delta,C}$	modification of $m_{\Delta,C}^{\text{qpol}}$ (replace Ver(-) with $V(-)$) (4.4.3)
$Q_{\Delta,C}$	inverse Fourier transform of $m_{\Delta,C}$ (4.4.3)
$\xi_{F,\Delta}$	choice of point in $(\Delta - \phi_F) \cap B^{\flat}(\Lambda)/\operatorname{ann}(\mathfrak{t}_F)$, subspace $\Delta = \Delta_0 +$
	$\phi_F + \xi_{F,\Delta} \ (4.4.3)$
ϕ_{Δ}	image of $\phi_F + \xi_{F,\Delta}$ under quotient map $\mathfrak{t}_F^* \to \mathfrak{t}_{\Delta}^*$ (4.4.3)
φ_Δ	image of $\varphi_F + \mathbf{h}^{\vee} \xi_{F,\Delta}$ under quotient map $\mathfrak{t}_F^* \to \mathfrak{t}_{\Delta}^*$ (4.4.3)
s_{Δ}^+	sum of γ_{Δ}^+ -polarized weights of T_{Δ} on ν_C (4.4.3)
$S_{k,\Delta,C}$	$T_{\ell}T_{\Delta}$ -equivariant induced spinor bundle on C (4.4.3)
$\mathcal{L}_{k,\Delta,C}$	determinant line bundle for $S_{k,\Delta,C}$ (4.4.3)
Λ_{Δ}	intersection $\Lambda \cap \Delta_0$ where Δ_0 is the subspace through 0 parallel
	to Δ_0 (4.4.3)
\mathcal{C}_{Δ}	fibre product $C \times_T \Delta$ (4.4.3)
$\nu(\gamma^+)$	vector bundle ν equipped with a complex structure such that
	weights of T-action are γ^+ -polarized (pair positively with the vec-
	tor γ^+) (4.2.2)
$S\nu(\gamma^+)$	symmetric algebra bundle of the complex vector bundle $\nu(\gamma^+)$
	(4.2.2)

4.1 Verlinde Series

In this section we introduce *Verlinde series* (called *rational trigonometric series* in [59]), the basic combinatorial objects which appear in the fixed-point expressions for the quantization of quasi-Hamiltonian G-spaces. These series originally appeared in the closely related Verlinde formulas [62]; special cases of these series compute dimensions of spaces of holomorphic sections for line bundles over moduli spaces of flat connections on Riemann surfaces. We describe their quasi-polynomial behaviour, giving a brief exposition of Szenes' residue formula, and then prove an unpublished result of A. Boysal and M. Vergne providing a 'decomposition formula' for Verlinde series.

4.1.1 Definition and quasi-polynomial behavior of Verlinde series.

Consider 4-tuples $(\mathfrak{t}, \Xi, \Lambda, \boldsymbol{\alpha})$, where \mathfrak{t} is a finite-dimensional vector space of dimension $r, \Lambda \subset \Xi$ are full-rank lattices in \mathfrak{t} , and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ is a list of affine linear functions on \mathfrak{t} . ($\boldsymbol{\alpha}$ will always denote a *list* below, although we will occasionally use set notation.) The quotient $T := \mathfrak{t}/\Lambda$ is a torus. For an affine linear function α , we write $\overline{\alpha} = \alpha - \alpha(0)$ for the linear part and set $b_{\alpha} = e^{2\pi i \alpha(0)}$, thus:

$$\alpha(\xi) = \langle \overline{\alpha}, \xi \rangle + \alpha(0), \qquad e^{2\pi i \alpha(\xi)} = b_{\alpha} e^{2\pi i \langle \overline{\alpha}, \xi \rangle}$$

We require that the α are rational, in the sense that the $\overline{\alpha}$ are elements of Λ^* , and $\alpha(0) \in \mathbb{Q}$.

Definition 4.1.1. The Verlinde series associated to a 4-tuple $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ as above, is the function $V_{\alpha} = V(\alpha) : \Lambda^* \times \mathbb{N} \to \mathbb{C}$ defined by

$$V_{\alpha}(\lambda,\ell) = \sum_{\xi \in \ell^{-1}\Xi/\Lambda}^{\prime} \frac{e^{-2\pi i \langle \lambda,\xi \rangle}}{\prod_{\alpha} 1 - e^{2\pi i \alpha(\xi)}}$$

The prime next to the summation symbol means that we only sum over elements $\xi \in \ell^{-1}\Xi/\Lambda$ such that the denominator does not vanish. We will refer to the natural number $\ell \in \mathbb{N}$ as the *level*. It is convenient to define the trivial Verlinde series, $V_{\emptyset} = 0$. Note that we have suppressed $\mathfrak{t}, \Xi, \Lambda$ from the notation, as usually these will be clear from the context.

Remark 4.1.2. Taking into account the behavior of Verlinde series as the level ℓ is varied (which re-scales the lattice Ξ) will play an important role. On the other hand, it is enough to prove certain results for $\ell = 1$, since changing ℓ just involves replacing Ξ with the finer lattice $\ell^{-1}\Xi$.

The quotient $T_{\ell} = \ell^{-1}\Xi/\Lambda$ is a finite subgroup of the torus T. Suppose we fix the level ℓ . Then it is clear from the definition that $V_{\alpha}(-,\ell)$ is $\ell\Xi^*$ -periodic, and thus descends to a function on the finite group $\Lambda^*/\ell\Xi^*$. We will refer to the set $(\ell^{-1}\Xi)_{\text{reg}}$ of elements $\xi \in \ell^{-1}\Xi$ such that $\alpha(\xi) \notin \mathbb{Z}$ as regular elements. Then $V_{\alpha}(-,\ell)$ is the finite Fourier transform of the function $\check{V}_{\alpha} : \ell^{-1}\Xi/\Lambda = T_{\ell} \to \mathbb{C}$,

$$\check{V}_{\alpha}(\xi) = \begin{cases} \frac{1}{\prod_{\alpha} 1 - e^{2\pi i \alpha(\xi)}} & \text{if } \xi \in (\ell^{-1}\Xi)_{\text{reg}} \\ 0 & \text{else.} \end{cases}$$

Definition 4.1.3. Let Γ be a lattice in a vector space V. A function

$$f:\Gamma\to\mathbb{C}$$

is called *quasi-polynomial* if there is a sublattice $\Gamma' \subset \Gamma$ of finite index such that f is given by a polynomial on each coset of Γ' . More generally if f is quasi-polynomial and f_1 is some other function such that $f = f_1$ on some subset $\Gamma_1 \subset \Gamma$, then we will say that f_1 is quasi-polynomial on Γ_1 . Remark 4.1.4. In some cases we will not mention the subset Γ_1 explicitly. For example, one of the main results of this chapter is that a function m(0,k) defined below is a quasi-polynomial function of k. This function will only be defined for k of the form $k = nk_1$ where k_1 is a fixed positive integer and n is an arbitrary positive integer. So in this case, what we mean more precisely is that there is a quasi-polynomial function of $k \in \mathbb{Z}$ which agrees with m(0, k) on the subset $\Gamma_1 = \{k = nk_1 | n \in \mathbb{N}\} \subset \mathbb{Z}$.

In [59], A. Szenes proved a remarkable residue formula for Verlinde series. We will not use the details of the formula, but will use the quasi-polynomial properties of Verlinde series which follow as a corollary. We give a brief exposition of Szenes' formula in the next subsection.

Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series. Let

$$\Box \boldsymbol{\alpha} = \{ \sum t_i \overline{\alpha}_i | 0 < t_i < 1 \}.$$

This is called the *zonotope* of the list of elements α .

Consider the list $\overline{\alpha} = (\overline{\alpha}_1, ..., \overline{\alpha}_n)$ of elements of \mathfrak{t}^* . We obtain a periodic collection of affine subspaces \mathcal{H} in \mathfrak{t}^* by taking translates by elements of Ξ^* of all proper subspaces spanned by subsets of $\overline{\alpha}$. We will refer to connected components \mathfrak{c} of the complement of \mathcal{H} in \mathfrak{t}^* as *chambers*.

Theorem 4.1.5 (Szenes, [59]). Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be as above, and let \mathfrak{c} be a chamber in \mathfrak{t}^* . There is a unique quasi-polynomial function on $\Lambda^* \times \mathbb{N}$ which agrees with V_{α} on the set

$$igcup_{\ell\in\mathbb{N}}(\ell\mathfrak{c}-\Boxm{lpha}) imes\{\ell\}\subset\mathfrak{t}^* imes\mathbb{N}$$

Remark 4.1.6. Suppose that the $\overline{\alpha}$ span \mathfrak{t}^* . Then \mathfrak{c} is a relatively compact polyhedral region containing only finitely many elements of Λ^* , so it would be meaningless to ask whether V_{α} is quasi-polynomial on \mathfrak{c} . On the other hand, the set $\bigcup_{\ell} \ell \mathfrak{c} \times \{\ell\}$ is an open cone in $\mathfrak{t}^* \times \mathbb{N}$, so that if there is a quasi-polynomial function agreeing with V_{α} on this set, it is unique. If the $\overline{\alpha}$ span \mathfrak{t}^* , then the regions in the theorem above, as \mathfrak{c} varies over the different chambers, give an open cover of $\mathfrak{t}^* \times \mathbb{N}$.

Remark 4.1.7. Let $f : \mathbb{N} \times \Lambda^* \to \mathbb{C}$ be quasi-polynomial (i.e. there is a $p \in \mathbb{N}$ such that $f|_{n+p\mathbb{N}}$ is quasi-polynomial for each $n \in \mathbb{N}$). The function

$$(fV_{\alpha})(\lambda,\ell) = f(\lambda,\ell)V_{\alpha}(\lambda,\ell)$$

is quasi-polynomial on the same regions as in Theorem 4.1.5. Two examples are: (1) $f(\ell) = \mathbf{1}_{p\mathbb{N}}$, where $\mathbf{1}_{p\mathbb{N}}(\ell) = 1$ for *n* congruent to 0 modulo *p* and $\mathbf{1}_{p\mathbb{N}}(\ell) = 0$ otherwise; (2) $f(\lambda, \ell) = \zeta^{\ell} e^{2\pi i \langle \lambda, \xi \rangle}$ where ζ is a root of unity and $\xi \in \Lambda \otimes \mathbb{Q}$.

Remark 4.1.8. In Section 4.4 we will express the multiplicity function $m(\lambda, k)$ for the quantization $Q(\mathcal{M}, k)$ of a Hamiltonian LG-space in terms of translations of Verlinde series (with $\ell = k + h^{\vee}$, h^{\vee} the dual Coxeter number of G). Our goal ultimately will be to show that m(0, k) is a quasi-polynomial function of the level k. In the case G = SU(2), Szenes' Theorem 4.1.5 already implies this result. For groups of higher rank, the situation is more complicated. This is related to the fact that m is expressed in terms of translations of Verlinde series. These translations depend on the level k, and hence a priori, as k varies, there can be 'wall-crossing' between the different quasi-polynomial regions in Theorem 4.1.5. It seems that for higher rank groups this wall-crossing can indeed happen for the individual Verlinde series appearing in the formula for $m(\lambda, k)$; the quasi-polynomial behaviour of m(0, k) will be a consequence

of certain cancellations between the different Verlinde series.

4.1.2 Szenes' Theorem.

In [59] (see also [58], [57]), A. Szenes proved a remarkable residue formula for Verlinde series, which we give a brief exposition of here. Throughout this subsection, let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series $V_{\alpha}(\lambda, \ell)$.

The constant term functional.

The formula in [59] is expressed in terms of iterated 'constant-term' functionals. For a meromorphic function of a single variable f(z) on a neighbourhood of a point $a \in \mathbb{C}$, the constant term

$$\operatorname{CT}_{z=a}(f)$$

is the constant term in the Laurent expansion of f(z) about the point a. Note that $\operatorname{CT}_{z=0}(f)$ does not change if we make a change of coordinates w = cz, $c \in \mathbb{C}^{\times}$. The constant term can be computed by the same methods used to compute residues. For example, if f(z) has a pole of order less than or equal to n at a, then

$$\operatorname{CT}_{z=a}(f) = \frac{1}{n!} \frac{d^n}{dz^n} \bigg|_0 (z^n f(z+a)).$$

More generally let W be a complex affine space of dimension r, and suppose $\mathbf{a} = (a_1, ..., a_r)$ is a list of affine linear functions such that the intersection of the $H_{a_k} = \{a_k = 0\}$ is a single point p. Let f be a meromorphic function on a neighbourhood p. The *iterated constant term functional*, denoted

$$\operatorname{iCT}_{\boldsymbol{a}}(f)$$

is defined as follows. The functions $a_1, ..., a_r$ determine a coordinate system $z_1, ..., z_r$ for the affine space W in which p is at the origin. Thus $f = f(z_1, ..., z_r)$, and $iCT_a(f)$ is computed by taking a sequence of 1-variable constant terms, while treating the other variables as auxiliary parameters. One first treats $z_1, ..., z_{r-1}$ as auxiliary parameters and computes $iCT_{z_r=0}(f)$. The result can be viewed as a meromorphic function of $z_1, ..., z_{r-1}$, and one proceeds inductively. More concretely, for sufficiently large n we have

$$\operatorname{iCT}_{\boldsymbol{a}}(f) = \frac{1}{n!} \frac{\partial^n}{\partial z_1^n} \bigg|_{z_1=0} z_1^n \cdots \frac{1}{n!} \frac{\partial^n}{\partial z_r^n} \bigg|_{z_r=0} z_r^n f.$$
(4.3)

In general this depends on the order $a_1, ..., a_r$ (the function f is allowed to be singular, so the order of the partial derivatives matters).

The hyperplane arrangement \mathcal{A} in t.

Let \mathfrak{t} be a real vector space. For $\alpha \in \alpha$, let $H_{\alpha} = \{X \in \mathfrak{t} | \alpha(X) = 0\}$. Let \mathcal{A} denote the set of affine hyperplanes which are translates of the H_{α} by elements of Λ . This is a periodic affine hyperplane arrangement in \mathfrak{t} .

For $p \in \mathfrak{t}$, let \mathcal{A}_p be the subset of hyperplanes containing the point p. The set of vertices of \mathcal{A} , denoted vx(\mathcal{A}), is the set of points $p \in \mathfrak{t}$ such that $\cap \mathcal{A}_p = \{p\}$. The set vx(\mathcal{A}) is invariant under Λ , and we write vx(\mathcal{A}/Λ) for a (finite) set of representatives of elements of vx(\mathcal{A}). If $p \in vx(\mathcal{A})$, we follow [59] in defining an *orthogonal basis* for \mathcal{A}_p to be a collection B_p of r-tuples \mathbf{a} of elements of \mathcal{A}_p such that $\cap \mathbf{a} = \{p\}$, and which is minimal with respect to the following property: for any r-tuple \mathbf{b} of elements of \mathcal{A}_p with $\cap \mathbf{b} = \{p\}$, there is a permutation $\sigma \in S_r$ and an r-tuple $\mathbf{a} \in B_p$ such that the complete flags determined by $\sigma(\mathbf{a})$ and \mathbf{b} are the same. An equivalent definition can be found, e.g., in [16], where it is shown that B_p corresponds naturally to a vector space basis for the subspace of simple fractions, in the ring of rational functions generated by the inverses of the affine linear functions defining the hyperplanes in \mathcal{A}_p . There exists a simple algorithm for generating such a set B_p , c.f. [58] and [59] for further details.

Statement of Szenes' theorem.

Choose representatives $p \in vx(\mathcal{A}/\Lambda)$ and let B_p denote an orthogonal basis of \mathcal{A}_p . For each hyperplane in \boldsymbol{a} where $\boldsymbol{a} \in B_p$, choose a representative affine linear function $\boldsymbol{a} = \overline{\boldsymbol{a}} + \boldsymbol{a}(0)$ with $\overline{\boldsymbol{a}} \in \Xi^*$ (we also use \boldsymbol{a} to denote this collection of affine linear functions).

The lattice Ξ^* determines a normalized Lebesgue measure on \mathfrak{t}^* such that \mathfrak{t}^*/Ξ^* has volume 1; let $\operatorname{vol}_{\Xi^*}(a)$ denote the volume of $\Box a$ with respect to this measure. Let \mathfrak{c} denote a chamber of the hyperplane arrangement \mathcal{H} in \mathfrak{t}^* (see Section 4.1.1 for the definition of \mathcal{H}). Let $\lambda \in \Lambda^*$ and $\ell \in \mathbb{N}$. The data \mathfrak{c} , a, λ , ℓ define a meromorphic function $T_{\mathfrak{c},a}(\lambda,\ell)(X)$ on $W = \mathfrak{t} \otimes \mathbb{C}$ by the formula:

$$T_{\mathfrak{c},\boldsymbol{a}}(\lambda,\ell)(X) = \frac{1}{\operatorname{vol}_{\Xi^*}(\boldsymbol{a})} e^{-2\pi i \langle \lambda, X \rangle} \sum_{\boldsymbol{\xi}^* \in \Xi^* \cap (\mathfrak{c} + \Box \overline{\boldsymbol{a}})} e^{2\pi i \ell \langle \boldsymbol{\xi}^*, X \rangle} \prod_{a \in \boldsymbol{a}} \frac{2\pi i \ell a(X)}{1 - e^{2\pi i \ell \langle \overline{\boldsymbol{a}}, X \rangle}}.$$

(One can check that for $\mu \in \mathfrak{c}$, the set $\Xi^* \cap (\mu + \Box \overline{a})$ does not depend on the choice of μ , and thus equals $\Xi^* \cap (\mathfrak{c} + \Box \overline{a})$ for any $\mu \in \mathfrak{c}$.)

Theorem 4.1.9 (Szenes [59]). Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series $V_{\alpha}(\lambda, \ell)$. Assume that the elements $\overline{\alpha} = (\overline{\alpha}_1, ..., \overline{\alpha}_n)$ span \mathfrak{t}^* and are minimal in Λ^* . Let f be the meromorphic function on $W = \mathfrak{t} \otimes \mathbb{C}$ defined by

$$f(X) = \frac{1}{\prod_{\alpha} 1 - e^{2\pi i \alpha(X)}}$$

Let $\ell \in \mathbb{N}$ and $\lambda \in \ell \mathfrak{c} - \Box \alpha$. Then

$$V_{\boldsymbol{\alpha}}(\lambda,\ell) = \sum_{p \in vx(\mathcal{A}/\Lambda)} \sum_{\boldsymbol{a} \in B_p} \operatorname{iCT}_{\boldsymbol{a}} \left(T_{\mathfrak{c},\boldsymbol{a}}(\lambda,\ell) f \right).$$
(4.4)

Example 1. Let $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}$ with pairing given by multiplication, and let $\mathbb{Z} = \Lambda$.

(i) Take $\Xi = \mathbb{Z}$ and $\alpha_1 = \{1\}$. Then $\mathcal{H} = \Xi^* = \mathbb{Z}$ and $T_{\ell} = \ell^{-1}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}_{\ell}$. The open interval $\mathfrak{c} = (-1, 0)$ is a chamber of \mathcal{H} . Applying (4.4) to $V_1 := V(\alpha_1)$ gives

$$V_1(\lambda,\ell) = -\frac{\ell}{2} - \lambda - \frac{1}{2}, \qquad \lambda \in (-\ell - 1, 0) \cap \mathbb{Z}$$

(ii) Take $\Xi = \frac{1}{2}\mathbb{Z}$ and $\alpha_2 = \{1, -1\}$. Then $\mathcal{H} = \Xi^* = 2\mathbb{Z}$ and $T_{\ell} = (2\ell)^{-1}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}_{2\ell}$. The open interval $\mathfrak{c} = (0, 2)$ is a chamber of \mathcal{H} . Applying (4.4) to $V_2 := V(\alpha_2)$ gives

$$V_2(\lambda, \ell) = \frac{1}{3}\ell^2 - \ell\lambda + \frac{1}{2}\lambda^2 - \frac{1}{12}, \qquad \lambda \in (-1, 2\ell + 1) \cap \mathbb{Z}.$$

(iii) Take $\Xi = \frac{1}{2}\mathbb{Z}$ and $\alpha_3 = \{-1, a\}$ where $a(x) = -x + \frac{1}{2}$. As above, $\mathcal{H} = \Xi^* = 2\mathbb{Z}$ and $T_{\ell} = (2\ell)^{-1}\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}_{2\ell}$. The open interval $\mathfrak{c} = (0, 2)$ is a chamber of \mathcal{H} . In this case the set of vertices (up to translation by Λ) is $vx(\mathcal{A}/\Lambda) = \{0, \frac{1}{2}\}$, and thus Szenes' formula for $V_3 := V(\alpha_3)$ has two contributions. The contribution from the vertex 0 is

$$-\tfrac{\ell}{2} + \tfrac{\lambda}{2} - \tfrac{1}{2},$$

while for the vertex $\frac{1}{2}$ we obtain

$$(-1)^{\lambda} \left(-\frac{\ell}{2} + \frac{\lambda}{2} - \frac{1}{2} \right).$$

The result is

$$V_3(\lambda,\ell) = (-\ell + \lambda - 1)\delta_{2\mathbb{Z}}(\lambda), \qquad \lambda \in (0, 2\ell + 2) \cap \mathbb{Z}.$$

We now explain why Theorem 4.1.5 follows from Theorem 4.1.9. The restrictions in the statement of the result are relatively minor. The condition that the $\overline{\alpha}$ are minimal elements of Λ^* can always be achieved by factoring:

$$(1-z^m) = \prod_{\zeta^m=1} (1-\zeta z)$$

In case the $\overline{\alpha}$ do not span \mathfrak{t}^* , Proposition 4.1.15 (and Remark 4.1.16, see below) show that the support of V_{α} is contained in the collection of affine subspaces \mathcal{H} , which has codimension at least 1, and so Theorem 4.1.5 follows in this case (with the quasi-polynomial being 0). The non-trivial case is when the $\overline{\alpha}$ span \mathfrak{t}^* , which is the case considered in Szenes' Theorem.

In order to compute the constant term functional (using for example (4.3)), one would first change variables, setting X' = X - p. In terms of the new variables, $T_{\mathfrak{c},\mathfrak{a}}(\lambda,\ell)f$ becomes

$$\frac{1}{\operatorname{vol}_{\Xi^*}(\boldsymbol{a})} \sum_{\boldsymbol{\xi}^* \in \Xi^* \cap (\mathfrak{c} + \Box \overline{\boldsymbol{a}})} e^{-2\pi i \langle \lambda, p \rangle + 2\pi i \ell \langle \boldsymbol{\xi}^*, p \rangle} \Biggl(e^{-2\pi i \langle \lambda, X' \rangle} e^{2\pi i \ell \langle \boldsymbol{\xi}^*, X' \rangle} \prod_{\boldsymbol{a}} \frac{2\pi i \ell \langle \overline{\boldsymbol{a}}, X' \rangle}{1 - u_a^{-\ell} e^{2\pi i \ell \langle \overline{\boldsymbol{a}}, X' \rangle}} f(X' + p) \Biggr).$$

Note that because $a(0) \in \mathbb{Q}$ by assumption, the phase $u_a^{-\ell} = e^{2\pi i \ell a(0)}$ is *periodic in* ℓ . It is enough to check quasi-polynomial behavior by checking on arithmetic subsequences of \mathbb{N} , and so without loss of generality, we can assume $u_a^{-\ell} = u$ is a constant.

The meromorphic function $T_{\mathbf{c},\mathbf{a}}(\lambda,\ell)(X')$ is in fact holomorphic on a neighbourhood of X' = 0 (a possible zero of the denominator $1 - ue^{2\pi i \ell \langle \overline{a}, X' \rangle}$ is cancelled by the zero in the numerator $2\pi i \ell \langle \overline{a}, X' \rangle$). Thus when

$$e^{-2\pi i \langle \lambda, X' \rangle} e^{2\pi i \ell \langle \xi^*, X' \rangle} \prod_{a} \frac{2\pi i \ell \langle \overline{a}, X' \rangle}{1 - u e^{2\pi i \ell \langle \overline{a}, X' \rangle}}$$

is expanded in power series about X' = 0, only positive powers of λ and ℓ will appear. Applying the constant term functional (using e.g. (4.3)) to the expression in the brackets thus results in an expression which is *polynomial* in λ, ℓ . The only other dependence on λ, ℓ is from the exponential expressions, $e^{-2\pi i \langle \lambda, p \rangle + 2\pi i \ell \langle \xi^*, p \rangle}$, which are quasi-polynomial.

4.1.3 Deleting a hyperplane.

In this subsection we describe the result of applying a finite-difference operator to V_{α} . The result is a sum of two less complicated Verlinde series. This will play a role in the proof by induction of the decomposition formula in a later subsection. Similar inductive arguments can be found in [15], [58], [59].

For an affine linear function $\alpha = \overline{\alpha} + \alpha(0)$, $b_{\alpha} = e^{2\pi i \alpha(0)}$, define the finite difference operator:

$$\nabla_{\alpha} f(\lambda) = f(\lambda) - b_{\alpha} f(\lambda - \overline{\alpha}).$$

This has the effect of multiplying $\check{f}(\xi)$ (the inverse Fourier transform) by $(1 - e^{2\pi i \alpha(\xi)})$.

Definition 4.1.10. Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be data for a Verlinde series, and let $\alpha \in \alpha$. The data for the Verlinde series $V_{\alpha \setminus \alpha}$ is $(\mathfrak{t}, \Xi, \Lambda, \alpha \setminus \alpha)$, where $\alpha \setminus \alpha$ denotes the list obtained by removing one copy of α from α .

If α contains exactly one affine linear function α vanishing identically on $H_{\alpha} = \{\alpha = 0\} \subset \mathfrak{t}$, then the sums $\nabla_{\alpha} V_{\alpha}$ and $V_{\alpha \setminus \alpha}$ differ by a sum over regular lattice points (modulo Λ) in the affine hyperplane H_{α} . The latter sum is a shift (by some $\xi' \in \Xi \cap H_{\alpha}$) of a lower-dimensional Verlinde series for the vector space $H_{\overline{\alpha}}$. More precisely, we have the following 'deletion formula'.

Proposition 4.1.11. Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series V_{α} , let $\alpha \in \alpha$ and set $H = H_{\overline{\alpha}}$ (a vector subspace). Let $\pi : \Lambda^* \to \Lambda^*/\langle \overline{\alpha} \rangle$. Then

$$\nabla_{\alpha} V_{\alpha} = V_{\alpha \setminus \alpha} - \mathbf{1}_{p \mathbb{N}} e(-\xi') \pi^* V_{\alpha'}, \qquad (4.5)$$

where α' , $p \in \mathbb{N}$ are defined as follows. If $\ell^{-1}\Xi \cap H_{\alpha} = \emptyset$ for all ℓ , then set $\alpha' = \emptyset$. Otherwise let $p \in \mathbb{N}$ be minimal such that $p^{-1}\Xi \cap H_{\alpha} \neq \emptyset$, and choose $\xi' \in p^{-1}\Xi \cap H_{\alpha}$. Define $V_{\alpha'}$ to be the Verlinde series associated to the 4-tuple

$$(H, \Xi \cap H, \Lambda \cap H, \alpha')$$

where

$$\boldsymbol{\alpha}' = \{\beta|_H + \langle \overline{\beta}, \xi' \rangle | \beta \in \boldsymbol{\alpha} \setminus \alpha\}$$

- Remark 4.1.12. 1. The shift of the constant parts by $\langle \overline{\beta}, \xi' \rangle$ is to compensate for restricting to $H = H_{\overline{\alpha}}$ instead of H_{α} .
 - 2. $V_{\alpha'}$ depends on the choice of $\xi' \in \Xi \cap H_{\alpha}$, but the product $\mathbf{1}_{p\mathbb{N}}e(-\xi')\pi^*V_{\alpha'}$ does not.
 - 3. $\ell^{-1}\Xi \cap H_{\alpha} \neq \emptyset$ if and only if $\ell \in p\mathbb{N}$.
 - 4. If $\boldsymbol{\alpha}$ contains a second affine linear function β vanishing identically on H_{α} , then $\beta|_{H} + \langle \overline{\beta}, \xi' \rangle$ is identically zero, and thus $V_{\boldsymbol{\alpha}'} = 0$ (no regular points to sum over at all).

4.1.4 Partition functions.

Let $\boldsymbol{\alpha}$ be a list of affine linear functions on \mathfrak{t} . Fix a vector $\gamma^+ \in \mathfrak{t}$ such that $\langle \overline{\alpha}, \gamma^+ \rangle \neq 0$ for each $\alpha \in \boldsymbol{\alpha}$, and let $\alpha^{\pm} = \pm \mu_{\alpha} \alpha$, where $\mu_{\alpha} = \pm 1$ is chosen such that $\mu_{\alpha} \langle \overline{\alpha}, \gamma^+ \rangle > 0$. Let $\boldsymbol{\alpha}^-$ be the sublist of $\boldsymbol{\alpha}$ for which $\mu_{\alpha} = -1$. We define $\check{P}(\boldsymbol{\alpha}; \gamma^+)$ to be given by the expression

$$\check{P}(\boldsymbol{\alpha};\gamma^{+}) = \left(\prod_{\boldsymbol{\alpha}^{-}} -e_{-\boldsymbol{\alpha}}\right) \prod_{\boldsymbol{\alpha}} \sum_{n \ge 0} e_{n\boldsymbol{\alpha}^{+}},$$

where the series converges in the sense of generalized functions, hence $\check{P}(\boldsymbol{\alpha};\gamma^+)$ can be thought of as a periodic generalized function on \mathfrak{t} . It can be thought of as the 'formal expansion' of the function

$$\prod_{\alpha} \frac{1}{1 - e_{\alpha}}$$

in the direction γ^+ . An equivalent definition (c.f. [23]) is as the periodic generalized function

$$\check{P}(\boldsymbol{\alpha};\gamma^{+})(X) = \lim_{\epsilon \to 0^{+}} \prod_{\boldsymbol{\alpha}} \frac{1}{1 - e_{\boldsymbol{\alpha}}(X + i\epsilon\gamma^{+})}$$

(meaning: take the limit $\epsilon \to 0^+$ only after pairing with a test measure).

Definition 4.1.13. The Fourier transform of $\check{P}(\alpha; \gamma^+)$ is a generalized partition function

$$P(\boldsymbol{\alpha}; \gamma^+) : \Lambda^* \to \mathbb{C}.$$

An alternate description is as follows (c.f. [23]). The elements $\overline{\alpha}_1^+, ..., \overline{\alpha}_n^+ \in \overline{\alpha}$ define a map

$$r: \mathbb{R}^n \to \mathfrak{t}^*, \qquad (a_1, ..., a_n) \mapsto \sum a_k \overline{\alpha}_k^+.$$

Let $T_{\sigma}: \mathfrak{t}^* \to \mathfrak{t}^*$ be translation by $\sigma = \sum_{\alpha^-} -\overline{\alpha}$, and let u be the phase factor $\prod_{\alpha^-} (-e^{-2\pi i\alpha(0)})$. Define a distribution φ on \mathbb{R}^n by

$$\varphi = \sum_{m \in \mathbb{Z}_+^n} b_{\alpha_1}^{m_1} \cdots b_{\alpha_n}^{m_n} \delta_m$$

where $\mathbb{Z}_{+}^{n} = \{(m_{1}, ..., m_{n}) \in \mathbb{Z}^{n} | m_{k} \geq 0\}$. Then

$$P(\boldsymbol{\alpha};\gamma^{+}) = uT_{\sigma}(r_{*}\varphi). \tag{4.6}$$

Since the finite-difference operator ∇_{α} has the effect of multiplying the inverse Fourier transform by $(1 - e_{\alpha})$, we see that for $\alpha \in \alpha$

$$\nabla_{\alpha} P(\boldsymbol{\alpha}; \gamma^{+}) = P(\boldsymbol{\alpha} - \boldsymbol{\alpha}; \gamma^{+}).$$
(4.7)

Another simple property is that multiplying $P(\alpha; \gamma^+)$ by an exponential has the effect of shifting the constant parts of the affine linear functions $\alpha \in \alpha$:

$$e(\xi')P(\boldsymbol{\alpha};\gamma^+) = P(\tilde{\boldsymbol{\alpha}};\gamma^+), \qquad \tilde{\boldsymbol{\alpha}} = \{\alpha + \langle \overline{\alpha},\xi' \rangle | \alpha \in \boldsymbol{\alpha}\}.$$
(4.8)

Note that $P(\boldsymbol{\alpha}; \gamma^+)$ is supported in an acute cone contained in the half-space $\{\gamma^+ \ge 0\} \subset \mathfrak{t}^*$ (we say that $P(\boldsymbol{\alpha}; \gamma^+)$ is γ^+ -polarized). Suppose $0 \neq \beta \in \Lambda^*$, set $H = H_\beta$, and let

$$\pi:\Lambda^* o \Lambda^*/\langleeta
angle\simeq (\Lambda\cap H)^*$$

be the quotient map (we have used the natural identification $H^* \simeq \mathfrak{t}^*/\langle\beta\rangle$). Note that if $\langle\beta,\gamma^+\rangle = 0$, then (1) γ^+ descends to a linear function on $H^* = \mathfrak{t}^*/\langle\beta\rangle$, and (2) π is proper on the support of $P(\alpha;\gamma^+)$, so the push-forward $\pi_*P(\alpha;\gamma^+)$ is defined. Moreover we have

$$\pi_* P(\boldsymbol{\alpha}; \gamma^+) = P(\boldsymbol{\alpha}|_H; \gamma^+), \qquad \boldsymbol{\alpha}|_H = \{\alpha|_H | \alpha \in \boldsymbol{\alpha}\} = \{\pi(\overline{\alpha}) + \alpha(0) | \alpha \in \boldsymbol{\alpha}\}.$$
(4.9)

This follows from equation (4.6), since the map r_* for the elements $\pi(\overline{\alpha}^+)$ factors through π_* .

4.1.5 The decomposition formula.

Choose an inner product on \mathfrak{t} , which we use to identify $\mathfrak{t} \simeq \mathfrak{t}^*$. If $\gamma \in \mathfrak{t}^*$ and Δ is an affine subspace in \mathfrak{t}^* , let $\operatorname{pr}_{\Delta}(\gamma)$ denote the orthogonal projection of γ onto Δ , and set $\gamma_{\Delta}^+ = \operatorname{pr}_{\Delta}(\gamma) - \gamma$.

Definition 4.1.14. Let $S \subset \mathfrak{t}^*$ be a rational subspace (S is cut out by elements of Λ). Let T_S be the subtorus $\exp(\operatorname{ann}(S)) \subset T = \mathfrak{t}/\Lambda$. Define

$$\boldsymbol{\alpha}_S = \{ \alpha \in \boldsymbol{\alpha} | \overline{\alpha} \in S \}, \qquad \boldsymbol{\alpha}_S^{\perp} = \boldsymbol{\alpha} \setminus \boldsymbol{\alpha}_S.$$

More generally, if Δ is a translate of S, we write $\overline{\Delta} = S$ and define

$$\alpha_{\Delta} = \alpha_S, \qquad \alpha_{\Delta}^{\perp} = \alpha_S^{\perp}$$

The translates $\Delta = \xi_{\Delta} + S$ of S by elements $\xi_{\Delta} \in \Xi^*$, are parametrized by Ξ^*/S .

Proposition 4.1.15. The Verlinde series $V(\alpha_S)$ is supported on the collection of sets of the form

$$\cup_{\ell \in \mathbb{N}} \ell \Delta \times \{\ell\} = \iota^{\Delta}(S \times \mathbb{N}), \tag{4.10}$$

where $\Delta \in \Xi^*/S$ runs over all translates of S, and $\iota^{\Delta} : S \times \mathbb{N} \hookrightarrow \mathfrak{t}^* \times \mathbb{N}$ is the ('shear') embedding

$$\iota^{\Delta}(\lambda,\ell) = (\lambda + \ell\xi_{\Delta},\ell),$$

(the map depends on the choice of $\xi_{\Delta} \in \Delta \cap \Xi^*$). Define $V(\boldsymbol{\alpha}_{\Delta},;\Delta)$ to be the restriction of $V(\boldsymbol{\alpha}_S)$ to (4.10), hence

$$V(\boldsymbol{\alpha}_S) = \sum_{\Delta \in \Xi^*/S} V(\boldsymbol{\alpha}_{\Delta}; \Delta).$$
(4.11)

Let $V_{\alpha,S}$ be the (lower-dimensional) Verlinde series defined by the 4-tuple

$$(\mathfrak{t}/\operatorname{ann}(S), q(\Xi), q(\Lambda), \boldsymbol{\alpha}_S), \qquad q: \mathfrak{t} \to \mathfrak{t}/\operatorname{ann}(S).$$

Then

$$V(\boldsymbol{\alpha}_{\Delta}; \Delta) = |T_S \cap T_\ell| \, \iota_*^{\Delta} V_{\boldsymbol{\alpha}, S}.$$

Proof. Fix $\ell \in \mathbb{N}$. Let $U = \operatorname{ann}(S) \subset \mathfrak{t}$, and note that $(\mathfrak{t}/U)^* \simeq S$ naturally. We have a short exact sequence of finite groups

$$1 \to T_S \cap T_\ell = (\ell^{-1}\Xi \cap U)/(\Lambda \cap U) \to \ell^{-1}\Xi/\Lambda \to q(\ell^{-1}\Xi)/q(\Lambda) \to 1$$

Choose representatives $\xi_q \in \ell^{-1}\Xi/\Lambda$ for the elements of $q(\ell^{-1}\Xi)/q(\Lambda)$. The affine linear functions α_S are constant on translates of U. Hence $\alpha(\xi_q + \xi) = \alpha(\xi_q)$ for $\xi \in \ell^{-1}\Xi \cap U$, $\alpha \in \alpha_S$. This means in particular that whether $\alpha(\xi_q + \xi)$ is an integer (i.e. whether $\xi_q + \xi$ is regular) depends only on ξ_q , not

on $\xi \in \ell^{-1}\Xi \cap U$. Thus we can split up the sum over regular elements of $\ell^{-1}\Xi/\Lambda$ into an iterated sum

$$V(\boldsymbol{\alpha}_S)(\lambda,\ell) = \sum_{\xi_q}' \frac{e^{-2\pi i \langle \lambda,\xi_q \rangle}}{\prod_{\boldsymbol{\alpha}_S} 1 - e^{2\pi i \alpha(\xi_q)}} \sum_{\xi \in (\ell^{-1} \Xi \cap U)/(\Lambda \cap U)} e^{-2\pi i \langle \lambda,\xi \rangle}.$$

The inner sum is

$$\sum_{\xi \in (\ell^{-1} \Xi \cap U)/(\Lambda \cap U)} e^{-2\pi i \langle \lambda, \xi \rangle} = \begin{cases} |T_S \cap T_\ell| & \text{if } \lambda \in \ell \Xi^* + S \\ 0 & \text{else,} \end{cases}$$

since $T_S \cap T_\ell = (\ell^{-1}\Xi \cap U)/(\Lambda \cap U)$ and $(\ell^{-1}\Xi)^* = \ell\Xi^*$. This shows that $V(\boldsymbol{\alpha}_S)(-,\ell)$ is supported on $\Lambda^* \cap (\ell\Xi^* + S)$, which is the set of translates $\ell\xi_\Delta + S$ of S by elements of $\ell\Xi^*$. If $\lambda = \lambda' + \ell\xi_\Delta$, $\lambda' \in \Lambda^* \cap S$ is in the support, then $e^{-2\pi i \langle \ell\xi_\Delta, \xi_q \rangle} = 1$ hence the outer sum is $V_{\boldsymbol{\alpha},S}(\lambda',\ell)$.

Remark 4.1.16. A corollary of this result is that if $S = \operatorname{span}(\overline{\alpha})$ then V_{α} is supported on the Ξ^* -translates of S.

Recall that for each chamber of a hyperplane arrangement in $S \simeq (\mathfrak{t}/\operatorname{ann}(S))^*$, there is a quasipolynomial function agreeing with $V_{\alpha,S}$ on some open cone in $S \times \mathbb{N}$ (Theorem 4.1.5 and Remark 4.1.6). Since $|T_{\ell} \cap T_S| = \ell^{\dim(T_S)} |T_1 \cap T_S|$ is polynomial, it follows that $V(\alpha_{\Delta}; \Delta)$ agrees with a quasi-polynomial function on certain cones in $\iota^{\Delta}(S \times \mathbb{N}) = \bigcup_{\ell} \ell \Delta \times \{\ell\}$.

Definition 4.1.17. If $\gamma \in \mathfrak{t}^*$ is generic, then $(\mathrm{pr}_{\Delta}(\gamma), 1)$ lies in the interior of one of these cones and thus selects a quasi-polynomial function

$$\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) : \bigcup_{\ell} (\Lambda^* \cap \ell \Delta) \times \{\ell\} \to \mathbb{C},$$

which agrees with $V(\boldsymbol{\alpha}_{\Delta}; \Delta)$ on the cone containing $(\mathrm{pr}_{\Delta}(\gamma), 1)$. We will refer to $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \mathrm{pr}_{\Delta}(\gamma))$ as the quasi-polynomial germ of $V(\boldsymbol{\alpha}_{\Delta}; \Delta)$ at $(\mathrm{pr}_{\Delta}(\gamma), 1)$.

Definition 4.1.18. Let $S = S(\alpha, \Xi^*)$ denote the collection of all affine subspaces in \mathfrak{t}^* obtained by taking Ξ^* -translates of subspaces spanned by subsets of $\overline{\alpha}$. A subspace $\Delta \in S$ is called *admissible*. (This terminology is used in [15].)

Note that the points of Ξ^* are amongst the admissible subspaces. If $\overline{\alpha}$ spans \mathfrak{t}^* , then $\mathfrak{t}^* \in \mathcal{S}$.

Lemma 4.1.19. Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series, and let $\Delta \in S$. Suppose $\alpha \in \alpha$ has linear part $\overline{\alpha} \in \overline{\Delta}$. Let $H = H_{\overline{\alpha}}$ and let $\pi : \Lambda^* \to \Lambda^* / \langle \overline{\alpha} \rangle$ be the quotient map. Then

$$\nabla_{\alpha} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) = \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta} \setminus \alpha; \operatorname{pr}_{\Delta}(\gamma)) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^* \operatorname{Ver}(\boldsymbol{\alpha}_{\pi(\Delta)}'; \pi(\operatorname{pr}_{\Delta}(\gamma))),$$

with α', ξ', p defined as in Proposition 4.1.11.

Proof. Assume $\xi' \in p^{-1}\Xi^* \cap H_{\alpha}$, and let $S = \overline{\Delta}$. Using the deletion formula (4.5),

$$\nabla_{\alpha} V(\boldsymbol{\alpha}_{S}) = V(\boldsymbol{\alpha}_{S} \setminus \alpha) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^{*} V((\boldsymbol{\alpha}_{S})'),$$

where $V((\boldsymbol{\alpha}_S)')$ is the Verlinde series defined by the 4-tuple

$$(H, \Xi \cap H, \Lambda \cap H, (\boldsymbol{\alpha}_S)'), \qquad (\boldsymbol{\alpha}_S)' = \{\beta|_H + \langle \overline{\beta}, \xi' \rangle | \beta \in \boldsymbol{\alpha}_S \setminus \boldsymbol{\alpha}\}.$$

Since $\overline{\alpha} \in S$, $\overline{\beta} \in S$ if and only if $\pi(\overline{\beta}) \in \pi(S)$. It follows that

$$(\boldsymbol{\alpha}_S)' = (\boldsymbol{\alpha}')_{\pi(S)} =: \boldsymbol{\alpha}'_{\pi(S)}, \qquad \boldsymbol{\alpha}' = \{\beta|_H + \langle \overline{\beta}, \xi' \rangle | \beta \in \boldsymbol{\alpha} \setminus \boldsymbol{\alpha} \}$$

Restricting to $\cup_{\ell} \ell \Delta \times {\ell}$, the left side becomes $\nabla_{\alpha} V(\boldsymbol{\alpha}_{\Delta}; \Delta)$, and the Verlinde series on the right side become $V(\boldsymbol{\alpha}_{\Delta} \setminus \alpha; \Delta)$ and $V(\boldsymbol{\alpha}'_{\pi(\Delta)}; \pi(\Delta))$ respectively. Thus

$$\nabla_{\alpha} V(\boldsymbol{\alpha}_{\Delta}; \Delta) = V(\boldsymbol{\alpha}_{\Delta} \setminus \alpha; \Delta) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^* V(\boldsymbol{\alpha}_{\pi(\Delta)}'; \pi(\Delta)), \qquad (4.12)$$

Since the two sides of equation (4.12) agree in particular on the open cone in $\cup_{\ell} \ell \Delta \times \{\ell\}$ containing $(\mathrm{pr}_{\Delta}(\gamma), 1)$ used to define $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \mathrm{pr}_{\Delta}(\gamma))$, the corresponding quasi-polynomial functions agree on $\cup_{\ell} \ell \Delta \times \{\ell\}$.

We can now state the decomposition formula for Verlinde series. This is an unpublished result of A. Boysal and M. Vergne (see [15] for the analogous formula for Bernoulli series).

Theorem 4.1.20. Let $(\mathfrak{t}, \Xi, \Lambda, \alpha)$ be a 4-tuple defining a Verlinde series, and let $\gamma \in \mathfrak{t}^*$ be generic. Then

$$V(\boldsymbol{\alpha}) = \sum_{\Delta \in \mathcal{S}} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+})$$

(This is a convolution of functions on Λ^* .)

The proof is by induction on the number of elements in the list $\boldsymbol{\alpha}$. For the base case $\boldsymbol{\alpha} = \emptyset$, $\mathcal{S} = \Xi^*$. For $\Delta \in \Xi^*$ (a single point), $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) = V(\boldsymbol{\alpha}_{\Delta}; \Delta)$, so the result follows from equation (4.11).

The inductive step is based on the deletion relation (4.5), which we recall here for convenience. Let $\alpha \in \alpha$, $H = H_{\overline{\alpha}}$ and let $\pi : \Lambda^* \to \Lambda^* / \langle \overline{\alpha} \rangle$ be the quotient map. Then

$$\nabla_{\alpha} V(\boldsymbol{\alpha}) = V(\boldsymbol{\alpha} \setminus \alpha) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^* V(\boldsymbol{\alpha}'), \qquad (4.13)$$

where $V(\alpha')$ and ξ' are defined as in Proposition 4.1.11: either $\alpha' = \emptyset$ (if $\ell^{-1}\Xi \cap H_{\alpha} = \emptyset$ for all ℓ) or there is a minimal $p \in \mathbb{N}, \xi' \in p^{-1}\Xi^* \cap H_{\alpha}$ and $V(\alpha')$ is the Verlinde series defined by the 4-tuple

$$(H, \Xi \cap H, \Lambda \cap H, \alpha'), \qquad \alpha' = \{\beta|_H + \langle \overline{\beta}, \xi' \rangle | \beta \in \alpha \setminus \alpha\}.$$
(4.14)

Now let $\tilde{V}(\boldsymbol{\alpha})$ denote the right-side of the Theorem. The argument splits into two parts: (1) proving $\nabla_{\alpha}(V(\boldsymbol{\alpha}) - \tilde{V}(\boldsymbol{\alpha})) = 0$ for each $\alpha \in \boldsymbol{\alpha}$ (Lemma 4.1.21), and (2) showing that $V(\boldsymbol{\alpha})$, $\tilde{V}(\boldsymbol{\alpha})$ agree on sufficiently many lattice points (Lemma 4.1.22).

In the proof of Lemma 4.1.21, we use the inductive hypothesis, as well as some basic properties of convolution:

1. $\nabla_{\alpha}(f * g) = (\nabla_{\alpha} f) * g = f * (\nabla_{\alpha} g),$

2.
$$e_{\xi}(f * g) = (e_{\xi}f) * (e_{\xi}g)$$

3.
$$(\pi^* f) * g = \pi^* (f * (\pi_* g)).$$

Lemma 4.1.21. For each $\alpha \in \boldsymbol{\alpha}$, $\nabla_{\alpha}(V(\boldsymbol{\alpha}) - \tilde{V}(\boldsymbol{\alpha})) = 0$.

Proof. The admissible subspaces \mathcal{S} can be split into two subsets

$$\mathcal{S}_{\alpha} = \{\Delta | \overline{\alpha} \in \overline{\Delta}\}, \qquad \mathcal{S}'_{\alpha} = \mathcal{S} \setminus \mathcal{S}_{\alpha}.$$

Then

$$\nabla_{\alpha} \tilde{V}(\boldsymbol{\alpha}) = \sum_{\mathcal{S}_{\alpha}} \nabla_{\alpha} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}) + \sum_{\mathcal{S}_{\alpha}'} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * \nabla_{\alpha} P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}).$$
(4.15)

Consider first the sum over $\Delta \in \mathcal{S}'_{\alpha}$. Notice that since $\overline{\alpha} \notin \overline{\Delta}$ (for $\Delta \in \mathcal{S}'_{\alpha}$), $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}; \operatorname{pr}_{\Delta}(\gamma))$ can be replaced with $\operatorname{Ver}((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}, \operatorname{pr}_{\Delta}(\gamma))$, without change. By equation (4.7),

$$\nabla_{\alpha} P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}) = P(\boldsymbol{\alpha}_{\Delta}^{\perp} \setminus \alpha; \gamma_{\Delta}^{+}) = P((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}^{\perp}; \gamma_{\Delta}^{+}).$$

Thus the sum over \mathcal{S}'_{α} becomes

$$\sum_{\mathcal{S}'_{\alpha}} \operatorname{Ver}((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * P((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}^{\perp}; \gamma_{\Delta}^{+}).$$
(4.16)

Applying Lemma 4.1.19 to the first sum in (4.15) we obtain

$$\sum_{\mathcal{S}_{\alpha}} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta} \setminus \alpha; \operatorname{pr}_{\Delta}(\gamma)) * P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^{*} \operatorname{Ver}(\boldsymbol{\alpha}_{\pi(\Delta)}'; \pi(\operatorname{pr}_{\Delta}(\gamma))) * P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+}).$$
(4.17)

Note that $\boldsymbol{\alpha}_{\Delta} \setminus \boldsymbol{\alpha} = (\boldsymbol{\alpha} \setminus \boldsymbol{\alpha})_{\Delta}$, and that for $\Delta \in S_{\alpha}$, $\boldsymbol{\alpha}_{\Delta}^{\perp} = (\boldsymbol{\alpha} \setminus \boldsymbol{\alpha})_{\Delta}^{\perp}$. Making these replacements in the first term of (4.17) and combining it with (4.16) we obtain

$$\sum_{\mathcal{S}(\boldsymbol{\alpha},\Xi^*)} \operatorname{Ver}((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * P((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}^{\perp}; \gamma_{\Delta}^+),$$
(4.18)

where we have used $\mathcal{S}_{\alpha} \cup \mathcal{S}'_{\alpha} = \mathcal{S} = \mathcal{S}(\alpha, \Xi^*).$

The inductive hypothesis can almost be applied to (4.18) (and the list $\boldsymbol{\alpha} \setminus \alpha$), except that the sum is over a larger set of affine subspaces $(\mathcal{S}(\boldsymbol{\alpha}, \Xi^*))$ instead of $\mathcal{S}(\boldsymbol{\alpha} \setminus \alpha, \Xi^*)$). If $\Delta \in \mathcal{S}(\boldsymbol{\alpha}, \Xi^*) \setminus \mathcal{S}(\boldsymbol{\alpha} \setminus \alpha, \Xi^*)$, this means that the elements of $(\overline{\boldsymbol{\alpha}} \setminus \overline{\alpha}) \cap \overline{\Delta}$ do not span $\overline{\Delta}$. Proposition 4.1.15 (and Remark 4.1.16) showed that if the linear parts of the affine linear functions do not span the vector space, then the Verlinde series is supported on a subset of codimension at least 1, and consequently the corresponding quasi-polynomial vanishes identically. We apply this to $V_{\boldsymbol{\alpha}\setminus\alpha,S}$ (see Proposition 4.1.15) where $S = \overline{\Delta}$; this equals $V((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}; \Delta)$ up to a (level-dependent) translation and constant multiple (Proposition 4.1.15), and thus $\operatorname{Ver}((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) = 0$ for $\Delta \in \mathcal{S}(\boldsymbol{\alpha}, \Xi^*) \setminus \mathcal{S}(\boldsymbol{\alpha} \setminus \alpha, \Xi^*)$. Dropping these terms from equation (4.18), we obtain

$$\sum_{\mathcal{S}(\boldsymbol{\alpha} \setminus \alpha, \Xi^*)} \operatorname{Ver}((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}; \operatorname{pr}_{\Delta}(\gamma)) * P((\boldsymbol{\alpha} \setminus \alpha)_{\Delta}^{\perp}; \gamma_{\Delta}^+)$$

This equals $V(\boldsymbol{\alpha} \setminus \alpha)$ by the induction hypothesis.

We still have to deal with the second term in (4.17). Applying two of the basic properties of convo-

lutions, it becomes

$$-\mathbf{1}_{p\mathbb{N}}e(-\xi')\pi^*\sum_{\mathcal{S}_{\alpha}}\operatorname{Ver}(\boldsymbol{\alpha}_{\pi(\Delta)}';\pi(\operatorname{pr}_{\Delta}(\gamma)))*\pi_*e(\xi')P(\boldsymbol{\alpha}_{\Delta}^{\perp};\gamma_{\Delta}^{\perp}).$$
(4.19)

Using (4.8), $e(\xi')$ shifts the constant terms of the affine linear functions:

$$e(\xi')P(\boldsymbol{\alpha}_{\Delta}^{\perp};\gamma_{\Delta}^{+}) = P(\tilde{\boldsymbol{\alpha}}_{\Delta}^{\perp};\gamma_{\Delta}^{+}), \qquad \tilde{\boldsymbol{\alpha}} = \{\beta + \langle \overline{\beta}, \xi' \rangle | \beta \in \boldsymbol{\alpha} \}.$$

Then using (4.9), the pushforward π_* simply restricts the affine linear functions to $H = H_{\overline{\alpha}}$:

$$\pi_* P(\tilde{\boldsymbol{\alpha}}_{\Delta}^{\perp}; \gamma_{\Delta}^+) = P((\tilde{\boldsymbol{\alpha}}_{\Delta}^{\perp})|_H; \pi(\gamma_{\Delta}^+)).$$

We have

$$(ilde{m{lpha}}_{\Delta}^{\perp})|_{H} = \{eta|_{H} + \langle \overline{m{eta}}, m{\xi}'
angle | eta \in m{lpha}_{\Delta}^{\perp} \} = (m{lpha}')_{\pi(\Delta)}^{\perp}.$$

where we have used the fact that $\overline{\alpha} \in \overline{\Delta}$ in the second equality, and the definition of α' (equation (4.14)). Equation (4.19) becomes

$$-\mathbf{1}_{p\mathbb{N}}e(-\xi')\pi^*\sum_{\mathcal{S}_{\alpha}}\operatorname{Ver}(\boldsymbol{\alpha}_{\pi(\Delta)}';\pi(\operatorname{pr}_{\Delta}(\gamma)))*P((\boldsymbol{\alpha}')_{\pi(\Delta)}^{\perp};\pi(\gamma_{\Delta}^{+})).$$
(4.20)

The affine subspaces $\Delta \in S_{\alpha}$ (that is, those containing the direction $\overline{\alpha}$) are in one-one correspondence with the admissible subspaces for α' , via the quotient map $\pi : \mathfrak{t}^* \to \mathfrak{t}^*/\langle \overline{\alpha} \rangle$. Thus (4.20) can be written as a sum over the admissible subspaces $\pi(\Delta)$ for α' . Applying the inductive hypothesis to α' , equation (4.20) becomes

$$-\mathbf{1}_{p\mathbb{N}}e(-\xi')\pi^*V(\boldsymbol{\alpha}')$$

Combining this with the other term, we have shown that

$$\nabla_{\alpha} \tilde{V}(\boldsymbol{\alpha}) = V(\boldsymbol{\alpha} \setminus \alpha) - \mathbf{1}_{p\mathbb{N}} e(-\xi') \pi^* V(\boldsymbol{\alpha}') = \nabla_{\alpha} V(\boldsymbol{\alpha}).$$

To complete the proof, we must show that \tilde{V}_{α} and V_{α} agree on a sufficiently large set of lattice points. To streamline the discussion we work with fixed $\ell = 1$ (one then replaces Ξ with $\ell^{-1}\Xi$). Let $S = \operatorname{span}(\overline{\alpha})$ be the span of the linear parts of the affine linear functions. Proposition 4.1.15 showed that $V(\alpha)$ is supported on the Ξ^* -translates of S. This is also true for $\tilde{V}(\alpha)$ as all admissible subspaces are contained in translates of S (the translates of S are the top-dimensional admissible subspaces), and see equation (4.6) for the partition functions, for example. The \mathbb{Z} -linear combinations of the elements of $\overline{\alpha}$ generate a sublattice $\mathbb{Z}\overline{\alpha} \subset \Lambda^* \cap S$. By Lemma 4.1.21, it suffices to prove the following:

Lemma 4.1.22. $V(\alpha)$ and $\tilde{V}(\alpha)$ agree on a set of representatives for the $\mathbb{Z}\overline{\alpha}$ action on each Ξ^* -translate Δ' of S.

Proof. Since $\operatorname{pr}_{\Delta'}(\gamma)$ is generic in Δ' , the set

$$(\operatorname{pr}_{\Delta'}(\gamma) - \Box \boldsymbol{\alpha}) \cap \Lambda^*, \qquad \Box \boldsymbol{\alpha} = \left\{ \sum t_i \overline{\alpha}_i \Big| 0 < t_i < 1 \right\}$$

$$(4.21)$$

contains a set of representatives. Theorem 4.1.5 shows that $V(\boldsymbol{\alpha})$ and $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta'}; \operatorname{pr}_{\Delta'}(\gamma))$ agree on this set (we are applying Theorem 4.1.5 to $V_{S,\boldsymbol{\alpha}}$ from Proposition 4.1.15 here). Notice that $P(\boldsymbol{\alpha}_{\Delta'}^{\perp}; \gamma_{\Delta'}^{\perp}) = \delta_0$, hence the term $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta'}; \operatorname{pr}_{\Delta'}(\gamma)) * P(\boldsymbol{\alpha}_{\Delta'}^{\perp}; \gamma_{\Delta'}^{\perp})$ agrees with $V(\boldsymbol{\alpha})$ on the set $(\operatorname{pr}_{\Delta'}(\gamma) - \Box \boldsymbol{\alpha}) \cap \Lambda^*$.

On the other hand, the supports of all other terms in $\tilde{V}(\alpha)$ are outside the set $(\operatorname{pr}_{\Delta'}(\gamma) - \Box \alpha) \cap \Lambda^*$. To see this, consider an admissible subspace $\Delta \subsetneq \Delta'$ and let $\gamma^+ = \operatorname{pr}_{\Delta}(\gamma) - \operatorname{pr}_{\Delta'}(\gamma)$ (this is an outward normal vector for Δ viewed as an affine subspace of Δ'). Let $m : \Delta' \to \mathbb{R}$ be the function

$$m(\mu) = \langle \mu - \mathrm{pr}_{\Delta'}(\gamma), \gamma^+ \rangle.$$

We have

$$m|_{\mathrm{pr}_{\Delta'}(\gamma)-\Box(\boldsymbol{\alpha})} < \sum_{\langle \overline{\alpha}, \gamma^+ \rangle < 0} - \langle \overline{\alpha}, \gamma^+ \rangle.$$
(4.22)

From equation (4.6), recall that $P(\boldsymbol{\alpha}_{\Delta}^{\perp}; \gamma_{\Delta}^{+})$ has a translation outward by

$$\sigma = \sum_{(\pmb{\alpha}_{\Delta}^{\perp})^{-}} - \overline{\alpha}$$

where $(\boldsymbol{\alpha}_{\Delta}^{\perp})^{-}$ denotes the set of $\alpha \in \boldsymbol{\alpha}_{\Delta}^{\perp}$ such that $\langle \overline{\alpha}, \gamma_{\Delta}^{+} \rangle < 0$. Since $\overline{\alpha} \in S$ while $\gamma_{\Delta'}^{+}$ is orthogonal to S, we have $\langle \overline{\alpha}, \gamma_{\Delta'}^{+} \rangle = 0$ and thus

$$\langle \overline{\alpha}, \gamma_{\Delta}^+ \rangle = \langle \overline{\alpha}, \gamma^+ + \gamma_{\Delta'}^+ \rangle = \langle \overline{\alpha}, \gamma^+ \rangle.$$

Hence, thanks to the shift by σ we have

$$\operatorname{supp}\bigg(\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta};\operatorname{pr}_{\Delta}(\gamma))*P(\boldsymbol{\alpha}_{\Delta}^{\perp};\gamma_{\Delta}^{+})\bigg)\subset\bigg\{m\geq\sum_{\langle\overline{\alpha},\gamma^{+}\rangle<0}-\langle\overline{\alpha},\gamma^{+}\rangle\bigg\}.$$

Comparing with equation (4.22), this completes the argument.

Example 2. Let $\mathfrak{t} = \mathfrak{t}^* = \mathbb{R}$ with pairing given by multiplication, and let $\Lambda = \mathbb{Z}$. We give the decomposition formula for the examples 1.

(i) Take $\Xi = \mathbb{Z} = \Xi^*$ and $\alpha_1 = \{1\}$. The admissible subspaces Δ are \mathfrak{t}^* and $\{n\}$, $n \in \mathbb{Z}$. The decomposition formula for $V_1 = V(\alpha_1)$, $\alpha_1 = \{1\}$ about a center $\gamma \in (-1, 0)$ reads

$$V_1 = \left(-\frac{\ell}{2} - \lambda - \frac{1}{2}\right) + \sum_{n \ge 0} \ell \delta_{n\ell} * \mathbf{1}_{[0,\infty)} - \sum_{n < 0} \ell \delta_{n\ell} * \mathbf{1}_{(-\infty,-1]},$$

where $\delta_a(\lambda) = \delta(\lambda - a)$.

(ii) Take $\Xi = \frac{1}{2}\mathbb{Z} \Rightarrow \Xi^* = 2\mathbb{Z}$. The admissible subspaces Δ are \mathfrak{t}^* and $\{2n\}, n \in \mathbb{Z}$. Define functions

$$x_{+}(\lambda) = \begin{cases} \lambda, & \text{if } \lambda \ge 0\\ 0, & \text{else.} \end{cases} \qquad \qquad x_{-}(\lambda) = x_{+}(-\lambda)$$

The decomposition formula for $V_2 = V(\alpha_2)$, $\alpha_2 = \{1, -1\}$ about a center $\gamma \in (0, 1)$ is

$$V_2 = \left(\frac{1}{3}\ell^2 - \ell\lambda + \frac{1}{2}\lambda^2 - \frac{1}{12}\right) - \sum_{n>0} 2\ell\delta_{2n\ell} * x_+ - \sum_{n\le 0} 2\ell\delta_{2n\ell} * x_-.$$

(iii) Take $\Xi = \frac{1}{2}\mathbb{Z} \Rightarrow \Xi^* = 2\mathbb{Z}$. The admissible subspaces Δ are \mathfrak{t}^* and $\{2n\}, n \in \mathbb{Z}$. The decomposition formula for $V_3 = V(\boldsymbol{\alpha}_3), \, \boldsymbol{\alpha}_3 = \{-1, a\}, \, a(x) = -x + \frac{1}{2}$ about a center $\gamma \in (0, 1)$ is

$$V_3 = \left(-\ell + \lambda - 1 - \sum_{n>0} 2\ell \delta_{2n\ell} * \mathbf{1}_{[2,\infty)} + \sum_{n\le 0} 2\ell \delta_{2n\ell} * \mathbf{1}_{(-\infty,0]}\right) \delta_{2\mathbb{Z}}(\lambda).$$

4.2 Differential form-valued Verlinde series.

In this section we define Verlinde series and partition functions which are *differential form-valued*. These can be viewed as forms with generalized coefficients [33], although our use of the latter is only as a convenient way of expressing a version of the Boysal-Vergne decomposition formula involving both forms and Verlinde series.

4.2.1 Some characteristic classes.

To fix notation and conventions, we describe here the characteristic classes used in the paper (in terms of their Chern-Weil representatives). Helpful references include [34], [8], [10].

The eigenvalues of a transformation $A \in SO(2n)$ can be +1 or else come in complex conjugate pairs $e^{\pm i\theta}$ with $\theta \in [0, \pi]$, hence $\det_{\mathbb{R}}(1 - A) \ge 0$. For $R \in \mathfrak{so}(2n)$ and $A \in SO(2n)$, the function

$$D(R) = \det_{\mathbb{R}} \left(1 - Ae^{\frac{i}{2\pi}R} \right)$$

has a unique analytic square-root, denoted $\det_{\mathbb{R}}^{1/2} (1 - Ae^{\frac{i}{2\pi}R})$, such that $D(0)^{1/2}$ is the positive square-root of $\det_{\mathbb{R}}(1 - A)$.

Let $\nu \to F$ be a *G*-equivariant oriented real Euclidean vector bundle of even rank, equipped with a connection with curvature $R \in \Omega^2(F, \mathfrak{so}(\nu))$. Let $g \in G$ be such that $\nu^g = F$, so g defines an element $A(g) = A_{\nu}(g) \in SO(\nu)$. Define

$$\mathcal{D}_{\mathbb{R}}^{g}(\nu) = \mathcal{D}_{\mathbb{R}}(\nu, g) = i^{\mathrm{rk}_{\mathbb{R}}(\nu)/2} \mathrm{det}_{\mathbb{R}}^{1/2} \left(1 - A(g) e^{\frac{i}{2\pi}R} \right).$$

This can viewed as a power series in R with constant term $i^{\mathrm{rk}_{\mathbb{R}}(\nu)/2} \det_{\mathbb{R}}^{1/2}(1-A(g)) \neq 0$, and thus has a well-defined inverse. See also Remark 4.2.2 below for a description of this form in terms of the 'Chern roots'.

Let $\nu \to F$ be now a *G*-equivariant complex Hermitian vector bundle, equipped with a connection with curvature $R \in \Omega^2(F, \mathfrak{u}(\nu))$. Assume *g* preserves the complex structure, so *g* defines an element $A(g) \in U(\nu)$. Define

$$\mathcal{D}^{g}_{\mathbb{C}}(\nu) = \mathcal{D}_{\mathbb{C}}(\nu, g) = \det_{\mathbb{C}} \left(1 - A(g) e^{\frac{i}{2\pi}R} \right).$$

The twisted Chern character is the form

$$\operatorname{Ch}^{g}(\nu) = \operatorname{Ch}(\nu, g) = \operatorname{Tr}_{\mathbb{C}}\left(A(g)e^{\frac{i}{2\pi}R}\right).$$

For a complex line bundle L,

$$Ch(L,g) = A(g)e^{c_1(L)}, \qquad c_1(L) = \frac{i}{2\pi}R, \ A(g) \in U(1)$$

with $c_1(L)$ the first Chern class.

It is convenient to have slight generalizations $\operatorname{Ch}^{b}(\nu, g)$, $\mathcal{D}^{b}_{\mathbb{C}}(\nu, g)$ of these forms, involving an auxiliary unitary transformation $b \in U(\nu)$ commuting with A(g). The definition is as above but with A(g) replaced by bA(g).

For a unitary transformation A, we define $A^{1/2}$ to be the unique square-root with all eigenvalues in $\{e^{i\theta}|\theta\in[0,\pi)\}$, and $A^{-1/2}=(A^{1/2})^{-1}$.

Proposition 4.2.1. Let $\nu \to F$ be a *G*-equivariant complex Hermitian vector bundle with connection, and suppose $F = \nu^g$ for some $g \in G$. View ν as an oriented real Euclidean vector bundle, with the induced orientation and metric. We have the equality

$$\frac{1}{\mathcal{D}_{\mathbb{R}}(\nu,g)} = \frac{\operatorname{Ch}(\operatorname{det}(\nu),g)^{-1/2}}{\mathcal{D}_{\mathbb{C}}(\overline{\nu},g)},\tag{4.23}$$

where

Ch
$$(\det(\nu), g)^{-1/2} = \det_{\mathbb{C}}(A(g)^{-1/2})e^{-\frac{i}{4\pi}R},$$

and R is the curvature of $det(\nu)$.

Proof. This is an equality between analytic functions of the curvature R, and thus reduces to a computation for the vector space $\nu = \mathbb{C}^n$, a skew-Hermitian matrix $R \in \mathfrak{u}(n)$, and a unitary transformation $g \in U(n)$ that commutes with R. Then R, g have eigenvalues $R_k, e^{i\theta_k}, k = 1, 2, ..., n$ respectively, with $\theta_k \in [0, 2\pi)$. Acting on $\overline{\nu}$, the corresponding eigenvalues are $-R_k, e^{-i\theta_k}$. In terms of these

$$\mathcal{D}_{\mathbb{C}}(\overline{\nu},g) = \prod_{k} (1 - e^{-i\theta_k} e^{-\frac{i}{2\pi}R_k}).$$
(4.24)

View $\nu \simeq \mathbb{R}^{2n}$ as an oriented real vector space, so that R, g define elements of $\mathfrak{so}(2n)$, SO(2n) respectively. Then R, g have pairs of complex conjugate eigenvalues $\pm R_k$, $e^{\pm i\theta_k}$, k = 1, 2, ..., n respectively. To simplify notation, set $A_k = \theta_k + \frac{R_k}{2\pi}$. Thus

$$\mathcal{D}_{\mathbb{R}}(\nu, g) = i^n \left(\prod_k (1 - e^{iA_k})(1 - e^{-iA_k}) \right)^{1/2}$$
$$= i^n \prod_k (4\sin^2(A_k/2))^{1/2}.$$

To determine the sign of the square root, set $R_k = 0$. As $\theta_k \in [0, 2\pi)$, the positive square root of $\sin^2(\theta_k/2)$ is $\sin(\theta_k/2)$. The expression above becomes the product over k of $2i\sin(A_k/2)$, hence

$$\mathcal{D}_{\mathbb{R}}(\nu, g) = \prod_{k} e^{i\theta_{k}/2} e^{\frac{i}{4\pi}R_{k}} - e^{-i\theta_{k}/2} e^{-\frac{i}{4\pi}R_{k}}.$$
(4.25)

Comparing, we see that this differs from (4.24) by the factor

$$\prod_{k} e^{-i\theta_k/2} e^{-\frac{i}{4\pi}R_k}.$$
(4.26)

The action of $ge^{\frac{i}{2\pi}R}$ on the 1-dimensional complex vector space $det(\nu)$ is by the complex scalar

$$\prod_k e^{i\theta_k} e^{\frac{i}{2\pi}R_k}$$

Taking the square-root (using $\theta_k/2 \in [0, \pi)$) and then the inverse gives (4.26).

Remark 4.2.2. Equation (4.25) can be viewed as a formula for $\mathcal{D}_{\mathbb{R}}(\nu, g)$ in terms of the 'Chern roots'. More precisely, using the splitting principle one passes to the case that ν splits into a sum of *G*-equivariant oriented rank 2 sub-bundles ν_k , k = 1, ..., n. View ν_k as a complex line bundle (with complex structure compatible with the orientation), and first Chern class $x_k = c_1(\nu_k) = \frac{i}{2\pi}R_k$. Then equation (4.25) implies

$$\mathcal{D}_{\mathbb{R}}(\nu,g) = \prod_{k} e^{i\theta_k/2} e^{x_k/2} - e^{-i\theta_k/2} e^{-x_k/2}.$$

There is a similar interpretation of equations (4.24) and (4.26).

In later sections we use a few more characteristic classes. Let M be a Riemannian manifold, and R the curvature of TM, for some choice of compatible connection. The Å-class is the cohomology class of the differential form

$$\hat{A}(M) = \det_{\mathbb{R}}^{1/2} \left(j \left(\frac{i}{2\pi} R \right)^{-1} \right), \qquad j(x) = \frac{e^{x/2} - e^{-x/2}}{x}.$$

The square-root is chosen such that the constant term of the corresponding power series is positive. If M has an almost complex structure, so that TM is a complex vector bundle and the curvature R is $\mathfrak{u}(TM)$ -valued, then the Todd class is

$$\mathrm{Td}(M) = \mathrm{det}_{\mathbb{C}}\Big(\mathrm{td}\big(\tfrac{i}{2\pi}R\big)\Big), \qquad \mathrm{td}(x) = \frac{x}{1 - e^{-x}}.$$

If $E \to M$ is an oriented vector bundle with curvature R (for some choice of connection), then the Euler class is the cohomology class of the differential form

$$\operatorname{Eul}(E) = \operatorname{det}_{\mathbb{R}}^{1/2} \left(\frac{R}{2\pi} \right),$$

where here $\det_{\mathbb{R}}^{1/2}$ is the *Pfaffian* (with respect to the orientation of *E*). Let $\operatorname{Th}(E)$ be a Thom form for *E* (a closed differential form on the total space of *E* having compact support in the vertical direction and with integral over the fibres equal to 1) and $\iota: M \hookrightarrow E$ the zero section. Then $\operatorname{Eul}(E)$ and $\iota^*\operatorname{Th}(E)$ define the same cohomology class.

Remark 4.2.3. In terms of the 'Chern roots' (Remark 4.2.2)

$$\hat{A}(M) = \prod_{k} \frac{x_k}{e^{x_k/2} - e^{-x_k/2}}, \quad Td(M) = \prod_{k} \frac{x_k}{1 - e^{-x_k}}, \quad Eul(E) = \prod_k x_k.$$

4.2.2 Differential form-valued partition functions.

Let G = T be a torus and $\nu \to F$ a complex *T*-equivariant vector bundle over a connected base *F*. Assume $\nu^T = F$ and choose a vector $\gamma^+ \in \mathfrak{t}$ such that $\nu^{\gamma^+} = F$. The vector bundle ν decomposes into a direct sum of two sub-bundles ν_+ , ν_- where ν_+ (respectively ν_-) is the sub-bundle where γ^+ acts with positive weights (respectively negative weights). Define

$$\nu(\gamma^+) = \nu_+ \oplus \overline{\nu_-}$$

Thus γ^+ acts with positive weights on $\nu(\gamma^+)$.

Let $S\nu(\gamma^+)$ denote the complex symmetric algebra bundle, an infinite-dimensional graded vector bundle. Using the grading, we view the Chern character $\operatorname{Ch}(S\nu(\gamma^+), t)$ as an infinite sum of the Chern characters of the finite-dimensional fixed-degree sub-bundles. The rank of these sub-bundles grows like a polynomial in the degree, implying that the resulting infinite sum converges in the sense of (tempered) generalized functions on T, and hence $\operatorname{Ch}(S\nu(\gamma^+), t)$ defines a form with tempered generalized coefficients.

Definition 4.2.4. Define the $\Omega(F)$ -valued generalized function $\check{P}(\nu, b; \gamma^+)$ on T by the expression

$$\check{P}(\nu, b; \gamma^+)(t) = (-1)^{\operatorname{rk}(\nu_-)} \operatorname{Ch}^b(S\nu(\gamma^+) \otimes \det(\nu_-)^{-1}, t)$$

The corresponding $\Omega(F)$ -valued generalized partition function

$$P(\nu, b; \gamma^+) : \Lambda^* \to \Omega(F)$$

is its Fourier transform.

To see the relation between this definition and that of Section 4.1.4, consider first the special case where ν is a complex line bundle. Then the action of T on ν is determined by a weight $\overline{\alpha} \in \Lambda^*$, thus

$$\mathcal{D}^b_{\mathbb{C}}(\nu, t) = 1 - bt^{\overline{\alpha}} e^{\frac{i}{2\pi}R}$$

If $\langle \overline{\alpha}, \gamma^+ \rangle > 0$, then

$$\check{P}(\nu,b;\gamma^+)(t) = \sum_{n\geq 0} b^n t^{n\overline{\alpha}} e^{n\frac{i}{2\pi}R}$$

Otherwise $\langle \overline{\alpha}, \gamma^+ \rangle < 0$, and

$$\check{P}(\nu,b;\gamma^+)(t) = -b^{-1}t^{-\overline{\alpha}}e^{-\frac{i}{2\pi}R}\sum_{n\geq 0}b^{-n}t^{-n\overline{\alpha}}e^{-n\frac{i}{2\pi}R}.$$

We see that $\check{P}(\nu, b; \gamma^+)$ is the 'formal expansion' (in t) of

$$\frac{1}{\mathcal{D}^b_{\mathbb{C}}(\nu,t)} = \frac{1}{1 - bt^{\overline{\alpha}} e^{\frac{i}{2\pi}R}}.$$
(4.27)

in the direction γ^+ . Since both the Chern character and $\mathcal{D}^b_{\mathbb{C}}(\nu, t)$ are multiplicative, one sees more generally that if ν splits into a sum of complex line bundles, $\check{P}(\nu, b; \gamma^+)$ is the 'formal expansion' of $\mathcal{D}^b_{\mathbb{C}}(\nu, t)^{-1}$ in this sense.

Now suppose ν splits into a sum of complex line bundles $\nu_1, ..., \nu_m$. In the expansion of $\dot{P}(\nu, b; \gamma^+)$ in terms of the curvatures $R_1, ..., R_m$, the coefficients will be various partition functions of the kind introduced in Section 4.1.4. To determine precisely which partition functions appear, we Taylor expand $\mathcal{D}^b_{\mathbb{C}}(\nu, t)^{-1}$ in the curvatures *first*, and then take the 'formal expansion' (in t) in the sense above. The Taylor expansion about x = 0 of the function

$$f(v,x) = \frac{1}{1 - ve^x}$$

takes the form

$$\sum_{k\geq 0} \sum_{s=0}^{k} \frac{a_{ks}}{(1-v)(1-v^{-1})^s} x^k, \qquad a_{ks} \in \mathbb{C}$$

Applying this to $\mathcal{D}^b_{\mathbb{C}}(\nu, t)^{-1}$ we obtain

$$\frac{1}{\mathcal{D}^{b}_{\mathbb{C}}(\nu,t)} = \sum_{K \ge 0} \sum_{S=0}^{K} \frac{a_{KS}}{(1 - bt^{\overline{\alpha}})^{1} (1 - b^{-1}t^{-\overline{\alpha}})^{S}} \left(\frac{i}{2\pi}R\right)^{K},$$
(4.28)

where $K = (k_1, ..., k_m)$, $S = (s_1, ..., s_m)$ are multi-indices, $\mathbf{1} = (1, ..., 1)$, $a_{KS} = \prod_i a_{k_i s_i}$, $R^K = R_1^{k_1} \cdots R_m^{k_m}$, and $(1 - bt^{\overline{\alpha}})^S = \prod_i (1 - b_i t^{\overline{\alpha}_i})^{s_i}$.

Write $b_i t^{\overline{\alpha}_i}$ as t^{α_i} , where α_i is an affine linear function with linear part $\overline{\alpha}_i$. Let α be the list of the α_i . Taking the 'formal expansion' (in t) in the direction γ^+ , and then taking the Fourier transform yields

$$P(\nu, b; \gamma^{+}) = \sum_{K \ge 0} \sum_{S=0}^{K} a_{KS} P(\boldsymbol{\alpha}^{(S)}; \gamma^{+}) \left(\frac{i}{2\pi} R\right)^{K}, \qquad (4.29)$$

where

$$\boldsymbol{\alpha}^{(S)} = \boldsymbol{\alpha} \cup (-\boldsymbol{\alpha})^S$$

and $(-\alpha)^S$ denotes the list with s_i copies of $-\alpha_i$, i = 1, ..., m.

4.2.3 Decomposition formula.

Let $T = \mathfrak{t}/\Lambda$ be a torus, and $\Xi \supset \Lambda$ a lattice. Define the finite subgroups

$$T_{\ell} = \ell^{-1} \Xi / \Lambda.$$

Let $\nu \to F$ be a *T*-equivariant complex vector bundle over a compact connected base *F*, and suppose $\nu^T = F$. Let $b \in U(\nu)$ be an auxiliary endomorphism commuting with the *T*-action.

Definition 4.2.5. The Verlinde series associated to the data $(\mathfrak{t}, \Xi, \Lambda, \nu, b)$ is the function

$$V_{\nu,b} = V(\nu, b) : \Lambda^* \times \mathbb{N} \to \Omega(F),$$

defined by

$$V_{\nu,b}(\lambda,\ell) = \sum_{t\in T_{\ell}}' \frac{t^{-\lambda}}{\mathcal{D}^b_{\mathbb{C}}(\nu,t)}.$$

The prime next to the summation sign means omit terms from the sum such that $1 - bA(t) \in \text{End}(\nu)$ is

not invertible.

The set of weights of the *T*-action on ν and the lattice Ξ determine a set of admissible subspaces S in \mathfrak{t}^* . Let *S* be a rational subspace of \mathfrak{t}^* . Replace the bundle ν (respectively, endomorphism *b*) with the subbundle ν_{Δ} whose weights lie in *S* (respectively, $b|_{\nu_{\Delta}}$). Exactly as in equation (4.11), the formula

$$V(\nu_{\Delta}, b|_{\nu_{\Delta}}) = \sum_{\Delta \in \Xi^*/S} V(\nu_{\Delta}, b; \Delta)$$

defines the functions

$$V(\nu_{\Delta}, b; \Delta) : \Lambda^* \times \mathbb{N} \to \Omega(F).$$

This function is supported on $\cup_{\ell} \ell \Delta \times \{\ell\}$ and is quasi-polynomial on certain open cones in this set. To see this, assume first that the bundle ν splits into a sum of line bundles ν_i . Then Taylor expanding in the curvatures R_i shows that $V(\nu_{\Delta}, b; \Delta)$ is a finite linear combination of \mathbb{C} -valued Verlinde series, weighted by various monomials in the curvatures (we do this more explicitly below). The result is symmetric in the curvatures, and thus can be expressed in terms of the Chern classes, with coefficients that are linear combinations of \mathbb{C} -valued Verlinde series. The general result follows from the splitting principle.

Given a generic $\gamma \in \mathfrak{t}$, define

$$\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) : \Lambda^* \times \mathbb{N} \to \Omega(F)$$

to be the unique quasi-polynomial supported on $\cup_{\ell} \ell \Delta \times \{\ell\}$ agreeing with $V(\nu_{\Delta}, b; \Delta)$ on the open cone containing $(\operatorname{pr}_{\Delta}(\gamma), 1)$.

Let ν_{Δ}^{\perp} denote the invariant complement of the sub-bundle ν_{Δ} . The endomorphism *b* commutes with the *T*-action, and thus induces endomorphisms of ν_{Δ} , ν_{Δ}^{\perp} , which we continue to denote by *b*. The $\Omega(F)$ -valued partition function $P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+})$ was defined in the previous subsection. We can now state the version of the decomposition formula for differential form-valued Verlinde series.

Proposition 4.2.6. Let $(\mathfrak{t}, \Xi, \Lambda, \nu, b)$ data defining a differential form-valued Verlinde series, and $\gamma \in \mathfrak{t}^*$ a generic vector. Then

$$V(\nu, b) = \sum_{\Delta \in \mathcal{S}} \operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+}).$$
(4.30)

Proof. For fixed λ, ℓ , both sides are characteristic classes, thus by the splitting principle, it suffices to consider the case that ν splits into a sum of complex line bundles ν_i , i = 1, ..., m. Taylor expand $\mathcal{D}^b_{\mathbb{C}}(\nu, t)^{-1}$ in the curvatures as in (4.28),

$$\frac{1}{\mathcal{D}^b_{\mathbb{C}}(\nu,t)} = \sum_{K \ge 0} \sum_{S=0}^K \frac{a_{KS}}{(1-bt^{\overline{\alpha}})^{\mathbf{1}}(1-b^{-1}t^{-\overline{\alpha}})^S} \left(\frac{i}{2\pi}R\right)^K.$$

It follows that $V(\nu, b)$ is a finite sum

$$V(\nu, b) = \sum_{K,S} a_{KS} V(\boldsymbol{\alpha}^{(S)}) \left(\frac{i}{2\pi} R\right)^{K},$$

with notation as in (4.29). Now apply the decomposition formula to each $V(\boldsymbol{\alpha}^{(S)})$:

$$V(\nu, b) = \sum_{\Delta \in \mathcal{S}} \sum_{K,S} a_{KS} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}^{(S)}; \operatorname{pr}_{\Delta}(\gamma)) * P(\boldsymbol{\alpha}_{\Delta}^{(S)\perp}; \gamma_{\Delta}^{+}) \left(\frac{i}{2\pi}R\right)^{K}.$$

The subspace Δ determines a splitting $\nu = \nu_{\Delta} \oplus \nu_{\Delta}^{\perp}$ and a corresponding partition of the subbundles $\nu_1, ..., \nu_m$, and hence of the multi-indices $K = (K_1, K_2)$, $S = (S_1, S_2)$, i.e. K_1, S_1 (respectively K_2, S_2) are the multi-indices for the subbundles contained in ν_{Δ} (respectively ν_{Δ}^{\perp}). Note that $\operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}^{(S)}, \operatorname{pr}_{\Delta}(\gamma))$ only depends on the multi-index S_1 (in its definition, we remove all α_i with linear part parallel to Δ , and these correspond to the multi-index S_2). Similarly, $P(\boldsymbol{\alpha}_{\Delta}^{(S)\perp}, \gamma_{\Delta}^{+})$ only depends on the multi-index S_2 . Also $a_{KS} = a_{K_1S_1}a_{K_2S_2}$ (see (4.28)). Thus we can write

$$V(\nu,b) = \sum_{\Delta \in \mathcal{S}} \left(\sum_{K_1,S_1} a_{K_1S_1} \operatorname{Ver}(\boldsymbol{\alpha}_{\Delta}^{(S_1)}; \operatorname{pr}_{\Delta}(\gamma)) \left(\frac{i}{2\pi}R\right)^{K_1} \right) * \left(\sum_{K_2,S_2} a_{K_2S_2} P(\boldsymbol{\alpha}_{\Delta}^{(S_2)\perp}; \gamma_{\Delta}^+) \left(\frac{i}{2\pi}R\right)^{K_2} \right).$$

The expression in the first bracket is $\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma))$ and that in the second bracket is $P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{\perp})$ (see (4.29)).

Later on we will encounter functions which are not quite Verlinde series, but instead translations of Verlinde series. These will be of the form

$$T_{\ell\phi+\sigma}V(\nu,b),\tag{4.31}$$

where $\phi, \sigma \in \mathfrak{t}^*$ are independent of ℓ , and T_{ζ} denotes translation by ζ . (Explicitly $(T_{\ell\phi+\sigma}V)(\lambda,\ell) = V(\lambda - \ell\phi - \sigma, \ell)$.) Formula (4.31) makes sense for arbitrary $\phi, \sigma \in \mathfrak{t}^*$, as the definition of a function on the subset $\bigcup_{\ell} T_{\ell\phi+\sigma}(\Lambda^*) \times \{\ell\} \subset \mathfrak{t}^* \times \mathbb{N}$. For such translated Verlinde series, it is convenient to adapt the notation in the decomposition formula so as to absorb the linear part of the shift $\ell\phi$.

Remark 4.2.7. Later on we will only apply the formulas to a subset of $\ell \in \mathbb{N}$, namely the subset of $\ell = k + h^{\vee}$ where h^{\vee} is the dual Coxeter number of G, $k = nk_1$ with $n \in \mathbb{N}$, and $k_1 \in \mathbb{N}$ is smallest natural number such that the quasi-Hamiltonian space M under consideration is prequantizable at level k_1 . For such ℓ the combination $\ell \phi + \sigma$ will be in the lattice Λ^* again, hence (4.31) will define a function on a subset of $\Lambda^* \times \mathbb{N}$. However this is not necessary for the discussion in this section.

Apply the decomposition formula to $V(\nu, b)$, choosing the centre of expansion to be $\tilde{\gamma} = -\phi + \gamma$, where γ is a small perturbation such that $\tilde{\gamma}$ is sufficiently generic. Using the proposition above,

$$V(\nu, b) = \sum_{\tilde{\Delta} \in \mathcal{S}(\Xi^*, \nu)} \operatorname{Ver}(\nu_{\tilde{\Delta}}, b; \operatorname{pr}_{\tilde{\Delta}}(\tilde{\gamma})) * P(\nu_{\tilde{\Delta}}^{\perp}, \tilde{\gamma}_{\tilde{\Delta}}^{+}).$$

Let $\mathcal{S}(\Xi^*, \nu, \phi)$ denote the set of affine subspaces obtained by translating those in $\mathcal{S}(\Xi^*, \nu)$ by ϕ . If $\Delta = \tilde{\Delta} + \phi$, let

$$V(\nu_{\Delta}, b; \Delta)(\lambda, \ell) = V(\nu_{\tilde{\Delta}}, b; \tilde{\Delta})(\lambda - \ell \phi, \ell),$$

which is supported on $\cup_{\ell} (\ell \tilde{\Delta} + \ell \phi) \times \{\ell\} = \cup_{\ell} \ell \Delta \times \{\ell\}$. Let $\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma))$ denote the quasi-polynomial germ of $V(\nu_{\Delta}, b; \Delta)$ at $(\operatorname{pr}_{\Delta}(\gamma), 1)$; at level ℓ , it equals the translation of $\operatorname{Ver}(\nu_{\tilde{\Delta}}, b; \operatorname{pr}_{\tilde{\Delta}}(\tilde{\gamma}))$ by $\ell \phi$.

Applying the translation to the decomposition formula for $V(\nu, b)$ yields:

$$T_{\ell\phi+\sigma}V(\nu,b) = \sum_{\Delta\in\mathcal{S}(\Xi^*,\nu,\phi)} T_{\sigma}\operatorname{Ver}(\nu_{\Delta},b;\operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp},b;\gamma_{\Delta}^{+}).$$
(4.32)

For reasons to become clear as we proceed, we have built part of the shift $\ell \phi$ directly into the notation, while we choose to keep the level-independent shift by σ explicit.

We record the basic properties of each summand for future reference:

- 1. Ver $(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma))$ is quasi-polynomial and supported on $\cup_{\ell} \ell \Delta \times \{\ell\}$. It agrees with $V(\nu_{\Delta}, b; \Delta)$ in a neighbourhood of $\operatorname{pr}_{\Delta}(\gamma)$.
- 2. $P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+})$ is γ_{Δ}^{+} -polarized, so that the convolution (before translation by σ) is supported in the half-space $\langle \gamma_{\Delta}^{+}, \xi \mathrm{pr}_{\Delta}(\gamma) \rangle \geq 0$.
- 3. The convolution (before translation by σ) is quasi-polynomial on each Λ^* -translate of $\cup_{\ell} \ell \Delta \times \{\ell\}$. One way to see this is to note that the convolution is annihilated after applying appropriate finite difference operators (in directions parallel to $\cup_{\ell} \ell \Delta \times \{\ell\}$); this is true because it is true of Ver($\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)$), and one has the choice to apply these finite difference operators to either function in the convolution. The translation by σ does not effect this property, hence

$$T_{\sigma} \operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+})$$

is quasi-polynomial on each Λ^* -translate of $\cup_{\ell} \ell \Delta \times \{\ell\}$. Consequently the Δ -summand is strongly determined by its asymptotic behaviour (large ℓ); for example, if it decays asymptotically on a translate $\cup_{\ell} (\ell \Delta + \lambda) \times \{\ell\}$, then it must actually vanish identically on that translate (a quasi-polynomial cannot decay asymptotically unless it is 0).

4.2.4 The decomposition formula for large ℓ .

For the application we have in mind later, it will be important to consider the large ℓ behavior of the terms in the decomposition formula (4.32).

Consider the summand

$$T_{\sigma} \operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+}).$$

Recall that $\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma))$ agrees with $V(\nu_{\Delta}, b; \Delta)$ on an open cone in $\cup_{\ell} \ell \Delta \times \{\ell\}$ containing $(\operatorname{pr}_{\Delta}(\gamma), 1)$. This cone is of the form $\cup_{\ell} \ell C \times \{\ell\}$, where C is a convex polyhedral set in Δ containing $\operatorname{pr}_{\Delta}(\gamma)$. On the other hand $P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{\perp})$ is supported on a *pointed cone* formed by the γ_{Δ}^{\perp} -polarized weights of the T-action on ν_{Δ}^{\perp} (in fact, it might be supported on a strict subset of this cone, because of the extra outward shift from the $\det(\nu_{-})^{-1}$ factor). It follows that the convolutions

$$\operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+}), \qquad V(\nu_{\Delta}, b; \Delta) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+})$$

agree on a set of the form $\cup_{\ell} \ell D \times \{\ell\}$ where D is the set of points in \mathfrak{t}^* which cannot be reached from $\Delta \setminus C$ by a positive linear combination of the polarized weights (for ν_{Δ}^{\perp}). Applying the translation T_{σ} , we find that the two expressions agree on $\cup_{\ell} (\ell(D + \ell^{-1}\sigma)) \times \{\ell\}$. For large ℓ , $\ell^{-1}\sigma$ becomes negligible, and this set contains $\ell \operatorname{pr}_{\Delta}(\gamma)$. We have shown the following.

Proposition 4.2.8. With notation as above, the two functions

$$V_1 = T_{\sigma} \operatorname{Ver}(\nu_{\Delta}, b; \operatorname{pr}_{\Delta}(\gamma)) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+}), \qquad V_2 = T_{\sigma} V(\nu_{\Delta}, b; \Delta) * P(\nu_{\Delta}^{\perp}, b; \gamma_{\Delta}^{+})$$

agree on $\cup_{\ell} (\ell D + \sigma) \times \{\ell\} \subset \mathfrak{t}^* \times \mathbb{N}$, where D is an open neighbourhood of $\operatorname{pr}_{\Delta}(\gamma)$. This set contains an open cone around the ray $\{(\ell \operatorname{pr}_{\Delta}(\gamma), \ell) : \ell \geq \ell_0\}$ for ℓ_0 sufficiently large. In particular, for each λ there is $N \in \mathbb{N}$ such that $V_1(\lambda, \ell) = V_2(\lambda, \ell)$ for $\ell \geq N$.

It follows from the Proposition that asymptotic properties of the summand V_1 can be determined by studying V_2 . For example, suppose that V_2 decays asymptotically on a certain translate $R_{\lambda} = \bigcup_{\ell} (\ell \Delta + \lambda) \times \{\ell\}$. It follows that V_1 decays asymptotically on an open cone in R_{λ} , and hence vanishes identically on R_{λ} (V_1 is quasi-polynomial on R_{λ}).

It is useful to record an expression for the inverse Fourier transform of V_2 . Assume $\ell \phi + \sigma \in \Lambda^*$, so that the inverse Fourier transform of V_2 can be thought of as a distribution on the torus T. Let $\delta_{T_\ell T_\Delta}$ denote the delta distribution supported on $T_\ell T_\Delta$, defined using the normalized Haar measure for the (disconnected) subgroup $T_\ell T_\Delta$. In other words, for $f \in C^\infty(T)$, $\langle \delta_{T_\ell T_\Delta}, f \rangle$ is the integral of f over the subset $T_\ell T_\Delta$, with normalization such that

$$\langle \delta_{T_\ell T_\Delta}, 1 \rangle = 1.$$

Proposition 4.2.9. The inverse Fourier transform $V_2(t, \ell)$ is the push-forward of a distribution on $T_\ell T_\Delta \subset T$. Let $i(\nu_\Delta; bt)$ denote the indicator function on $T_\ell T_\Delta$ which is 0 on the coset tT_Δ $(t \in T_\ell)$ if $1 - bA(t)|_{\nu_\Delta}$ is not invertible, and is 1 otherwise. Then

$$\check{V}_{2}(t,\ell) = |T_{\ell}|(-1)^{\operatorname{rk}(\nu_{\Delta,-}^{\perp})} \frac{t^{\ell(\phi+\xi_{\Delta})+\sigma} \operatorname{Ch}^{b}(S\nu_{\Delta}^{\perp}(\gamma_{\Delta}^{+}) \otimes \det(\nu_{\Delta,-}^{\perp})^{-1}, t)}{\mathcal{D}_{\mathbb{C}}^{b}(\nu_{\Delta}, t)} i(\nu_{\Delta}; bt) \delta_{T_{\ell}T_{\Delta}}(t),$$
(4.33)

where $\xi_{\Delta} \in (\Delta - \gamma) \cap \Xi^*$ (the choice does not matter), and the infinite sum in the numerator is understood to converge in the sense of generalized functions.

Proof. To prove the Proposition, we describe the Fourier coefficients of the distribution described in the statement and show they agree with V_2 . Recall first that the inverse Fourier transform of a convolution is the product of the inverse Fourier transforms. The factor $(-1)^{\operatorname{rk}(\nu_{\Delta,-}^{\perp})}\operatorname{Ch}^{b}(S\nu_{\Delta}^{\perp}(\gamma_{\Delta}^{+})\otimes \det(\nu_{\Delta,-}^{\perp})^{-1},t)$ comes from the definition of $P(\nu_{\Delta}^{\perp},b;\Delta)$. Thus it suffices to show that the remaining part of the expression (4.33),

$$|T_{\ell}| \frac{t^{\ell(\phi+\xi_{\Delta})+\sigma}}{\mathcal{D}^{b}_{\mathbb{C}}(\nu_{\Delta},t)} i(\nu_{\Delta};bt)\delta_{T_{\ell}T_{\Delta}}(t),$$
(4.34)

has Fourier coefficients given by $T_{\sigma}V(\nu_{\Delta}, b; \Delta)$.

The λ Fourier coefficient of (4.34) is

$$|T_{\ell} \cap T_{\Delta}| \sum_{t \in T_{\ell}/T_{\Delta}}' \int_{T_{\Delta}} dh \frac{(th)^{\ell(\phi+\xi_{\Delta})+\sigma-\lambda}}{\mathcal{D}^b_{\mathbb{C}}(\nu_{\Delta},th)},$$

where the sum is over a set of representatives $t \in T_{\ell}$ for the finite group T_{ℓ}/T_{Δ} , the prime next to the summation restricts the sum (because of the indicator function $i(\nu_{\Delta}; bt)$), and dh denotes the Haar measure on T_{Δ} whose total integral is 1. We have divided by $|T_{\ell}/T_{\Delta}|$ (obtaining $|T_{\ell}| \cdot |T_{\ell}/T_{\Delta}|^{-1} =$ $|T_{\ell} \cap T_{\Delta}|$) to account for the difference between normalized Haar measure on $T_{\ell}T_{\Delta}$ and the unique measure on $T_{\ell}T_{\Delta}$ which is the product of the measure dh on T_{Δ} with the counting measure on the finite subgroup T_{ℓ}/T_{Δ} .

The denominator does not depend on h (i.e. $\mathcal{D}^b_{\mathbb{C}}(\nu_{\Delta}, th) = \mathcal{D}^b_{\mathbb{C}}(\nu_{\Delta}, t)$) since ν_{Δ} is fixed by T_{Δ} . The integral

$$\int_{T_{\Delta}} h^{\ell(\phi+\xi_{\Delta})+\sigma-\lambda} dh$$

gives 1 if $\ell(\phi + \xi_{\Delta}) + \sigma - \lambda \in \Lambda^* \cap \operatorname{ann}(\mathfrak{t}_{\Delta})$ and gives 0 otherwise. Since $\phi + \xi_{\Delta} \in \Delta$, it is equivalent to say that the integral gives 1 if $\lambda \in \Lambda^* \cap T_{\sigma}(\ell\Delta)$ and gives 0 otherwise; note that this agrees with the support of $T_{\sigma}V(\nu_{\Delta}, b; \Delta)$. Note also that $t^{\ell(\phi + \xi_{\Delta})} = 1$ since $t \in T_{\ell}$. Thus if $\lambda \in \Lambda^* \cap T_{\sigma}(\ell\Delta)$ the λ Fourier coefficient is

$$|T_{\ell} \cap T_{\Delta}| \sum_{t \in T_{\ell}/T_{\Delta}}^{\prime} \frac{t^{\sigma-\lambda}}{\mathcal{D}^{b}_{\mathbb{C}}(\nu_{\Delta}, t)},$$
(4.35)

and is 0 otherwise.

Apart from the factor $|T_{\ell} \cap T_{\Delta}|$ and the shift by σ , the expression (4.35) is a (lower-dimensional) differential form-valued Verlinde series. Recall the formula $V(\boldsymbol{\alpha}_{\Delta}; \Delta) = |T_{\ell} \cap T_{\Delta}| \iota_*^{\Delta} V_{\boldsymbol{\alpha},S}$ from Proposition 4.1.15, which expressed $V(\boldsymbol{\alpha}_{\Delta}; \Delta)$ in terms of the lower-dimensional Verlinde series $V_{\boldsymbol{\alpha},S}$. Using this formula and the definition of $T_{\sigma}V(\nu_{\Delta}, b; \Delta)$, one sees that (4.35) coincides with $(T_{\sigma}V(\nu_{\Delta}, b; \Delta))(\lambda, \ell)$. \Box

4.3 Quantization of q-Hamiltonian G-spaces.

We recall the definition of the quantization of a quasi-Hamiltonian G-space given in [44], where the reader is referred for further details. Our goal is to state the fixed-point formula of Atiyah-Segal-Singer-type for the quantization derived in [44], and describe its behaviour under changes in the level.

A Dixmier-Douady bundle $\mathcal{A} \to M$ is a bundle of C^* -algebras, with fibres modelled on the algebra of compact operators $\mathbb{K}(H)$ on a separable Hilbert space H. Continuous sections of \mathcal{A} vanishing at infinity form a C^* -algebra denoted $\Gamma_0(\mathcal{A})$. A Morita morphism \mathcal{E} between Dixmier-Douady bundles \mathcal{A}_0 , \mathcal{A}_1 , denoted

$$\mathcal{E}: \mathcal{A}_0 \dashrightarrow \mathcal{A}_1,$$

is a $\mathcal{A}_0 - \mathcal{A}_1$ bimodule $\mathcal{E} \to M$, with fibres modelled on the $\mathbb{K}(H_0) - \mathbb{K}(H_1)$ bimodule $\mathbb{K}(H_0, H_1)$. In the special case $\mathcal{A}_1 = \underline{\mathbb{C}}$, \mathcal{E} is a bundle of Hilbert spaces, and is called a *Morita trivialization* of \mathcal{A}_0 . If $L \to M$ is a line bundle, then $\mathcal{E} \otimes L$ is again a Morita morphism, and any two Morita morphisms $\mathcal{A}_1 - - \to \mathcal{A}_2$ are related in this way.

By a theorem of Dixmier-Douady, Morita isomorphism classes of Dixmier-Douady bundles are classified by a degree 3 class

$$DD(\mathcal{A}) \in H^3(M,\mathbb{Z})$$

known as the Dixmier-Douady class. More generally, given a map $\Phi : M_1 \to M_2$, Morita equivalence classes of Dixmier-Douady bundles $\mathcal{A} \to M_2$, together with Morita trivializations of the pullback $\Phi^*\mathcal{A}$, are classified by elements of the relative group $H^3(\Phi, \mathbb{Z})$. A Morita morphism

$$\mathcal{E}: \mathcal{A}_0 \dashrightarrow \mathcal{A}_1$$

is equivalent to a Morita trivialization of $\mathcal{A}_0 \otimes \mathcal{A}_1^{op}$, where \mathcal{A}_1^{op} denotes the same bundle but with the opposite algebra structure. The corresponding relative class expresses the difference $DD(\mathcal{A}_0) - DD(\mathcal{A}_1)$ as a coboundary, modulo degree 2 cocycles. A pair of Morita morphisms $\mathcal{E}, \mathcal{E}'$ from \mathcal{A}_0 to \mathcal{A}_1 gives two relative classes whose difference is $c_1(L) \in H^2(M, \mathbb{Z})$, where L is the line bundle relating $\mathcal{E}, \mathcal{E}'$. There are further generalizations to graded G-equivariant Dixmier-Douady bundles, classified by $DD(\mathcal{A}) \in$ $H^3_G(M, \mathbb{Z}) \times H^1_G(M, \mathbb{Z}_2)$.

A Dixmier-Douady bundle $\mathcal{A} \to M$ gives rise to a corresponding twisted K-homology group ([56]):

$$\mathrm{K}_{0}^{G}(M,\mathcal{A}) := \mathrm{K}\mathrm{K}_{G}(\Gamma_{0}(\mathcal{A}),\mathbb{C}),$$

where the right-hand-side is a Kasparov KK-group (c.f. [30]). Morita isomorphisms as above give rise to maps between twisted K-homology groups. More generally, one gets a map

$$(\Phi, \mathcal{E})_* : \mathrm{K}^G_0(M_0, \mathcal{A}_0) \to \mathrm{K}^G_0(M_1, \mathcal{A}_1)$$

from a pair (Φ, \mathcal{E}) consisting of a G-equivariant continuous map $\Phi: M_0 \to M_1$ and Morita morphism

$$\mathcal{E}:\mathcal{A}_0\dashrightarrow \Phi^*\mathcal{A}_1.$$

For G simple and simply connected, acting on itself by conjugation, the G-equivariant cohomology group $H^3_G(G,\mathbb{Z})$ is canonically isomorphic to \mathbb{Z} , while $H^1_G(G,\mathbb{Z}_2) = 0$. Thus Morita isomorphism classes of G-equivariant Dixmier-Douady bundles $\mathcal{A} \to G$ are classified by an integer known as the *level*. Fix a generator \mathcal{A} at level 1, and let \mathcal{A}^{ℓ} denote its ℓ -th tensor power (a level ℓ Dixmier-Douady bundle). The image of DD(\mathcal{A}) in $H^3_G(G,\mathbb{R})$ is the equivariant Cartan 3-form η_G . Let M be a quasi-Hamiltonian G-space. The relation

$$d_G\omega = -\Phi^*\eta_G$$

says that the pair (ω, η_G) is a de Rham representative for a class in the relative group $H^3_G(\Phi, \mathbb{R})$. A level k prequantization of M is an integral lift of the class $k[(\omega, \eta_G)]$. By the discussion above, this is equivalent to a choice of Morita trivialization

$$\mathcal{E}^{\mathrm{preq}}: \underline{\mathbb{C}} \dashrightarrow \Phi^* \mathcal{A}^k.$$

If M is prequantizable at level k, then it is prequantizable at level nk, for all $n \in \mathbb{N}$. In this chapter, k_1 will denote the smallest positive integer such that M is prequantizable at level k_1 .

It is known that for G compact, simply connected, a quasi-Hamiltonian G-space is necessarily evendimensional. For any even-dimensional manifold, the Clifford algebra bundle Cliff(TM) is a (graded) Dixmier-Douady bundle. In [3] it is shown that for any q-Hamiltonian G-space, there is a canonical Morita morphism

$$\mathcal{E}^{\mathrm{spin}}:\mathrm{Cliff}(TM) \dashrightarrow \Phi^*\mathcal{A}^{\mathrm{h}}$$

where h^{\vee} is the dual Coxeter number. In [3] this Morita morphism is referred to as a *twisted spin-c struc*ture (by analogy with spin-c structures, which can be viewed as Morita trivializations $\operatorname{Cliff}(TM) = - \rightarrow \underline{\mathbb{C}}$).

The fundamental class of M is the class $[M] \in K_0^G(M, \operatorname{Cliff}(TM))$ defined by the de Rham-Dirac operator acting on differential forms. Tensoring the Morita morphisms $\mathcal{E}^{\operatorname{preq}}$ and $\mathcal{E}^{\operatorname{spin}}$ together, one

obtains a Morita morphism

$$\mathcal{E}^{\mathrm{preq}}\otimes\mathcal{E}^{\mathrm{spin}}:\mathrm{Cliff}(TM)\dashrightarrow\mathcal{A}^{k+\mathrm{h}^{\vee}}.$$

The corresponding level k quantization of M, defined in [44], is the image of [M] under the corresponding pushforward map:

$$Q(M,k) = (\Phi, \mathcal{E}^{\text{preq}} \otimes \mathcal{E}^{\text{spin}})_*[M] \in K_0^G(G, \mathcal{A}^{k+h^{\vee}})$$

A special case of the Freed-Hopkins-Teleman Theorem ([20]) states that one has an isomorphism

$$K_0^G(G, \mathcal{A}^{k+\mathrm{h}^{\vee}}) \simeq R_k(G)$$

where $R_k(G)$ denotes the ring of level k positive energy representations of LG.

4.3.1 Spin-c structures and moment maps.

In this article, by spin-c structure on an oriented even-dimensional Euclidean vector bundle V, we mean a \mathbb{Z}_2 -graded spinor bundle $S \to M$ for V. That is, a complex \mathbb{Z}_2 -graded vector bundle with a fibre-wise action of the complex Clifford algebra bundle $\operatorname{Cliff}(V)$ compatible with the grading of $\operatorname{Cliff}(V)$ induced by the orientation, and such that $\operatorname{Cliff}(V) \simeq \operatorname{End}(S)$. Equivalently S defines a Morita trivialization $\operatorname{Cliff}(V) \to \underline{\mathbb{C}}$.

If S_1, S_2 are spinor bundles, the space of intertwining operators

$$L = \operatorname{Hom}_{\operatorname{Cliff}(V)}(S_1, S_2)$$

is a \mathbb{Z}_2 -graded line bundle, and $S_2 \simeq L \otimes S_1$. As a special case, if S is a spinor bundle then the dual S^* is also a spinor bundle, and the space of intertwining operators

$$\det(S) := \operatorname{Hom}_{\operatorname{Cliff}(V)}(S^*, S)$$

is called the *determinant line bundle* of the spin-c structure. In the special case that V is a complex vector bundle and $S = \wedge_{\mathbb{C}} V$ is the associated spinor bundle, then $\det(S) \simeq \det(V) = \wedge_{\mathbb{C}}^{\operatorname{top}} V$ is the complex determinant line bundle. If S is the spinor bundle associated to a *spin structure* (i.e. a lift of the principal SO(V) frame bundle to Spin(V)) then $\det(S) = \underline{\mathbb{C}}$.

Let $E \to M$ be a G-equivariant vector bundle. Then G acts on the space of sections $\Gamma(E)$ by

$$(g \cdot \sigma)(m) := g \cdot \sigma(g^{-1}m). \tag{4.36}$$

For $X \in \mathfrak{g}$, we denote by \mathcal{L}_X^E the operator on sections of E obtained by differentiating (4.36):

$$(\mathcal{L}_X^E \sigma)(m) := \frac{d}{dt}\Big|_0 \exp(tX) \cdot \sigma(\exp(-tX)m).$$

Note that if $X_M(m) = 0$ then \mathcal{L}_X^E is the linear endomorphism of the fibre E_m induced by X.

Suppose M is a G-manifold and $S \to M$ is a G-equivariant spinor bundle, equipped with a G-invariant Hermitian metric. Choose a connection $\nabla^{\det(S)}$ on $\det(S)$ compatible with the metric. For $X \in \mathfrak{g}$, the difference

$$\mathcal{L}_X^{\det(S)} - \nabla_{X_M}^{\det(S)}$$

defines a bundle endomorphism of det(S), i.e. a function $M \to \mathfrak{u}(1) = i\mathbb{R}$.

Definition 4.3.1. The *spin-c moment map* determined by the connection $\nabla^{\det(S)}$ is the map $\varphi: M \to \mathfrak{g}^*$ defined by

$$2\langle \varphi, X \rangle := \frac{i}{2\pi} \bigg(\nabla_{X_M}^{\det(S)} - \mathcal{L}_X^{\det(S)} \bigg).$$
(4.37)

The spin-c moment map is an example of an *abstract moment map* (c.f. [21]), that is, a *G*-equivariant smooth map $\varphi: M \to \mathfrak{g}^*$ such that for each $X \in \mathfrak{g}$, the pairing $\langle \varphi, X \rangle$ is locally constant on M^X .

Remark 4.3.2. Apart from the additional factor of 2 on the left side, equation (4.37) has the same form as the Kostant condition for a *G*-equivariant prequantum line bundle.

Proposition 4.3.3. Let $S \to M$ be a *G*-equivariant spin-*c* structure, and choose a connection on det(*S*) as above. Let $\beta \in \mathfrak{g}$ and suppose $\beta_M \equiv 0$. Then β acts fibrewise on *S* by scalar multiplication by $2\pi i \langle \varphi, \beta \rangle$.

Proof. Let $m \in M$. β acts on the fibre S_m by the linear endomorphism $\mathcal{L}^S_\beta|_m$:

$$\mathcal{L}_{\beta}^{S}\big|_{m} v := \frac{d}{dt}\Big|_{0} \exp(t\beta) \cdot v, \qquad v \in S_{m}$$

Since β acts trivially on $T_m M$, the action of β on S_m commutes with the Clifford action. Since S_m is an irreducible $\operatorname{Cliff}(T_m M)$ -module, the action is by a scalar. By the definition of $\det(S)$, this scalar is one half the scalar by which β acts on $\det(S_m)$. Independent of the choice of connection we have

$$\mathcal{L}_{\beta}^{\det(S)}\big|_{m} = 4\pi i \langle \varphi(m), \beta \rangle,$$

hence $\mathcal{L}^{S}_{\beta}|_{m} = 2\pi i \langle \varphi(m), \beta \rangle.$

4.3.2 Morita trivialization of $\mathcal{A}|_U$.

It is known that the restriction of \mathcal{A}^{ℓ} to a tubular neighbourhood U of T in G admits a Morita trivialization (only equivariant for a finite subgroup $T_{\ell} \subset T$), and this fact is crucial in the derivation of the Atiyah-Segal-Singer-type fixed-point formula for the quantization ([44]). We will explain a particular viewpoint on the Morita trivialization in this subsection, which emphasizes the behaviour when the level k is changed.

Since the action of T on itself by conjugation is trivial

$$H^3_T(T,\mathbb{Z}) \simeq H^3(T,\mathbb{Z}) \oplus H^1(T,\mathbb{Z}) \otimes H^2_T(\mathrm{pt},\mathbb{Z})$$

by the Kunneth theorem. There is an isomorphism

$$H^1(T,\mathbb{Z})\otimes H^2_T(\mathrm{pt},\mathbb{Z})\simeq \Lambda^*\otimes \Lambda^*.$$

As there is no torsion, we can work with de Rham cocycles. Restricting η_G to the maximal torus (and restricting the action from G to T) gives

$$\iota_T^* \eta_G(\xi) = -B(\theta_T, \xi), \qquad \xi \in \mathfrak{t}.$$

Thus the restriction is contained entirely in the $H^1(T,\mathbb{Z}) \otimes H^2_T(\text{pt},\mathbb{Z})$ component of $H^3_T(T,\mathbb{Z})$, and corresponds to the class defined by minus the basic inner product $-B^{\flat} \in \text{Hom}(\Lambda, \Lambda^*) \simeq \Lambda^* \otimes \Lambda^*$.

Let $\pi_T : U \to T$ be a *T*-invariant tubular neighbourhood of *T*, and let $\hat{U} = \mathfrak{t} \times_T U$ be the fibre product, $\pi : \hat{U} \to U = \hat{U}/\Lambda$ the quotient map. Consider the Hilbert space $L^2(T)$ with the action of *T* given by

$$(h \cdot f)(t) = f(ht).$$

The integral lattice Λ acts on $L^2(T)$ by multiplication operators

$$\lambda \cdot f = e(B^{\flat}(\lambda))f.$$

The actions of T, Λ on $L^2(T)$ only commute up to a scalar

$$h\lambda h^{-1}\lambda^{-1} = h^{B^{\flat}(\lambda)}.$$
(4.38)

Recall the finite subgroup

$$T_{\ell} = \ell^{-1} B^{\sharp}(\Lambda^*) / \Lambda \subset T.$$

The scalar (4.38) is trivial exactly for elements h in the finite subgroup $T_1 \subset T$. The induced actions of T, Λ on $\mathbb{K}(L^2(T))$ commute, thus

$$\mathcal{A}' := \hat{U} \times_{\Lambda} \mathbb{K}(L^2(T))$$

is a T-equivariant Dixmier-Douady bundle over U. The bundle of Hilbert spaces

$$\mathcal{E}' := \hat{U} \times_{\Lambda} L^2(T)$$

is a T_1 -equivariant Morita trivialization of \mathcal{A}' . Similarly $(\mathcal{E}')^{\ell}$ is a T_{ℓ} -equivariant Morita trivialization of $(\mathcal{A}')^{\ell}$; the only change is in the commutation relation (4.38), with the right side becoming $h^{-\ell B^{\flat}(\lambda)}$.

Since \mathcal{A}' is non-equivariantly Morita trivial, the class

$$\iota_T^* \mathrm{DD}(\mathcal{A}') \in H^3_T(T,\mathbb{Z}) \simeq H^3(T,\mathbb{Z}) \oplus H^1(T,\mathbb{Z}) \otimes H^2_T(\mathrm{pt},\mathbb{Z})$$

is concentrated in $H^1(T, \mathbb{Z}) \otimes H^2_T(\text{pt}, \mathbb{Z}) \simeq \Lambda^* \otimes \Lambda^*$. In fact it corresponds again to minus the basic inner product $-B^{\flat} \in \Lambda^* \otimes \Lambda^* = \text{Hom}_{\mathbb{Z}}(\Lambda, \Lambda^*)$ ([41]). As explained above, this is the same as the image of the generator of $H^3_G(G, \mathbb{Z})$ under the restriction map. Thus there exists a *T*-equivariant Morita morphism

$$\tilde{\mathcal{E}}:\mathcal{A}|_U \dashrightarrow \mathcal{A}'.$$

There is a canonical choice of Morita morphism, up to tensoring by a line bundle over U with trivial T action ([44]). Let $\mathcal{E} = \tilde{\mathcal{E}} \otimes_{\mathcal{A}'} \mathcal{E}'$ denote the composition of these Morita morphisms:

$$\mathcal{E}:\mathcal{A}|_U \dashrightarrow \underline{\mathbb{C}}.$$

The pullback $\pi^* \mathcal{A}'$ has a *T*-equivariant Morita trivialization $\pi^* \mathcal{E}' = \hat{U} \times L^2(T)$.³ This is also clear in cohomology, since the pullback $-\exp^* B(\theta_T, \xi)$ has primitive $-B(\operatorname{id}_t, \xi)$, where $\operatorname{id}_t : \mathfrak{t} \to \mathfrak{t}$ is the identity

³On the pullback $\pi^* \mathcal{E}'$, the T_1 -action extends to a *T*-action. This notation $\pi^* \mathcal{E}'$ is slightly misleading, as it hides this extension.

map.

Summarizing some consequences of this discussion, we have the following.

Proposition 4.3.4. The pullback $\pi^* \mathcal{E}^{\ell}$ is a *T*-equivariant Morita trivialization of $\pi^* \mathcal{A}^{\ell}|_U$, which also admits an action of the lattice Λ . The quotient $\pi^* \mathcal{E}^{\ell} / \Lambda = \mathcal{E}^{\ell}$ is a T_{ℓ} -equivariant Morita trivialization of $\mathcal{A}^{\ell}|_U$.

Remark 4.3.5. It is also possible to show that $\mathcal{A}^{\ell}|_{U}$ admits a T_{ℓ} -equivariant Morita trivialization using a cohomology calculation ([44]).

4.3.3 Spin-c structure on the abelianization.

The Morita trivialization described in the previous subsection implies the existence of spin-c structures on the abelianization $N = \Phi^{-1}(U)$, crucial for the Atiyah-Segal-Singer-type fixed-point formula ([44]). In this section we describe the spin-c structure from a viewpoint that emphasizes the behaviour when the level k is changed. This behaviour is made explicit by passing to covering spaces (c.f. Proposition 4.3.4); this extra structure will play an important role in Section 4.4.

Let $N = \Phi^{-1}(U)$ be the abelianization. The composition of Morita morphisms $\mathcal{E}^{\text{spin}}$ and $\Phi^* \mathcal{E}^{h^{\vee}}$

$$\operatorname{Cliff}(TM)|_N = \operatorname{Cliff}(TN) \dashrightarrow \Phi^* \mathcal{A}^{h^{\vee}}|_U \dashrightarrow \mathfrak{Q}$$

defines a $T_{\mathbf{h}^{\vee}}$ -equivariant spin-c structure $S = S_0 \to N$. Let $\mathcal{N} = \mathfrak{t} \times_T N$ be the fibre product, and $\pi : \mathcal{N} \to N = \mathcal{N}/\Lambda$ the quotient map. The discussion in the previous subsection shows that the pullback π^*S has a *T*-equivariant extension, still denoted π^*S , and furthermore π^*S admits an action of Λ , such that *S* is the quotient π^*S/Λ .

Choosing a Λ -invariant connection on π^*S we obtain a spin-c moment map

$$\varphi: \mathcal{N} \to \mathfrak{t}^*.$$

It follows from the commutation relation (4.38) that

$$\varphi(\lambda \cdot x) = \varphi(x) + \mathbf{h}^{\vee} B^{\flat}(\lambda), \qquad \lambda \in \Lambda.$$

In a later section we need some precise information about the spin-c moment map (equation (4.40) below), which is difficult to extract directly using what we have discussed so far. It can be found in [47] (in terms of a 'canonical bundle' for Hamiltonian *LG*-spaces); in [36] the spin-c structure π^*S is constructed explicitly and related to the canonical bundle of [47].

Let σ be a face of the fundamental alcove, and $w \in W$. Consider the corresponding cross-section $Y_{w\sigma} = \Phi^{-1}(U_{w\sigma})$, with inverse image

$$\pi^{-1}(Y_{w\sigma} \cap N) \simeq \Lambda \times (Y_{w\sigma} \cap N)$$

Identify $Y_{w\sigma} \cap N$ with the component whose image under ϕ intersects the face $w\sigma$ of the Stiefel diagram. According to [47], [36], on $Y_{w\sigma} \cap N$, the determinant line bundle is *T*-equivariantly isomorphic to

$$\det_{\mathbb{C}}(TY_{w\sigma}) \otimes \mathbb{C}_{2w \cdot (\rho - \rho'_{\sigma})}$$

where we use an almost complex structure compatible with the symplectic form on $Y_{w\sigma}$, and ρ'_{σ} is the half-sum of a certain set of positive roots $\mathcal{R}'_{\sigma,+}$ for \mathfrak{g}_{σ} , namely those which are positive on the cone over the translate of the alcove $\mathfrak{a} - \zeta$, where $\zeta \in \sigma$. Thus at a point $x \in N^{\beta} \cap Y_{w\sigma}$ we have

$$\langle \varphi(x), \beta \rangle = -\frac{i}{4\pi} \operatorname{Tr}_{\mathbb{C}}^{T_x Y_{w\sigma}} \left(\mathcal{L}_{\beta}^{TY_{w\sigma}} \right) + \langle w(\rho - \rho_{\sigma}), \beta \rangle.$$
(4.39)

Let $\mathbb{T}_{\beta} = \overline{\exp(\mathbb{R}\beta)} \subset T$ be the torus generated by β . Let $s_{w\sigma}$ be the sum of the complex weights for the action of \mathbb{T}_{β} on $T_x Y_{w\sigma}$. Then equation (4.39) can be re-written

$$\langle \varphi(x), \beta \rangle = \frac{1}{2} \langle s_{w\sigma}, \beta \rangle + \langle w(\rho - \rho_{\sigma}), \beta \rangle.$$
(4.40)

Suppose that M is prequantizable at level $k_1 \ge 1$, and let $\mathcal{E}^{\text{preq}}$ be a choice of Morita morphism

$$\mathcal{E}^{\mathrm{preq}}:\mathbb{C}\dashrightarrow\Phi^*\mathcal{A}^{k_1}$$

The composition of the Morita morphisms $\mathcal{E}^{\text{preq}}$ and $\Phi^* \mathcal{E}^{k_1}$

$$\underline{\mathbb{C}} \dashrightarrow \Phi^* \mathcal{A}^{k_1} |_U \dashrightarrow \underline{\mathbb{C}}$$

defines a T_{k_1} -equivariant Morita morphism $\underline{\mathbb{C}} \to - \to \underline{\mathbb{C}}$, that is, a T_{k_1} -equivariant line bundle $L \to N$. The discussion in the previous subsection shows that the pullback π^*L possesses a T-equivariant extension still denoted π^*L . Furthermore, π^*L admits an action of Λ , and L is the quotient π^*L/Λ . The line bundle π^*L is the line bundle relating the trivial Morita morphism $\mathbb{C} : \underline{\mathbb{C}} \to - \to \underline{\mathbb{C}}$ with the Morita morphism $\pi^*L : \underline{\mathbb{C}} \to - \to \underline{\mathbb{C}}$. From the calculation

$$d_T\omega_a = -(\pi_T \circ \Phi)^* \eta_T \quad \Rightarrow \quad d_T \pi^* \omega_a = \phi^* dB(\mathrm{id}_{\mathfrak{t}}, -) = d\langle \phi, - \rangle$$

it follows that the T-equivariant first Chern class of π^*L has de Rham representative $k_1(\omega_a - \phi)$.

Set

 $k = nk_1$

where $n \in \mathbb{N}$, and consider the tensor product

$$S_k = S \otimes L^n$$
.

Theorem 4.3.6. The pullback π^*S_k is a *T*-equivariant spin-*c* structure on \mathcal{N} , which also admits an action of Λ , and the quotient S_k is a $T_{k+h^{\vee}}$ -equivariant spin-*c* structure on N. The spin-*c* moment map of π^*S_k is

$$k\phi + \varphi$$

where φ is the spin-c moment map for π^*S and $\phi: \mathcal{N} \to \mathfrak{t}^*$ is the moment map for the (degenerate) Hamiltonian T-space $(\mathcal{N}, \pi^*\omega_a, \phi)$.

4.3.4 Atiyah-Segal-Singer formula.

From now on, we set

$$\ell := k + \mathbf{h}^{\vee}$$

where k is the level of the prequantization. It is known that $R_k(G)$ is isomorphic to a quotient $R(G)/I_k(G)$ where R(G) is the representation ring, and $I_k(G)$ is the *fusion ideal* consisting of characters vanishing at the regular points T_{ℓ}^{reg} of T_{ℓ} . Therefore, for each $t \in T_{\ell}^{\text{reg}}$ the evaluation map ev_t descends to the quotient

$$\operatorname{ev}_t : R_k(G) \to \mathbb{C}.$$

In [44], it is shown that $Q(M,k)(t) = ev_t Q(M,k)$ can be computed from an Atiyah-Segal-Singer-type fixed-point formula, involving the T_{ℓ} -equivariant spin-c structure S_k :

$$Q(M,k)(t) = \sum_{F \subset M^t} \int_F \frac{\hat{\mathcal{A}}(F) \mathrm{Ch}(\mathcal{L}_k, t)^{1/2}}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_F, t)},$$

for $t \in T_{\ell}^{\text{reg}}$, where $\tilde{\nu}_F$ is the normal bundle to F and $\mathcal{L}_k = \det(S_k)$. The square root $\operatorname{Ch}(\mathcal{L}_k, t)^{1/2}$ is given by the expression

$$Ch(\mathcal{L}_k, t)^{1/2} = \kappa_{F,k}(t)^{1/2} e^{\frac{1}{2}c_1(\mathcal{L}_k)},$$

where $\kappa_{F,k}(t)$ is the phase factor for the action of t on $\mathcal{L}_k|_F$, and we need to specify the sign of the square root. This is explained in [44] (p. 26; see also [4]). Choose a complex structure on $T_x M$ for some $x \in F$. The $\text{Cliff}(T_x M)$ -representations $S_k|_x$ and $\wedge_{\mathbb{C}} T_x M$ are related by a Hermitian line, on which t acts by some phase $\rho_{F,k}(t) \in U(1)$. Then

$$\kappa_{F,k}(t)^{1/2} = \rho_{F,k}(t) \det_{\mathbb{C}}(A(t)^{1/2})$$

where $A(t) \in U(T_x M)$ is the action of t on $T_x M$ and $A(t)^{1/2}$ is the unique square root with all eigenvalues in $\{e^{i\phi} | \phi \in [0, \pi)\}$.

Remark 4.3.7. As commented already above, the Morita trivialization \mathcal{E} , and consequently the spin-c structure S_k , is not canonical. In Section 5.4 of [44] it was noted that det (S_k) can be replaced with

$$\det(S_k) \otimes \Phi^* R^{-(k+h^{\vee})}. \tag{4.41}$$

in the fixed-point expression above, where $R \to T$ is a line bundle with trivial *T*-action. This replacement could affect the contributions of the individual fixed-point components $F \subset M^t$, but not their sum. In particular, by making a particular choice of R, it is possible to obtain a 'normalized' form for the fixed-point expressions, invariant under the Weyl group, and no longer dependent on the choice of \mathcal{E} .

Recall that irreducible representations of LG at level k are in 1-1 correspondence with the dominant weights λ satisfying $B(\lambda, \theta) \leq k$, where θ is the highest root. Equivalently, with those weights lying in $k\mathfrak{a}$ where \mathfrak{a} is the fundamental alcove (viewed as a subset of \mathfrak{t}^* using the basic inner product). Any $\chi \in R_k(G)$ has a multiplicity function

$$m:\Lambda^*\cap k\mathfrak{a}\to\mathbb{Z}$$

counting the multiplicity of each irreducible positive energy representation in χ . Let J(t) denote the Weyl denominator, χ_{λ} the character of the irreducible representation of G corresponding to the dominant

weight λ . The multiplicity function $m(\lambda, k)$ is obtained from $Q(M, k) : T_{\ell}^{\text{reg}} \to \mathbb{C}$ by finite Fourier transform ([44] or [4]):

$$m(\lambda,k) = \frac{1}{|T_{\ell}|} \sum_{t \in T_{\ell}^{\text{reg}}/W} |J(t)|^2 \overline{\chi_{\lambda}}(t) Q(M,k)(t).$$
(4.42)

Of course this is only defined at values of $k \in \mathbb{N}$ such that M is prequantizable at level k, that is, $k = nk_1$ where $k_1 \in \mathbb{N}$ is the smallest level at which M is prequantizable. We extend the definition of $m(\lambda, k)$ trivially to all $k \in \mathbb{N}$, by setting it to be zero for k not equal to a positive integer multiple of k_1 .

It is convenient to extend m(-,k) to a function

$$m(-,k):\Lambda^*\to\mathbb{Z}$$

which agrees with the definition above on $\Lambda^* \cap k\mathfrak{a}$, and is extended to Λ^* so that it is *alternating* under the shifted level $\ell = (k + h^{\vee})$ action of the affine Weyl group $W_{\text{aff}} := W \ltimes \Lambda$, given by

$$(w, \gamma) \cdot_{\ell} \lambda = w \cdot (\lambda + \rho) - \rho + \ell B^{\flat}(\gamma).$$

This determines the extension uniquely. Equivalently, the extended function m(-,k) gives the Fourier coefficients of the distributional character obtained by multiplying Q(M,k) by the Weyl-Kac denominator followed by restriction to T.

Remark 4.3.8. A simple example is the multiplicity function for S^4 , a quasi-Hamiltonian SU(2)-space (c.f. [42]), at level k: it is a function $m(-,k): \mathbb{Z} \to \mathbb{Z}$ with $m(0,k) = m(1,k) = \cdots = m(\ell-1,k) = 1$, where $\ell = k + h^{\vee} = k + 2$, and the rest of the values of m are determined by the affine Weyl action.

4.4 Norm-square localization formula

In this section we will derive the norm-square localization formula for the quantization Q(M, k) (more precisely, its multiplicity function $m(\lambda, k)$, see (4.42)). This involves several steps. First we rearrange the fixed-point expressions (4.42) to show that $m(\lambda, k)$ can be expressed in terms of finitely many Verlinde series. Then we apply the decomposition formula for Verlinde series, organizing the terms according to the affine subspaces $\Delta \in S$. We study the asymptotics $(k \to \infty)$ of the resulting formula, proving that the non-zero terms correspond to the critical values of the norm-square of the moment map $\|\Psi\|^2$. We emphasize that throughout this section and the next one, we set

$$\ell = k + \mathbf{h}^{\vee},$$

where k is the level of the prequantization.

4.4.1 Rearrangement of the fixed-point expressions.

Using $J(t) = t^{\rho} \prod_{\alpha < 0} (1 - t^{\alpha})$ and the Weyl character formula, one finds

$$m(\lambda,k) = \frac{1}{|T_{\ell}|} \sum_{t \in T_{\ell}} t^{-\lambda} \prod_{\alpha < 0} (1 - t^{\alpha}) Q(M,k)(t) = \frac{1}{|T_{\ell}|} \prod_{\alpha < 0} \nabla_{\alpha} \sum_{t \in T_{\ell}} t^{-\lambda} Q(M,k)(t).$$
(4.43)

We have allowed the sum to run over all of T_{ℓ} , since $\prod_{\alpha < 0} (1 - t^{\alpha})$ vanishes at all non-regular elements in any case. For the second equality, we have used the fact that the finite Fourier transform of $(1 - t^{\alpha})f(t)$ is equal to the finite difference operator ∇_{α} applied to the finite Fourier transform of f. Equation (4.43) expresses $m(\lambda, k)$ as the finite Fourier transform of a function on T_{ℓ} . Equivalently, identifying distributions and (generalized) functions on T_{ℓ} using normalized Haar measure on T_{ℓ} , we can view $m(\lambda, k)$ as the Fourier transform of the distribution supported on $T_{\ell} \subset T$ obtained by pushforward along $T_{\ell} \hookrightarrow T$ of

$$\prod_{\alpha<0} (1-t^{\alpha})Q(M,k)(t), \qquad t \in T_{\ell}.$$

Our goal is to understand the dependence of m(0, k) on the level k (or ℓ), and ultimately show that it is quasi-polynomial. There are two sources of dependence: (1) the set T_{ℓ} indexing the sum, and (2) the form $\operatorname{Ch}(\mathcal{L}_k, t)^{1/2}$ for the T_{ℓ} -equivariant Spin^c line bundle \mathcal{L}_k .

The first step is to re-order the sums in the formula, so that one first sums over orbit-types of T, followed by a sum over a subset of T_{ℓ} . Let \mathcal{F} be the collection, for all regular elements $t \in T$, of all connected components $F \subset M^t$. For each $F \in \mathcal{F}$, let $\tilde{T}_F \subset T$ be the stabilizer of F in T. Let $T_{F,a}$, a = 0, 1, 2, ... be the connected components of \tilde{T}_F with $T_{F,0} = T_F = \exp(\mathfrak{t}_F)$ the identity component. The elements $t \in T_{\ell}$ for which the corresponding term in (4.43) involves an integral over F are those $t \in T_{\ell}$ such that $t \in \tilde{T}_F$, but $t \notin \tilde{T}_{F'}$ for any $F \subsetneq F' \in \mathcal{F}$. Equivalently, this is the set of $t \in T_{\ell} \cap \tilde{T}_F$ such that $1 - A(t) \in \operatorname{End}(\tilde{\nu}_F)$ is invertible, where A(t) denotes the action of t on $\tilde{\nu}_F$. Equation (4.43) becomes

$$m(\lambda,k) = \frac{1}{|T_{\ell}|} \prod_{\alpha < 0} \nabla_{\alpha} \sum_{F \in \mathcal{F}} \int_{F} \hat{A}(F) \sum_{t \in T_{\ell} \cap \tilde{T}_{F}} \frac{\operatorname{Ch}(\mathcal{L}_{k},t)^{1/2} t^{-\lambda}}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F},t)},$$
(4.44)

where the prime next to the summation means omit terms such that $1 - A(t) \in \text{End}(\tilde{\nu}_F)$ is not invertible.

For $F \in \mathcal{F}$, let ν'_F be the normal bundle to F inside $M^{\mathfrak{t}_F}$; thus only the finite group \tilde{T}_F/T_F can act non-trivially on ν'_F . Let ν_F be its T_F -invariant complement in $\tilde{\nu}_F$, so that \mathfrak{t}_F acts with non-zero weights on ν_F and

$$\tilde{\nu}_F = \nu_F \oplus \nu'_F.$$

Using the \mathfrak{t}_F -action, the vector bundles ν_F can be equipped with complex structures. We do this by fixing a "generic" element $0 \neq \delta \in \mathfrak{t}$, and then choose the complex structure on ν_F such that the complex weights $\alpha \in (\Lambda \cap \mathfrak{t}_F)^*$ of the \mathfrak{t}_F -action on ν_F are δ -polarized:

$$\langle \alpha, \operatorname{pr}_{\mathfrak{t}_{F}}(\delta) \rangle > 0$$

where $pr_{\mathfrak{t}_F}$ denotes orthogonal projection to \mathfrak{t}_F .

- Remark 4.4.1. 1. Below we treat ν_F (and sub-bundles of ν_F) as complex vector bundles with this δ -polarized complex structure, unless otherwise specified. To indicate a different polarization, we write the polarization in brackets, for example $\nu_F(\gamma^+)$ as in Section 4.2.2. Otherwise the δ -polarized complex structure is the "default".
 - 2. If one assumes that the stabilizers are connected, then $\nu'_F = 0$ and several of the equations below simplify.

Applying Proposition 3.1.4 to $V = \mathfrak{t}/\mathfrak{t}_F$, $\Xi = B^{\#}(\Lambda^*)/\mathfrak{t}_F$, $\Gamma = \Lambda/\mathfrak{t}_F$, shows that for each component
$T_{F,a}$ there is a minimal $\ell_{F,a}$ such that

$$T_{F,a} \cap T_{\ell} \neq \emptyset \quad \iff \quad \ell \equiv 0 \pmod{\ell_{F,a}}.$$

For each (F, a) choose $t_{F,a} \in T_{F,a} \cap T_{\ell_{F,a}}$.

In order to state the formula expressing m in terms of Verlinde series, we choose a point $x \in F$ for each F, and a lift $\hat{x} \in \pi^{-1}(x) \subset \mathcal{N}$. Let

$$\phi_F = \operatorname{pr}_{\mathfrak{t}_F^*}(\phi(\hat{x})), \qquad \varphi_F = \operatorname{pr}_{\mathfrak{t}_F^*}(\varphi(\hat{x})),$$

where $\operatorname{pr}_{\mathfrak{t}_{F}^{*}}: \mathfrak{t}^{*} \to \mathfrak{t}^{*}/\operatorname{ann}(\mathfrak{t}_{F})$ is the quotient map. (Recall: $\phi: \mathcal{N} \to \mathfrak{t}^{*}$ is a moment map, $k_{1}\phi$ is the moment map for the line bundle $\pi^{*}L \to \mathcal{N}$, and $\varphi: \mathcal{N} \to \mathfrak{t}^{*}$ was the spin-c moment map for S.) We also choose a complex structure on $T_{x}N \simeq T_{\hat{x}}\mathcal{N}$ extending the δ -polarized complex structure on $\nu_{F}|_{x}$. Then T_{F} acts on $\operatorname{det}_{\mathbb{C}}(T_{x}N)$ by some weight $s_{F} \in (\Lambda \cap \mathfrak{t}_{F})^{*}$.

Theorem 4.4.2. Let $\sigma_F = \varphi_F - \frac{1}{2}s_F - h^{\vee}\phi_F$, with s_F , ϕ_F , φ_F as defined above. For each (F, a) there is a (differential form-valued) quasi-polynomial function

$$(\lambda, k) \mapsto t_{F,a}^{-\lambda} f_{F,a}(k)$$

(see equation (4.49) for the definition of $f_{F,a}$) such that

$$m = \frac{1}{|T_\ell|} \prod_{\alpha < 0} \nabla_\alpha \sum_{F,a} \int_F f_{F,a} t_{F,a}^{-\lambda} T_{\ell\phi_F + \sigma_F} V(\overline{\nu}_F, t_{F,a}).$$
(4.45)

 $V(\overline{\nu}_F, t_{F,a})$ is a Verlinde series with data $(\mathfrak{t}_F, \Xi_F, \Lambda_F, \overline{\nu}_F, t_{F,a})$, where $\Xi = B^{\#}(\Lambda^*)$, $\Lambda_F = \Lambda \cap \mathfrak{t}_F$, $\Xi_F = \Xi \cap \mathfrak{t}_F$.

- Remark 4.4.3. 1. $V(\overline{\nu}_F, t_{F,a})$ initially has domain $(\Lambda \cap \mathfrak{t}_F)^* = \Lambda^*/\operatorname{ann}(\mathfrak{t}_F)$. In the equation above, we have omitted a pullback under the map $\Lambda^* \to \Lambda^*/\operatorname{ann}(\mathfrak{t}_F)$ from the notation.
 - 2. The shift σ_F depends on the choice of complex structure on $T_x N$ (through s_F), but not on the choice of lift $\hat{x} \in \pi^{-1}(x) \subset \mathcal{N}$. This is because if one chooses instead $\lambda \cdot \hat{x}$, then φ_F and $\mathbf{h}^{\vee} \phi_F$ both change by $\mathbf{h}^{\vee} B^{\flat}(\lambda)$.

Proof. Applying equation (4.23) relating $\mathcal{D}_{\mathbb{R}}$ and $\mathcal{D}_{\mathbb{C}}$ we find

$$m(\lambda,k) = \frac{1}{|T_{\ell}|} \prod_{\alpha < 0} \nabla_{\alpha} \sum_{F \in \mathcal{F}} \int_{F} \hat{A}(F) \sum_{t \in T_{\ell} \cap \tilde{T}_{F}}' \frac{\operatorname{Ch}(\mathcal{L}_{k},t)^{1/2} \operatorname{Ch}(\operatorname{det}(\nu_{F}),t)^{-1/2} t^{-\lambda}}{\mathcal{D}_{\mathbb{R}}(\nu_{F}',t) \mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F},t)}.$$
(4.46)

Recall the definition of the square-root $Ch(\mathcal{L}_k, t)^{1/2}$:

$$\operatorname{Ch}(\mathcal{L}_k, t)^{1/2} = \rho_{F,k}(t) \operatorname{det}_{\mathbb{C}}(A(t)^{1/2}) \operatorname{Ch}(\mathcal{L}_k)^{1/2}$$

where $\rho_{F,k}(t)$ is the phase factor for the action of t on the Hermitian line relating $S_k|_x$ and $\wedge_{\mathbb{C}} T_x N$. Since

$$Ch(\det(\nu_F), t)^{-1/2} = \det_{\mathbb{C}}(A_{\nu_F}(t)^{-1/2})Ch(\det(\nu_F))^{-1/2},$$

we have

$$Ch(\mathcal{L}_k, t)^{1/2} Ch(\det(\nu_F), t)^{-1/2} = \rho_{F,k}(t) \det_{\mathbb{C}}(A_{\nu'_F}(t)^{1/2}) Ch(\mathcal{L}_k \otimes \det(\nu_F)^{-1})^{1/2}, \qquad (4.47)$$

where we have used $TN|_F = TF \oplus \nu_F \oplus \nu'_F$ and the fact that t acts trivially on TF.

For each k, $\rho_{F,k}$ is a homomorphism

$$\rho_{F,k}: \tilde{T}_F \cap T_\ell \to U(1)$$

Suppose $t \in T_{F,a}$, then $t_{F,a}^{-1}t \in T_F$ and therefore acts trivially on ν'_F . Hence (4.47) can be re-written

$$Ch(\mathcal{L}_k, t)^{1/2} Ch(\det(\nu_F), t)^{-1/2} = \rho_{F,k}(t_{F,a}^{-1}t) Ch(\mathcal{L}_k, t_{F,a})^{1/2} Ch(\det(\nu_F), t_{F,a})^{-1/2}.$$

To describe the homomorphism $\rho_{F,k}: T_F \to U(1)$, recall that $k = nk_1$ and

$$\pi^* S_k = \pi^* S \otimes \pi^* L^n$$

where $\pi^* S$, $\pi^* L$ are both T-equivariant. Using the chosen complex structure on $T_{\hat{x}} \mathcal{N} \simeq T_x N$, we have

$$\pi^* S|_{\hat{x}} = E \otimes \wedge_{\mathbb{C}} T_{\hat{x}} \mathcal{N}_{\hat{x}}$$

for some Hermitian line E. Then

$$\pi^* \mathcal{L}_k|_{\hat{x}} = \pi^* L_{\hat{x}}^{2n} \otimes E^2 \otimes \det(T_{\hat{x}} \mathcal{N}).$$
(4.48)

The action of T_F on L is given by the weight $k_1\phi_F \in \Lambda^*/\operatorname{ann}(\mathfrak{t}_F)$. Let $\mu_F \in \Lambda^*/\operatorname{ann}(\mathfrak{t}_F)$ be the weight for the action of T_F on E. The homomorphism $t \in T_F \mapsto \operatorname{det}_{\mathbb{C}}(A(t)) \in U(1)$ is given by the weight s_F . From (4.48) (setting k = 0) we have

$$\mu_F = \varphi_F - \frac{1}{2}s_F$$

Thus $\rho_{F,k}|_{T_F}$ is given by the weight

$$k\phi_F + \varphi_F - \frac{1}{2}s_F.$$

Let

$$f_{F,a}(k) = \frac{\hat{A}(F)Ch(\mathcal{L}_k, t_{F,a})^{1/2}Ch(\det(\nu_F), t_{F,a})^{-1/2}}{\mathcal{D}_{\mathbb{R}}(\nu'_F, t_{F,a})} \mathbf{1}_{\ell_{F,a}\mathbb{N}}(\ell).$$
(4.49)

The factor $\mathbf{1}_{\ell_{F,a}\mathbb{N}}(\ell)$ is used here as the component $t_{F,a}T_F$ only intersects T_ℓ non-trivially when $\ell = k + \mathbf{h}^{\vee} \in \ell_{F,a}\mathbb{N}$. Note that $\mathrm{Ch}(\mathcal{L}_k)^{1/2}$ is polynomial in k. Moreover, the element $t_{F,a} \in T_\ell$ has finite order, and so acts on $\pi^*L_{\hat{x}}$ by multiplication by a root of unity. Hence $f_{F,a}(k)$ is quasi-polynomial.

It follows from our definitions that

$$\sum_{t\in T_{\ell}\cap\tilde{T}_{F}}^{\prime}\frac{\hat{A}(F)\mathrm{Ch}(\mathcal{L}_{k},t)^{1/2}\mathrm{Ch}(\det(\nu_{F}),t)^{-1/2}t^{-\lambda}}{\mathcal{D}_{\mathbb{R}}(\nu_{F}^{\prime},t)\mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F},t)}=\sum_{a}f_{F,a}(k)t_{F,a}^{-\lambda}\sum_{t\in T_{\ell}\cap T_{F}}^{\prime}\frac{t^{k\phi_{F}+\varphi_{F}-\frac{1}{2}s_{F}-\lambda}}{\mathcal{D}_{\mathbb{C}}^{t_{F,a}}(\overline{\nu}_{F},t)}.$$

c		

4.4.2 Applying the decomposition formula.

Equation (4.32) applies to each Verlinde series V(-) in formula (4.45):

$$T_{\ell\phi_F+\sigma_F}V(\overline{\nu}_F, t_{F,a}) = \sum_{\Delta \in \mathcal{S}_F(\Xi_F^*, \nu_F, \phi_F)} T_{\sigma_F} \operatorname{Ver}(\overline{\nu}_{F,\Delta}, t_{F,a}; \operatorname{pr}_{\Delta}(\gamma)) * P(\overline{\nu}_{F,\Delta}^{\perp}, t_{F,a}; \gamma_{\Delta}^{\perp}).$$
(4.50)

One should view the affine subspaces Δ as subspaces of \mathfrak{t}^* , by taking inverse image under the map $\mathfrak{t}^* \to \mathfrak{t}^*_F = \mathfrak{t}^*/\operatorname{ann}(\mathfrak{t}_F)$.

Next we group the terms according to the affine subspaces Δ , or equivalently, we switch the order of the summation over Δ with that over (F, a). Let S be the union of all the sets S_F , $F \in \mathcal{F}$; note that we view all elements of S as affine subspaces of \mathfrak{t}^* , by taking inverse images under the maps $\mathfrak{t}^* \to \mathfrak{t}_F^*$. The terms in equations (4.45), (4.50) are indexed by triples (F, a, Δ) . The affine subspaces Δ are *rational* in the sense that the subalgebra \mathfrak{t}_{Δ} orthogonal to Δ exponentiates to a closed subtorus $T_{\Delta} \subset T$. We group the terms corresponding to (F, a, Δ) and (F', a', Δ') if

- 1. $\Delta = \Delta'$
- 2. F, F' belong to the same connected component $C \subset N^{\mathfrak{t}_{\Delta}}$.

Note that for such terms, the orthogonal projection $\operatorname{pr}_{\Delta}(\gamma)$ of γ to Δ is the same. Consequently for these terms, the quasi-polynomial "germs" $\operatorname{Ver}(-)$ are taken about the same point $(\operatorname{pr}_{\Delta}(\gamma))$ and the polarization direction $(\gamma_{\Delta}^{+} = \operatorname{pr}_{\Delta}(\gamma) - \gamma)$ for the partition function P(-) is the same. For $C \subset M^{\mathfrak{t}_{\Delta}}$, let ν_{C} be the normal bundle. If $F \in \mathcal{F}$ and $F \subset C$, then

$$\nu_{F,\Delta}^{\perp} = \nu_C|_F.$$

The normal bundle $\tilde{\nu}_{F,C}$ to F in C decomposes

$$\tilde{\nu}_{F,C} = \nu_{F,C} \oplus \nu'_F, \qquad \nu_{F,C} = \nu_{F,\Delta}$$

 (ν'_F) was defined already above).

After rearrangement, equation (4.45) takes the form

$$m = \sum_{\Delta,C} m_{\Delta,C}^{\text{qpol}} \tag{4.51}$$

where $C \subset M^{\mathfrak{t}_{\Delta}}$ and

$$m_{\Delta,C}^{\text{qpol}} = \frac{1}{|T_{\ell}|} \prod_{\alpha < 0} \nabla_{\alpha} \sum_{F \subset C,a} \int_{F} f_{F,a} t_{F,a}^{-\lambda} T_{\sigma_{F}} \operatorname{Ver}(\overline{\nu}_{F,C}, t_{F,a}; \operatorname{pr}_{\Delta}(\gamma)) * P(\overline{\nu}_{C}, t_{F,a}; \gamma_{\Delta}^{+}).$$
(4.52)

Some properties of the terms $m_{\Delta,C}^{\text{qpol}}$ follow immediately from the properties listed at the end of Section 4.2.3:

1. $|T_{\ell}|m_{\Delta,C}^{\text{qpol}}$ is quasi-polynomial on each translate of $\cup_{\ell}\ell\Delta \times \{\ell\}$. This is true of the terms of the decomposition formula $\operatorname{Ver}(-) * P(-)$, and is not destroyed by the translations T_{σ_F} , the quasi-polynomial pre-factors depending on (F, a) and k, or the finite difference operators ∇_{α} (which amount to taking certain finite linear combinations of translations of these quasi-polynomials).

This is our reason for using the superscript "qpol". Notice that if $0 \in \Delta$, it follows that $|T_{\ell}|m_{\Delta,C}^{\text{qpol}}$ is quasi-polynomial when restricted to $\Delta \times \mathbb{N} \subset \mathfrak{t}^* \times \mathbb{N}$, and in particular $|T_{\ell}|m_{\Delta,C}^{\text{qpol}}(0,k)$ is a quasi-polynomial function of k.

2. $m_{\Delta,C}^{\text{qpol}}(-,k)$ is supported in a half-space of the form $\langle -, \gamma_{\Delta}^+ \rangle \geq d \| \gamma_{\Delta}^+ \|$ where d is a constant depending on Δ , k, the shifts by σ_F and the finite difference operators.

Equations (4.51), (4.52) are the key formulas underlying this approach to the [Q, R] = 0 Theorem. Our next goal is to show that the terms in equation (4.51) which do not correspond to critical points of $\|\Psi\|^2$ vanish—this will justify calling (4.51), (4.52) a 'norm-square localization' formula.

4.4.3 Norm-square contributions.

Consider the related distribution $m_{\Delta,C}$ obtained from (4.52) by replacing Ver(-) with V(-), as in Section 4.2.4:

$$m_{\Delta,C} = \frac{1}{|T_{\ell}|} \prod_{\alpha < 0} \nabla_{\alpha} \sum_{F \subset C,a} \int_{F} f_{F,a} t_{F,a}^{-\lambda} T_{\sigma_{F}} V(\overline{\nu}_{F,C}, t_{F,a}; \Delta) * P(\overline{\nu}_{C}, t_{F,a}; \gamma_{\Delta}^{+}).$$
(4.53)

As explained in Section 4.2.4, for each $\lambda \in \Lambda^*$, $m_{\Delta,C}(\lambda, k) = m_{\Delta,C}^{\text{qpol}}(\lambda, k)$ for k sufficiently large.

We next compute the inverse Fourier transform $\check{m}_{\Delta,C}(t,k)$. Recall notation used in Section 4.2.4: $\delta_{T_{\ell}T_{\Delta}} \in \mathcal{D}'(T)$ denotes the delta distribution supported on $T_{\ell}T_{\Delta}$ defined using normalized Haar measure on the (disconnected) subgroup $T_{\ell}T_{\Delta}$.

Lemma 4.4.4. The inverse Fourier transform $Q_{\Delta,C} := \check{m}_{\Delta,C}$ of $m_{\Delta,C}$ is the push-forward of a distribution on $T_{\ell}T_{\Delta} \subset T$. Write an element of $T_{\ell}T_{\Delta}$ (non-uniquely) as a product the where $t \in T_{\ell}$, $h \in T_{\Delta}$. Let $s_{\Delta}^+ \in \Lambda^*/\operatorname{ann}(\mathfrak{t}_{\Delta})$ be the sum of the γ_{Δ}^+ -polarized weights of the T_{Δ} -action on the normal bundle ν_C . Choose $\xi_{F,\Delta} \in (\Delta - \phi_F) \cap (B^{\flat}(\Lambda)/\operatorname{ann}(\mathfrak{t}_F))$, and let

$$\phi_{\Delta} = \operatorname{pr}_{\mathfrak{t}_{\Delta}^{*}}(\phi_{F} + \xi_{F,\Delta}), \quad \varphi_{\Delta} = \operatorname{pr}_{\mathfrak{t}_{\Delta}^{*}}(\varphi_{F} + \operatorname{h}^{\vee}\xi_{F,\Delta}),$$

where $\operatorname{pr}_{\mathfrak{t}^*_{\Delta}}$ denotes the quotient map $\mathfrak{t}^*_F \to \mathfrak{t}^*_{\Delta}$. The distribution $Q_{\Delta,C}$ is given by the formula

$$Q_{\Delta,C}(th,k) = \prod_{\alpha<0} (1-(th)^{\alpha}) \sum_{F\subset C^{t}} (-1)^{\operatorname{rk}(\nu_{C})} h^{k\phi_{\Delta}+\varphi_{\Delta}+\frac{1}{2}s_{\Delta}^{+}} \\ \int_{F} \frac{\hat{A}(F)\operatorname{Ch}(\mathcal{L}_{k}\otimes \det(\nu_{C}(\gamma_{\Delta}^{+})),t)^{1/2}\operatorname{Ch}(S\nu_{C}(\gamma_{\Delta}^{+}),th)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} \delta_{T_{\ell}T_{\Delta}}(th).$$
(4.54)

Proof. Proposition 4.33 (applied to the torus T_F , the finite subgroup $T_{\ell} \cap T_F$, and so on) showed that the inverse Fourier transform of $T_{\sigma_F}V(\overline{\nu}_{F,C}, t_{F,a}; \Delta) * P(\overline{\nu}_C, t_{F,a}; \gamma_{\Delta}^+)$ is a distribution on T_F given by the formula

$$|T_{\ell} \cap T_{F}|(-1)^{\operatorname{rk}(\overline{\nu}_{C,-})} \frac{t^{\ell(\phi_{F}+\xi_{F,\Delta})+\sigma_{F}} \operatorname{Ch}^{t_{F,a}}(S\overline{\nu}_{C}(\gamma_{\Delta}^{+}) \otimes \det(\overline{\nu}_{C,-})^{-1}, t)}{\mathcal{D}_{\mathbb{C}}^{t_{F,a}}(\overline{\nu}_{F,C}, t)} i(\nu_{F,C}; t_{F,a}t) \delta_{(T_{\ell} \cap T_{F})T_{\Delta}}(t)$$
(4.55)

where $i(\nu_{F,C}; t_{F,a}t)$ is an indicator function on $(T_{\ell} \cap T_F)T_{\Delta}$ equal to 0 if $1 - A_{\nu_{F,C}}(t_{F,a}t)$ is not invertible and equal to 1 otherwise; note that it is constant on each coset $tT_{\Delta}, t \in T_{\ell} \cap T_F$. $\delta_{(T_{\ell} \cap T_F)T_{\Delta}}$ denotes the delta distribution for $(T_{\ell} \cap T_F)T_{\Delta} \subset T_F$, defined using normalized Haar measure for the subgroup $(T_{\ell} \cap T_F)T_{\Delta}$. Here $\xi_{F,\Delta} \in (\Delta - \phi_F) \cap B^{\flat}(\Lambda)/\operatorname{ann}(\mathfrak{t}_F)$ (the choice does not matter).

Extend the definition of the indicator function $i(\nu_{F,C}; t_{F,a}t)$ to be 0 away from T_F . The factor $|T_{\ell} \cap T_F|$ in (4.55) and the factor $|T_{\ell}|^{-1}$ in (4.53) combine to give $|T_{\ell}/T_F|^{-1}$. Since the indicator function vanishes away from T_F we have

$$i(\nu_{F,C}; t_{F,a}t)|T_{\ell}/T_{F}|^{-1}\delta_{(T_{\ell}\cap T_{F})T_{\Delta}} = i(\nu_{F,C}; t_{F,a}t)\delta_{T_{\ell}T_{\Delta}},$$

i.e. the factor $|T_{\ell}/T_F|^{-1}$ relates the two different normalizations for the Haar measure on $(T_{\ell} \cap T_F)T_{\Delta}$. Thus the distribution $|T_{\ell}|^{-1}T_{\sigma_F}V(\overline{\nu}_{F,C}, t_{F,a}; \Delta) * P(\overline{\nu}_C, t_{F,a}; \gamma_{\Delta}^+)$ on T_F is given by the formula

$$(-1)^{\operatorname{rk}(\overline{\nu}_{C,-})} \frac{t^{\ell(\phi_F+\xi_{F,\Delta})+\sigma_F} \operatorname{Ch}^{t_{F,a}}(S\overline{\nu}_C(\gamma_{\Delta}^+) \otimes \det(\overline{\nu}_{C,-})^{-1}, t)}{\mathcal{D}_{\mathbb{C}}^{t_{F,a}}(\overline{\nu}_{F,C}, t)} i(\nu_{F,C}; t_{F,a}t) \delta_{T_{\ell}T_{\Delta}}(t)$$

$$(4.56)$$

By definition $\overline{\nu}_C(\gamma_{\Delta}^+) = \nu_C(\gamma_{\Delta}^+)$ (the polarization γ_{Δ}^+ determines the complex structure on the same underlying real vector bundle). Note that $(\overline{\nu_C})_- = \overline{\nu_{C,+}}$ hence

$$\det(\overline{\nu}_{C,-})^{-1} = \det(\overline{\overline{\nu}_{C,-}}) = \det(\nu_{C,+}).$$

The factor $t_{F,a}^{-\lambda}$ in (4.53) means one must push-forward the distribution defined by (4.56) on T_F to a distribution on T by the map

$$\iota_{F,a}: T_F \hookrightarrow T, \qquad t \mapsto t_{F,a} t.$$

Hence $\operatorname{Ch}^{t_{F,a}}(-,t)$ is replaced with $\operatorname{Ch}^{t_{F,a}}(-,t_{F,a}^{-1}t) = \operatorname{Ch}(-,t)$ and similarly for $\mathcal{D}_{\mathbb{C}}^{t_{F,a}}(-,t)$. It is convenient to change variables, decomposing t into a product of $t \in T_{\ell}$ and $h \in T_{\Delta}$ (this decomposition is not unique). Note that h acts trivially on $\nu_{F,C}$. We have $t^{\ell\xi_{F,\Delta}} = 1$, and

$$h^{\ell(\phi_F + \xi_{F,\Delta}) + \sigma_F} = h^{k\phi_\Delta + \varphi_\Delta - \frac{1}{2}s_\Delta}.$$

where

$$\phi_{\Delta} = \mathrm{pr}_{\mathfrak{t}_{\Delta}^{*}}(\phi_{F} + \xi_{F,\Delta}), \quad \varphi_{\Delta} = \mathrm{pr}_{\mathfrak{t}_{\Delta}^{*}}(\varphi_{F} + \mathbf{h}^{\vee}\xi_{F,\Delta}), \quad s_{\Delta} = \mathrm{pr}_{\mathfrak{t}_{\Delta}^{*}}(s_{F}),$$

and $\operatorname{pr}_{\mathfrak{t}^*_{\Delta}}$ denotes the quotient map $\mathfrak{t}^*_F \to \mathfrak{t}^*_{\Delta}$. Therefore the inverse Fourier transform of $|T_\ell|^{-1} t_{F,a}^{-\lambda} T_{\sigma_F} V(\overline{\nu}_{F,C}, t_{F,a}; \Delta) * P(\overline{\nu}_C, t_{F,a}; \gamma^+_{\Delta})$ is given by

$$(-1)^{\operatorname{rk}(\nu_{C,+})} \frac{(t_{F,a}^{-1}t)^{\ell\phi_{F}+\sigma_{F}}h^{k\phi_{\Delta}+\varphi_{\Delta}-\frac{1}{2}s_{\Delta}}\operatorname{Ch}(S\nu_{C}(\gamma_{\Delta}^{+})\otimes\det(\nu_{C,+}),th)}{\mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F,C},t)}i(\nu_{F,C};t)\delta_{T_{\ell}T_{\Delta}}(th),$$

Recall from the proof of Theorem 4.4.2 that on $t_{F,a}T_F \cap T_\ell$

$$Ch(\mathcal{L}_k, t)^{1/2} Ch(\det(\nu_F), t)^{-1/2} = Ch(\mathcal{L}_k, t_{F,a})^{1/2} Ch(\det(\nu_F), t_{F,a})^{-1/2} (t_{F,a}^{-1}t)^{\ell\phi_F + \sigma_F}.$$

Using (4.23) and the multiplicative behaviour of $\mathcal{D}_{\mathbb{R}}(-)$, we arrive at the following formula for the inverse

Fourier transform of $|T_{\ell}|^{-1} f_{F,a} t_{F,a}^{-\lambda} T_{\sigma_F} V(\overline{\nu}_{F,C}, t_{F,a}; \Delta) * P(\overline{\nu}_C, t_{F,a}; \gamma_{\Delta}^+)$:

$$\mathbf{1}_{\ell_{F,a}}(-1)^{\mathrm{rk}(\nu_{C,+})} \hat{\mathcal{A}}(F) \mathrm{Ch}(\mathcal{L}_{k},t)^{1/2} \mathrm{Ch}(\det(\nu_{C}),t)^{-1/2} \\
\frac{h^{k\phi_{\Delta}+\varphi_{\Delta}-\frac{1}{2}s_{\Delta}} \mathrm{Ch}(S\nu_{C}(\gamma_{\Delta}^{+})\otimes\det(\nu_{C,+}),th)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} i(\nu_{F,C};t) \delta_{T_{\ell}T_{\Delta}}(th).$$

Finally note that

$$(-1)^{\operatorname{rk}(\nu_{C,+})}\operatorname{Ch}(\det(\nu_{C}),t)^{-1/2}h^{\varphi_{\Delta}-\frac{1}{2}s_{\Delta}}\operatorname{Ch}(\det(\nu_{C,+}),th)$$
$$= (-1)^{\operatorname{rk}(\nu_{C})}\operatorname{Ch}(\det(\nu_{C}(\gamma_{\Delta}^{+})),t)^{1/2}h^{\varphi_{\Delta}+\frac{1}{2}s_{\Delta}^{+}}$$

where s_{Δ}^+ is defined similarly to s_{Δ} , but with respect to the γ_{Δ}^+ -polarized complex structure on ν_C : it is the sum of the (γ_{Δ}^+ polarized) weights for the action of T_{Δ} on $\nu_C(\gamma_{\Delta}^+)$.

Assembling the contributions from different fixed-point sets $F \subset C$, we partially undo the separation into different orbit types, initially used in equation (4.44). By summing over all components of $F \subset C^t$ in the final result (4.54), the factor of $\mathbf{1}_{\ell_{F,a}}$ and the indicator function $i(\nu_{F,C};t)$ are incorporated automatically.

According to the lemma, $Q_{\Delta,C} = q_{\Delta,C} \delta_{T_{\ell}T_{\Delta}}$ where $q_{\Delta,C}$ is the generalized function on $T_{\ell}T_{\Delta}$ given by the formula

$$q_{\Delta,C}(th,k) = \prod_{\alpha<0} (1-(th)^{\alpha}) \sum_{F\subset C^t} (-1)^{\operatorname{rk}(\nu_C)} h^{k\phi_{\Delta}+\varphi_{\Delta}+\frac{1}{2}s_{\Delta}^+} \\ \int_F \frac{\hat{A}(F)\operatorname{Ch}(\mathcal{L}_k\otimes \det(\nu_C(\gamma_{\Delta}^+)),t)^{1/2}\operatorname{Ch}(S\nu_C(\gamma_{\Delta}^+),th)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)}$$

Formally, $q_{\Delta,C}$ is the sum of Atiyah-Segal-Singer fixed-point expressions for a $T_{\ell}T_{\Delta}$ -equivariant spinor bundle on C, tensored with the infinite-dimensional \mathbb{Z}_2 -graded vector bundle $\wedge_{\mathbb{C}} \mathfrak{n}_- \otimes S\nu_C(\gamma_{\Delta}^+)$. We now explain this interpretation.

Equipping the normal bundle ν_C with the *T*-equivariant spinor bundle $\wedge_{\mathbb{C}}\nu_C(-\gamma_{\Delta}^+)$, the 2-out-of-3 lemma for spin-c structures yields a T_{ℓ} -equivariant spinor bundle $S_{k,\Delta,C}$ for *TC*, with determinant line bundle $\mathcal{L}_{k,\Delta,C} = \mathcal{L}_k \otimes \det(\nu_C(\gamma_{\Delta}^+))$ having grading $(-1)^{\mathrm{rk}(\nu_C)}$ relative to the grading of \mathcal{L}_k .

Since $C \subset M^{T_{\Delta}}$, a $T_{\ell}T_{\Delta}$ -equivariant extension of the spinor bundle $S_{k,\Delta,C}$ amounts to the choice of a weight for the action of T_{Δ} on the fibres, compatible with the given action of $T_{\Delta} \cap T_{\ell}$. The choice of weight is suggested by the above expression for $Q_{\Delta,C}$: $k\phi_{\Delta} + \varphi_{\Delta} + \frac{1}{2}s_{\Delta}^{+}$.

Note that ϕ_{Δ} , φ_{Δ} depend only on the affine subspace Δ , and not on the earlier choice of lift $\hat{x} \in \pi^{-1}(x)$ $(\phi_{\Delta} = \operatorname{pr}_{\mathfrak{t}^{*}_{\Delta}}(\phi_{F} + \xi_{F,\Delta}))$ and changing the lift \hat{x} does not alter the sum, similarly for φ_{Δ}). Moreover, as ϕ , φ are abstract moment maps, the projections $\operatorname{pr}_{\mathfrak{t}^{*}_{\Delta}} \circ \phi$, $\operatorname{pr}_{\mathfrak{t}^{*}_{\Delta}} \circ \varphi$ are constant on connected components of $\mathcal{N}^{\mathfrak{t}_{\Delta}}$. Thus $k\phi_{\Delta} + \varphi_{\Delta} + \frac{1}{2}s_{\Delta}^{+}$ determines a $T_{\ell}T_{\Delta}$ -equivariant extension of $S_{k,\Delta,C}$. And with this choice we have

$$(-1)^{\operatorname{rk}(\nu_C)}h^{k\phi_{\Delta}+\varphi_{\Delta}+\frac{1}{2}s_{\Delta}^+}\operatorname{Ch}(\mathcal{L}_k\otimes\det(\nu_C(\gamma_{\Delta}^+)),t)^{1/2}=\operatorname{Ch}(\mathcal{L}_{k,\Delta,C},th)^{1/2}$$

The equivariant extension of $S_{k,\Delta,C}$ then determines a $T_{\ell}T_{\Delta}$ -equivariant extension of S_k itself, which we denote $S_{k,\Delta}$ (and $\mathcal{L}_{k,\Delta}$ for the determinant line bundle).

There is a second way to understand the resulting equivariant extension $S_{k,\Delta}$. View Δ as an affine

subspace of $\mathfrak{t}\simeq\mathfrak{t}^*$ and consider the fibre product

Then $\mathcal{C}_{\Delta} \hookrightarrow \mathcal{N}$ and $C = \mathcal{C}_{\Delta}/(\Lambda \cap \mathfrak{t}_{\Delta}^{\perp})$. The restriction $\pi^* S_k|_{\hat{C}_{\Delta}}$ is a *T*-equivariant spinor bundle for $T\mathcal{N}|_{\hat{C}_{\Delta}}$. Inspecting the commutation relation (4.38), the phase $h^{\ell B^{\flat}(\lambda)}$ is trivial for $h \in T_{\ell}T_{\Delta}$ and $\lambda \in \Lambda \cap \mathfrak{t}_{\Delta}^{\perp}$. It follows that the quotient

$$\pi^* S_k|_{\hat{C}_{\Lambda}}/(\Lambda \cap \mathfrak{t}_{\Delta}^{\perp})$$

is a $T_{\ell}T_{\Delta}$ -equivariant spinor bundle $S_{k,\Delta}$ for $TN|_C$. Since ϕ , φ are abstract moment maps, the projections $\operatorname{pr}_{\mathfrak{t}^*_{\Delta}} \circ \phi$, $\operatorname{pr}_{\mathfrak{t}^*_{\Delta}} \circ \varphi$ are constant on \mathcal{C}_{Δ} , and equal ϕ_{Δ} , φ_{Δ} respectively.

The product $\prod_{\alpha < 0} (1 - t^{\alpha})$ is the *T*-equivariant Chern character of the \mathbb{Z}_2 -graded vector space $\wedge_{\mathbb{C}} \mathfrak{n}_-$. We no longer need to factor $t \in T_{\ell}T_{\Delta}$, and thus the formula for $Q_{\Delta,C}$ becomes:

$$Q_{\Delta,C}(t,k) = \sum_{F \subset C^t} \int_F \frac{\hat{A}(F) \mathrm{Ch}(\mathcal{L}_{k,\Delta,C},t)^{1/2} \mathrm{Ch}(\wedge_{\mathbb{C}} \mathfrak{n}_- \otimes S\nu_C(\gamma_{\Delta}^+),t)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} \delta_{T_{\ell}T_{\Delta}}(t).$$
(4.58)

The integrand in (4.58) indeed takes the form of fixed-point contributions for a $T_{\ell}T_{\Delta}$ -equivariant spin-c structure on C, tensored with the infinite-dimensional \mathbb{Z}_2 -graded bundle $\wedge_{\mathbb{C}}\mathfrak{n}_-\otimes S\nu_C(\gamma^+_{\Delta})$.

Theorem 4.4.5. There is a $T_{\ell}T_{\Delta}$ -equivariant spin-c structure $S_{k,\Delta,C}$ on C with determinant line bundle $\mathcal{L}_{k,\Delta} \otimes \det(\nu_C(\gamma_{\Delta}^+))$. In terms of this spin-c structure, equation (4.58) defines a distribution supported on $T_{\ell}T_{\Delta} \subset T$. Its Fourier transform is the function $m_{\Delta,C}$. The contribution $m_{\Delta,C}^{\text{qpol}}$ of the pair Δ, C to equation (4.51) for m is obtained by taking the 'quasi-polynomial germ along Δ ' of $m_{\Delta,C}$ at $(\mathrm{pr}_{\Delta}(\gamma), 1) \in \mathfrak{t}^* \times \mathbb{N}$, i.e. making the replacement $V(-; \Delta) \to \operatorname{Ver}(-; \mathrm{pr}_{\Delta}(\gamma))$ in the formula for $m_{\Delta,C}$.

4.4.4 'Delocalized' index formula.

In this subsection we derive a Berline-Vergne-type 'delocalized' index formula (c.f. [12]) for the quantities $Q_{\Delta,C}$ defined in the previous subsection.

Let K be a compact Lie group acting on a manifold M. Let $E \to M$ be a K-equivariant vector bundle with K-invariant connection ∇^E . The group K acts on the space of sections $\Gamma(E)$ according to the formula

$$(k \cdot \sigma)(m) = k \cdot \sigma(k^{-1}m),$$

and the corresponding Lie derivative operator for an element $X \in \mathfrak{k}$ is

$$\mathcal{L}_X^E \sigma(m) := \frac{d}{dt} \Big|_0 \exp(tX) \cdot \sigma(\exp(-tX)m).$$

For any $X \in \mathfrak{k}$, the difference

$$\mu(X) = \nabla^E_{X_M} - \mathcal{L}^E_X$$

commutes with multiplication by $C^{\infty}(M)$, and thus defines a section of $\operatorname{End}(E)$; the map $\mu : \mathfrak{k} \to \mathfrak{k}$

 $\Gamma(\text{End}(E))$ is known as the moment map of (E, ∇^E) (see [10]). The K-equivariant curvature ([10]) of E is

$$R_K(X) = R - \mu(X),$$

where R is the ordinary curvature of E.⁴ One obtains d_K -closed equivariant extensions of the characteristic classes described in Section 4.2.1 by replacing R with R_K in the definitions. We use the notation Ch(E, X), $\hat{A}(E, X)$ and so on, for these extensions. Note however that $\hat{A}(E, X)$, Td(E, X) are only defined for X in a sufficiently small neighbourhood of $0 \in \mathfrak{k}$. Similarly, for $s \in K$ one defines extensions of the twisted characteristic classes $\mathcal{D}^s_{\mathbb{R}}(E, X)$, $Ch^s(E, X)$ and so on, except that one requires $X \in \mathfrak{k}^s$ (so that X_M is tangent to M^s).

We recall the usual Berline-Vergne formula for the *T*-equivariant index of a spin-c Dirac operator D for the spinor bundle S with determinant line bundle $\mathcal{L} = \operatorname{Hom}_{\operatorname{Cliff}}(S^*, S)$. For $t \in T$ and $X \in \mathfrak{t}$ sufficiently small, the Berline-Vergne formula is

$$\operatorname{index}(D)(t \exp(X)) = \sum_{F \subset M^t} \int_F \frac{\hat{A}(F, \frac{2\pi}{i}X) \operatorname{Ch}^t(\mathcal{L}, \frac{2\pi}{i}X)^{1/2}}{\mathcal{D}_{\mathbb{R}}^t(\nu_F, \frac{2\pi}{i}X)}.$$

This can be checked by applying abelian localization to the right-hand-side and comparing with the Atiyah-Segal-Singer fixed-point formula. The factor of $\frac{2\pi}{i}$ in the arguments is needed to cancel the $\frac{i}{2\pi}$ appearing in the definitions of the characteristic forms.

Recall (Chapter 1, Section 1.3.4) that we have an $N_G(T)$ -equivariant map

$$\phi_{\mathfrak{g}/\mathfrak{t}}: \mathcal{N} \to B \subset \mathfrak{g}/\mathfrak{t} \simeq \mathfrak{t}^{\perp},$$

where B is small ball around the origin. Let $b \in K_T^0(\mathfrak{g}/\mathfrak{t})$ denote the T-equivariant Bott class for the complex structure on $\mathfrak{g}/\mathfrak{t}$ determined by the negative roots. Its pull-back to $0 \in \mathfrak{g}/\mathfrak{t}$ is $\wedge_{\mathbb{C}}\mathfrak{n}_{-} \in K_T^0(\mathrm{pt})$. Let $\mathrm{Ch}^t(b) = \mathrm{Ch}(b,t) \in \Omega((\mathfrak{g}/\mathfrak{t})^t)$ denote a representative of the twisted Chern character of b with compact support contained in B^t . To simplify notation, we continue to denote the pull-back $\phi^*_{\mathfrak{g}/\mathfrak{t}}\mathrm{Ch}(b,t)$ by $\mathrm{Ch}(b,t)$.

Remark 4.4.6. Let V be a complex vector space with a linear action of a torus T, and assume $V^T = \{0\}$ for simplicity. Let $t \in T$ and let $b \in K_T^0(V)$ denote the Bott class of V determined by the complex structure. A representative for $\operatorname{Ch}^t(b) = \operatorname{Ch}(b, t)$ is the following compactly supported form on V^t

$$\operatorname{Ch}^{t}(b) = \operatorname{det}_{\mathbb{C}}^{V/V^{t}} (1 - A(t)) \tau(V^{t}),$$

where $\tau(V^t)$ is a Thom form on V^t for the orientation induced by the complex structure, and $A(t) \in U(V)$ denotes the action of t on V. This form has an equivariant extension given by the formula

$$\operatorname{Ch}^{t}(b, \frac{2\pi}{i}X) = \operatorname{det}_{\mathbb{C}}^{V/V^{t}} \left(1 - A(t)e^{A(X)}\right) \operatorname{det}_{\mathbb{C}}^{V^{t}} \left(\frac{1 - e^{A(X)}}{A(X)}\right) \tau(V^{t}, \frac{2\pi}{i}X),$$

where $X \in \mathfrak{t}$, $A(X) \in \mathfrak{u}(V)$ is the infinitesimal action of X:

$$A(X) = \mathcal{L}_X^V = \frac{d}{dt} \Big|_0 A(\exp(tX)),$$

⁴The sign convention for $\mu(X)$ is opposite that of [10], but the minus sign in the definition for $R_K(X)$ cancels this, hence $R_K(X)$ agrees with the equivariant curvature defined in [10].

and $\tau(V^t, X)$ is an equivariant Thom form. This formula can be checked by pulling back the right-side to $0 \in V^t$, where it becomes

$$\det_{\mathbb{C}}^{V} \left(1 - A(t)e^{A(X)} \right) = \operatorname{Ch}^{t}(\wedge V, \frac{2\pi}{i}X).$$

The construction above yields *T*-equivariant extensions for the characteristic classes appearing in (4.58), except for $\operatorname{Ch}^t(\mathcal{L}_{k,\Delta,C})^{1/2}$ —the obstacle being that, in general, $S_{k,\Delta,C}$ is only $T_{\ell}T_{\Delta}$ -equivariant. To resolve this difficulty we pass to the cover \mathcal{C}_{Δ} .

To simplify the notation, let

$$\mathcal{C} = \mathcal{C}_{\Delta}, \qquad \Lambda_{\Delta} = \Lambda \cap \mathfrak{t}_{\Delta}^{\perp}, \qquad \pi : \mathcal{C} \to C = \mathcal{C}/\Lambda_{\Delta}$$

Let $s \in T$. Since the pullback spinor bundle $\pi^* S_{k,\Delta,C}$ is *T*-equivariant, the form $\operatorname{Ch}^s(\pi^* \mathcal{L}_{k,\Delta,C})^{1/2}$ on \mathcal{C}^s has a d_T -closed equivariant extension $\operatorname{Ch}^s(\pi^* \mathcal{L}_{k,\Delta,C}, X)^{1/2}$; in terms of the moment maps ϕ , φ and a moment map s^+ for $\det(\nu_C(\gamma_{\Delta}^+))$ the extension is

$$\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2} = \operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C})^{1/2}e^{2\pi i \langle k\phi + \varphi + \frac{1}{2}s^{+}, X \rangle}$$

(in order to simplify notation, we omit pullback by π here and below). The form $\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C}, X)^{1/2}$ is nearly Λ_{Δ} -invariant; one has instead

$$\zeta^* \mathrm{Ch}^s(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2} = (s \exp(X))^{-\ell B^\flat(\zeta)} \mathrm{Ch}^s(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2}, \qquad \zeta \in \Lambda_\Delta.$$
(4.59)

This follows from equation (4.38) and the definition.

Let $s \in T$ and $X \in \mathfrak{t}$ sufficiently near $0 \in \mathfrak{t}$. For short, write

$$\alpha_s(X) = \operatorname{Ch}^s(\mathcal{L}_{k,\Delta,C}, X)^{1/2} \operatorname{Ch}^s(b \otimes S\nu_C(\gamma_{\Delta}^+), X).$$

By equation (4.59), we have

$$\zeta^* \alpha_s(\frac{2\pi}{i}X) = (s\exp(X))^{-\ell B^\flat(\zeta)} \alpha_s(\frac{2\pi}{i}X), \qquad \zeta \in \Lambda_\Delta.$$
(4.60)

The following is an adaptation of the Berline-Vergne 'delocalized' index formula (c.f. [12]).

Theorem 4.4.7. The expression

$$Q_{\Delta,C,s}(X,k) = \int_{\mathcal{C}^s} \frac{\hat{A}(\mathcal{C}^s, \frac{2\pi}{i}X)\alpha_s(\frac{2\pi}{i}X)}{\mathcal{D}^s_{\mathbb{R}}(\nu_{\mathcal{C}^s,\mathcal{C}}, \frac{2\pi}{i}X)}.$$
(4.61)

defines a distribution on a neighbourhood of $0 \in \mathfrak{t}$. Moreover, its push-forward under the map $s : X \in \mathfrak{t} \mapsto s \exp(X) \in T$ agrees with $Q_{\Delta,C}$ on a small neighbourhood of $s \in T$.

- *Remark* 4.4.8. 1. The integration is over a possibly non-compact manifold C^s , but the integral converges in the sense of distributions.
 - 2. Recall that C^s is a q-Hamiltonian G^s -space, and thus has a 'twisted' spin-c structure: a Morita morphism $\operatorname{Cliff}(TC^s) \to \mathcal{A}_{G^s}^{\operatorname{Spin}}$. The image of the moment map $\Phi(C^s)$ is contained in a tubular neighbourhood of the maximal torus in G^s , and thus $\mathcal{A}_{G^s}^{\operatorname{Spin}}$ has a Morita trivialization (not

equivariant). Thus the fixed-point set C^s is spin-c (in particular, it is oriented).

We outline the proof for completeness, omitting some details since a part of the argument is well-known (c.f. [10], [12]).

Proof. Choose a fundamental domain $\mathcal{C}_0^s \subset \mathcal{C}^s$ for the action of Λ_{Δ} . Then using (4.60),

$$Q_{\Delta,C,s}(X,k) = \sum_{\zeta \in \Lambda_{\Delta}} (s \exp(X))^{-\ell B^{\flat}(\zeta)} \int_{\mathcal{C}_{0}^{s}} \frac{\hat{A}(\mathcal{C}^{s}, \frac{2\pi}{i}X)\alpha_{s}(\frac{2\pi}{i}X)}{\mathcal{D}_{\mathbb{R}}^{s}(\nu_{\mathcal{C}^{s}, \mathcal{C}}, \frac{2\pi}{i}X)}.$$

Choose X_s such that $\exp(X_s) = s$. By the Poisson summation formula (c.f. Lemma 3.1.5 for the variation used here), this is equal to

$$\sum_{U \in \ell^{-1}B^{\sharp}(\Lambda^{*})/\mathfrak{t}_{\Delta}} \delta_{U}(X+X_{s}) \int_{\mathcal{C}_{0}^{s}} \frac{\hat{A}(\mathcal{C}^{s}, \frac{2\pi}{i}X)\alpha_{s}(\frac{2\pi}{i}X)}{\mathcal{D}_{\mathbb{R}}^{s}(\nu_{\mathcal{C}^{s}, \mathcal{C}}, \frac{2\pi}{i}X)}.$$
(4.62)

In this expression U is viewed as a coset $U = \mu + \mathfrak{t}_{\Delta}, \ \mu \in \ell^{-1}B^{\sharp}(\Lambda^*)$; the normalization of δ_U is induced from normalized Haar measure on the compact (disconnected) group $T_{\ell}T_{\Delta} = \exp(\ell^{-1}B^{\sharp}(\Lambda^*) + \mathfrak{t}_{\Delta})$.

To see that (4.62) defines a distribution on a neighbourhood of $0 \in \mathfrak{t}$, it is enough to show that a product of distributions of the form

$$\delta_{\mathfrak{t}_{\Delta}} \cdot \dot{P}(\boldsymbol{\alpha}; \gamma_{\Delta}^{+})$$

is defined. The Fourier transform is a convolution

$$\delta_{\mathfrak{t}_{\Delta}^{\perp}} * P(\boldsymbol{\alpha}; \gamma_{\Delta}^{+}).$$

The partition function $P(\boldsymbol{\alpha}; \gamma_{\Delta}^+)$ is supported in a translate of an acute cone $K_{\boldsymbol{\alpha}}$ contained in the half-space $\{\xi | \langle \xi, \gamma_{\Delta}^+ \rangle \geq 0\}$. Since γ_{Δ}^+ is orthogonal to $\mathfrak{t}_{\Delta}^\perp$, the map

$$K_{\alpha} \times \mathfrak{t}_{\Delta}^{\perp} \to \mathfrak{t}, \qquad (\xi_1, \xi_2) \mapsto \xi_1 + \xi_2$$

is proper.

The delta distributions in (4.62) force $X + X_s$ to lie in $\mathfrak{t}_{\Delta} + \ell^{-1}B^{\sharp}(\Lambda^*) \subset \mathfrak{t}$. For such X, $\ell B(\zeta, X + X_s)$ is an integer, for all $\zeta \in \Lambda_{\Delta} = \Lambda \cap \mathfrak{t}_{\Delta}^{\perp}$. Thus equation (4.60) gives

$$\zeta^* \alpha_s(\frac{2\pi}{i}X) = e^{-2\pi i \ell B(\zeta, X+X_s)} \alpha_s(\frac{2\pi}{i}X) = \alpha_s(\frac{2\pi}{i}X), \qquad \zeta \in \Lambda_\Delta$$

This shows that for $X + X_s \in \mathfrak{t}_{\Delta} + \ell^{-1}B^{\sharp}(\Lambda^*)$, the integrand descends to the quotient $\mathcal{C}^s/\Lambda_{\Delta} = C^s$. Hence (4.62) simplifies:

$$Q_{\Delta,C,s}(X,k) = \sum_{U \in \ell^{-1}B^{\sharp}(\Lambda^*)/\mathfrak{t}_{\Delta}} \delta_U(X+X_s) \int_{C^s} \frac{\dot{A}(C^s, \frac{2\pi}{i}X)\alpha_s(\frac{2\pi}{i}X)}{\mathcal{D}^s_{\mathbb{R}}(\nu_{C^s,C}, \frac{2\pi}{i}X)}.$$
(4.63)

The manifold C^s can still be non-compact, but the Chern character of the Bott element has compact support, ensuring that the integrand is compactly supported. The integrand is closed for the equivariant differential $d_{-2\pi i X} = d - \frac{2\pi}{i} \iota(X_{C^s})$, and thus localizes to the fixed-point set of the vector field generated by X on C^s . Let $t = s \exp(X)$. For X sufficiently small we have

$$(C^s)^X = C^t$$

Applying the abelian localization formula, the integral in (4.63) becomes

$$\sum_{F \subset C^t} \int_F \frac{\hat{A}(C^s, \frac{2\pi}{i}X)\alpha_s(\frac{2\pi}{i}X)}{\mathcal{D}^s_{\mathbb{R}}(\nu_{C^s, C}, \frac{2\pi}{i}X)\mathrm{Eul}(\nu_{F, C^s}, \frac{2\pi}{i}X)}.$$
(4.64)

One has

$$\frac{\iota_F^* \hat{\mathcal{A}}(C^s, \frac{2\pi}{i}X)}{\operatorname{Eul}(\nu_{F,C^s}, \frac{2\pi}{i}X)} = \frac{\hat{\mathcal{A}}(F)}{\mathcal{D}_{\mathbb{R}}(\nu_{F,C^s}, \frac{2\pi}{i}X)}$$

(This follows easily from the formulas in terms of 'Chern roots', Section 4.2.1.) We write $\tilde{\nu}_{F,C}$ for the normal bundle to F inside C, in order to agree with the notation used in Sections 4.4.1-4.4.3. Equation (4.64) becomes

$$\sum_{F \subset C^t} \int_F \frac{\hat{A}(F)\alpha_s(\frac{2\pi}{i}X)}{\mathcal{D}_{\mathbb{R}}^s(\tilde{\nu}_{F,C}, \frac{2\pi}{i}X)}.$$
(4.65)

Along the fixed-point set of X, the equivariant curvature $R_T(X)$ of a vector bundle E simplifies to $R + \mathcal{L}_X^E$. Hence

$$e^{\frac{i}{2\pi}R_T(\frac{2\pi}{i}X)} = e^{\frac{i}{2\pi}R}\exp(\mathcal{L}_X^E) = A_E(\exp(X))e^{\frac{i}{2\pi}R},$$

where $A_E(\exp(X)) \in \operatorname{End}(E)$ denotes the fibre-wise action of $\exp(X)$ on E. In particular, pulled back to F one has

$$\operatorname{Ch}^{s}(b \otimes S\nu_{C}(\gamma_{\Delta}^{+}), \frac{2\pi}{i}X) = \operatorname{Ch}(b \otimes S\nu_{C}(\gamma_{\Delta}^{+}), t).$$

For the forms $\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2}$ and $\mathcal{D}^{s}_{\mathbb{R}}(\tilde{\nu}_{F,C}, \frac{2\pi}{i}X)$ one must be more careful, because of the choices of square roots involved. Nevertheless for the quotient

$$\frac{\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2}}{\mathcal{D}_{\mathbb{R}}^{s}(\tilde{\nu}_{F,C}, \frac{2\pi}{i}X)} = \frac{\operatorname{Ch}(\mathcal{L}_{k,\Delta,C}, t)^{1/2}}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C}, t)}.$$
(4.66)

One can verify this by a calculation similar to those in Sections 4.4.1-4.4.3:

The normal bundle decomposes

$$\tilde{\nu}_{F,C} = \nu_{F,C} \oplus \nu'_{F,C}, \qquad \nu'_{F,C} = (\tilde{\nu}_{F,C})^X$$

The action of X on the sub-bundle $\nu'_{F,C}$ is trivial, hence

$$\mathcal{D}^s_{\mathbb{R}}(\nu'_{F,C}, \frac{2\pi}{i}X) = \mathcal{D}_{\mathbb{R}}(\nu'_{F,C}, s) = \mathcal{D}_{\mathbb{R}}(\nu'_{F,C}, t).$$

On the other hand, the sub-bundle $\nu_{F,C}$ can be equipped with a complex structure (for example, the complex structure can be chosen such that the complex eigenvalues for the action $A_{\nu_{F,C}}(X)$ of X on $\nu_{F,C}$ are of the form *ia* with a > 0). By (4.23), and using $\mathcal{D}^s_{\mathbb{C}}(\overline{\nu}_{F,C}, \frac{2\pi}{i}X) = \mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F,C}, t)$,

$$\frac{1}{\mathcal{D}^s_{\mathbb{R}}(\nu_{F,C}, \frac{2\pi}{i}X)} = \frac{\operatorname{Ch}^s(\det(\nu_{F,C}), \frac{2\pi}{i}X)^{-1/2}}{\mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F,C}, t)}.$$

Choose $x \in F$ and a lift $\hat{x} \in \pi^{-1}(x)$. Equip $T_x C \simeq T_{\hat{x}} \mathcal{C}$ with a complex structure extending the complex structure on $\nu_{F,C}|_x$. Then

$$\pi^* S_{k,\Delta,C}|_{\hat{x}} = E \otimes \wedge_{\mathbb{C}} T_{\hat{x}} \mathcal{C},$$

where E is a Hermitian line equipped with a 1-dimensional representation ρ_E of the stabilizer T_x of x in T. Let μ denote the spin-c moment map for $\pi^* S_{k,\Delta,C}$ and $\mu_F = \mu(\hat{x})$. Then

$$\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C},\frac{2\pi}{i}X)^{1/2} = \operatorname{Ch}(\mathcal{L}_{k,\Delta,C})^{1/2}\rho_{E}(s)e^{2\pi i \langle \mu_{F},X \rangle} \operatorname{det}_{\mathbb{C}}(A_{\tilde{\nu}_{F,C}}(s)^{1/2}).$$

Let s_F be the weight of the representation of the identity component of T_x on det_C($\nu_{F,C}$), hence

$$\operatorname{Ch}^{s}(\operatorname{det}_{\mathbb{C}}(\nu_{F,C}), \frac{2\pi}{i}X)^{-1/2} = \operatorname{Ch}(\operatorname{det}_{\mathbb{C}}(\nu_{F,C}))^{-1/2} e^{-2\pi i \langle \frac{1}{2}s_{F}, X \rangle} \operatorname{det}_{\mathbb{C}}(A_{\nu_{F,C}}(s)^{-1/2}).$$

Thus

$$\frac{\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2}}{\mathcal{D}_{\mathbb{R}}^{s}(\tilde{\nu}_{F,C}, \frac{2\pi}{i}X)} = \operatorname{Ch}(\mathcal{L}_{k,\Delta,C} \otimes \det(\nu_{F,C})^{-1})^{1/2} \\
\frac{\rho_{E}(s)e^{2\pi i\langle\mu_{F}} - \frac{1}{2}s_{F},X\rangle}{\mathcal{D}_{\mathbb{R}}(\nu_{F,C}', t)\mathcal{D}_{\mathbb{C}}(\overline{\nu}_{F,C}, t)}.$$
(4.67)

Since X acts trivially on $\nu'_{F,C}$

$$\det_{\mathbb{C}}(A_{\nu'_{F,C}}(s)^{1/2}) = \det_{\mathbb{C}}(A_{\nu'_{F,C}}(t)^{1/2}).$$

Also

$$e^{2\pi i \langle \mu_F - \frac{1}{2} s_F, X \rangle} = \rho_E(\exp(X)).$$

Hence equation (4.67) further simplifies

$$\frac{\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C},\frac{2\pi}{i}X)^{1/2}}{\mathcal{D}_{\mathbb{R}}^{s}(\tilde{\nu}_{F,C},\frac{2\pi}{i}X)} = \operatorname{Ch}(\mathcal{L}_{k,\Delta,C}\otimes\det(\nu_{F,C})^{-1})^{1/2}\frac{\rho_{E}(t)\det_{\mathbb{C}}(A_{\nu'_{F,C}}(t)^{1/2})}{\mathcal{D}_{\mathbb{R}}(\bar{\nu}'_{F,C},t)\mathcal{D}_{\mathbb{C}}(\nu_{F,C},t)}$$

This agrees with the right-side of equation (4.66) when the latter is expressed in terms of the same complex structure on $\nu_{F,C}$.

Thus after applying abelian localization, equation (4.63) simplifies to

$$Q_{\Delta,C,s}(X,k) = \sum_{U \in \ell^{-1}B^{\sharp}(\Lambda^*)/\mathfrak{t}_{\Delta}} \delta_U(X+X_s) \sum_{F \subset C^t} \int_F \frac{\hat{A}(F)\mathrm{Ch}(\mathcal{L}_{k,\Delta,C},t)^{1/2}\mathrm{Ch}(b \otimes S\nu_C(\gamma_{\Delta}^+),t)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)}.$$
 (4.68)

The remaining difference between equations (4.58) and (4.68) is the appearance of the Chern character of the Bott element b in (4.68).

Consider a coset $tT_{\Delta} \subset T_{\ell}T_{\Delta}$ with $t \in T_{\ell}$, and let $F \subset C^t$. If tT_{Δ} contains a regular element $h \in T$, then $F \subset C^h \subset \Phi^{-1}(G^h) = \Phi^{-1}(T)$. If this is the case then F is compact, and the Bott element b can be replaced by its pullback to $0 \in \mathfrak{g/t}$, which is $\wedge_{\mathbb{C}}\mathfrak{n}_{-} \in K^0_T(\operatorname{pt})$. Comparing equations (4.58) and (4.68), this shows that the distributions $s_*Q_{\Delta,C,s}$ and $Q_{\Delta,C}$ agree on tT_{Δ} , and hence on a neighbourhood of $t \in T$ (since both are delta-type distributions supported on $T_{\ell}T_{\Delta}$).

On the other hand, if tT_{Δ} does not contain any regular points, then t must be fixed by some non-

trivial reflection $w \in W$, which also fixes T_{Δ} . Choose a representative $g \in N_G(T)$ for w, which defines a map $\operatorname{Ad}_g : B^t \to B^t$ (recall B is an open ball around 0 in $\mathfrak{g}/\mathfrak{t}$). Recall that for G simply connected, the group G^h is connected for any $h \in G$. Thus G^t is a connected subgroup of G with maximal torus T, and $g \in G^t$. It follows that $\operatorname{Ad}_g : \mathfrak{g}^t \to \mathfrak{g}^t$ is orientation-preserving. As $\mathfrak{g}^t = \mathfrak{t} \oplus (\mathfrak{g}/\mathfrak{t})^t$ and since Ad_g acts as the reflection w on \mathfrak{t} , Ad_g is orientation-reversing on $(\mathfrak{g}/\mathfrak{t})^t$, hence also for the open subset B^t . Thus under pull-back by Ad_g , one has (c.f. Remark 4.4.6)

$$\operatorname{Ad}_{q}^{*}\operatorname{Ch}(b,t) = -\operatorname{Ch}(b,t), \qquad (4.69)$$

in compactly-supported cohomology.

The group $(G^t)^{T_{\Delta}}$ acts on C^t and is connected, since $(G^t)^{T_{\Delta}} = G^{tt'}$ with t' a topological generator of T_{Δ} . It follows that $(G^t)^{T_{\Delta}}$ acts trivially on compactly-supported cohomology on C^t . Since $g \in (G^t)^{T_{\Delta}}$ and as $\phi_{\mathfrak{g}/\mathfrak{t}}$ is $N_G(T)$ -equivariant, it follows that

$$\phi^*_{\mathfrak{g}/\mathfrak{t}}\mathrm{Ch}(b,t) = g^*\phi^*_{\mathfrak{g}/\mathfrak{t}}\mathrm{Ch}(b,t) = \phi^*_{\mathfrak{g}/\mathfrak{t}}\mathrm{Ad}^*_g\mathrm{Ch}(b,t) = -\phi^*_{\mathfrak{g}/\mathfrak{t}}\mathrm{Ch}(b,t)$$

in compactly-supported cohomology, where we used (4.69). Thus $\phi_{\mathfrak{g}/\mathfrak{t}}^* \mathrm{Ch}(b,t) = 0$ in compactlysupported cohomology. Applying this to the right-side of (4.68) shows that for such cosets tT_{Δ} , the right-side vanishes.

4.4.5 Cancellations.

Recall the description of the critical set of the norm-square of the moment map for the Hamiltonian LG-space $\Psi : \mathcal{M} \to L\mathfrak{g}^*$:

$$\operatorname{Crit}(\|\Psi\|^2) = G \cdot \bigcup_{\beta \in \mathcal{B}} \mathcal{M}^{\beta} \cap \Psi^{-1}(\beta),$$

where $\mathcal{B} = \{\beta \in \mathfrak{t}_+ | \mathcal{M}^\beta \cap \Psi^{-1}(\beta) \neq \emptyset\}$ is a discrete subset.

Equation (4.51)

$$m = \sum_{\Delta,C} m_{\Delta,C}^{\text{qpol}} \tag{4.70}$$

is the desired 'norm-square localization formula' for the multiplicity function. To justify this statement, we will show that the non-zero terms of (4.70) are indexed by the set $W \cdot \mathcal{B}$, where W is the Weyl group. More precisely, we show the following:

Theorem 4.4.9. Assume the perturbation γ used in the decomposition formula is sufficiently small. Let β be the nearest point to 0 on the affine subspace Δ . The contribution $m_{\Delta,C}^{\text{qpol}}$ is zero unless $\exp(\beta) \in \Phi(C)$.

Remark 4.4.10. Recall $X = \Phi^{-1}(T)$, and the covering space $\mathcal{X} = \mathfrak{t} \times_T X$ can be identified with $\Psi^{-1}(\mathfrak{t}^*) \subset \mathcal{M}$. Since $\beta \in \mathfrak{t}_{\Delta}$ (identifying $\mathfrak{t} \simeq \mathfrak{t}^*$ using the inner product), we have $C \cap X \subset X^{\beta}$. Thus Theorem 4.4.9 implies $\beta \in W \cdot \mathcal{B}$.

Following the strategy in [60], we deduce Theorem 4.4.9 from the 'delocalized' formulas derived in the previous subsection combined with a stationary phase argument $(k \to \infty)$. The first step is to convert the problem to one involving the $m_{\Delta,C}$ instead of the $m_{\Delta,C}^{\text{qpol}}$. This follows from three observations:

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1. It is enough to show that $m_{\Delta,C}^{\text{qpol}}$ vanishes on each subset of the form

$$\Delta(\delta) := \bigcup_{\ell} (\ell \Delta + \delta) \times \{\ell\} \subset \mathfrak{t}^* \times \mathbb{N}, \qquad \delta \in \Lambda^*.$$
(4.71)

On the other hand, $m_{\Delta,C}^{\text{qpol}}$ is quasi-polynomial on $\Delta(\delta)$. One advantage of restricting to $\Delta(\delta)$ is that only a finite-dimensional sub-bundle of $S\nu_C$ will contribute, i.e. there is some n (depending on δ) such that we might as well replace $S\nu_C$ with $S^n\nu_C$, the sum of the symmetric powers up to and including the n^{th} symmetric power. One way to see this is because the weights for the action of T_{Δ} on $\nu_C(\gamma_{\Delta}^+)$ form a pointed cone, and thus the minimum value of $\langle \alpha, \gamma_{\Delta}^+ \rangle$ where α ranges over the weights of the T_{Δ} -action on the n^{th} symmetric power, increases linearly with n. In particular, for sufficiently large n it is much larger than the constant $\langle \Delta + \delta, \gamma_{\Delta}^+ \rangle$, and this implies that the support of the contribution of the n^{th} symmetric power of $\nu_C(\gamma_{\Delta}^+)$ to $m_{\Delta,C}^{\text{qpol}}$ will not intersect $\Delta(\delta)$. New functions $m_{\Delta,C,n}^{\text{qpol}}$, $m_{\Delta,C,n}$ are defined as before, but replacing $S\nu_C$ with $S^n\nu_C$.

2. Suppose we are able to show that $m_{\Delta,C,n}^{\text{qpol}}$ decays as $k \to \infty$ on some open cone of the form

$$\bigcup_{\ell} \ell \mathfrak{b} \times \{\ell\},$$

where \mathfrak{b} is an open neighbourhood of γ_{Δ} in \mathfrak{t}^* . This open cone intersects $\Delta(\delta)$ in an open cone (in a lower dimensional subspace). Thus $m_{\Delta,C,n}^{\text{qpol}}$ decays as $k \to \infty$ on an open cone in $\Delta(\delta)$, and since $m_{\Delta,C,n}^{\text{qpol}}$ is quasi-polynomial on this subset, it must be identically zero.

3. There is some neighbourhood \mathfrak{b} of γ_{Δ} in \mathfrak{t}^* such that (see Proposition 4.2.8)

$$m_{\Delta,C,n}^{\text{qpol}}(\lambda,k) = m_{\Delta,C,n}(\lambda,k), \qquad \lambda \in \ell \mathfrak{b}, \quad k \ge 1.$$

So instead it is enough to show that $m_{\Delta,C,n}$ decays on an open cone of this type.

From the observations above, Theorem 4.4.9 follows from:

Lemma 4.4.11. Let $m_{\Delta,C,n}$ be as defined above. Assume the perturbation γ used in the decomposition formula is sufficiently small. Let β be the nearest point to 0 on the affine subspace Δ , and suppose $\exp(\beta) \notin \Phi(C)$. Then there is an open neighbourhood \mathfrak{b} of γ_{Δ} such that $m_{\Delta,C,n}$ decays on the set

$$\bigcup_{\ell\in\mathbb{N}}\ell\mathfrak{b}\times\{\ell\}\subset\mathfrak{t}^*\times\mathbb{N}.$$

Proof. Choose a finite open cover $\{U_s | s \in S\}$ of T by neighbourhoods as in Theorem 4.4.7, and let χ_s be bump functions on \mathfrak{t} such that $\{s_*\chi_s | s \in S\}$ is a partition of unity subordinate to this open cover. By Theorem 4.4.7

$$Q_{\Delta,C} = \sum_{s \in S} s_*(\chi_s Q_{\Delta,C,s})$$

Let dX denote Lebesgue measure on \mathfrak{t} , with normalization compatible with normalized Haar measure on T. For the multiplicity function

$$m_{\Delta,C}(\lambda,k) = \sum_{s \in S} \int_{\mathfrak{t}} \chi_s(X) (s \exp(X))^{-\lambda} Q_{\Delta,C,s}(X,k) dX.$$

We have the corresponding versions where $S\nu_C$ is replaced with $S^n\nu_C$. For short, define $\mathcal{E} = b \otimes S^n\nu_C(\gamma^+_{\Delta})$. Then

$$m_{\Delta,C,n}(\lambda,k) = \sum_{s \in S} \int_{\mathfrak{t}} \chi_s(X) s^{-\lambda} e^{-2\pi i \langle \lambda, X \rangle} Q_{\Delta,C,s,n}(X,k) dX,$$

where, as in the proof of Theorem 4.4.7,

$$Q_{\Delta,C,s,n}(X,k) = \sum_{\zeta \in \Lambda_{\Delta}} s^{\ell B^{\flat}(\zeta)} e^{2\pi i \ell B(\zeta,X)} \int_{\mathcal{C}_{0}^{s}} \frac{\hat{A}(C^{s}, \frac{2\pi}{i}X) Ch^{s}(\mathcal{L}_{k,\Delta,C}, \frac{2\pi}{i}X)^{1/2} Ch^{s}(\mathcal{E}, \frac{2\pi}{i}X)}{\mathcal{D}_{\mathbb{R}}^{s}(\nu_{C^{s},C}, \frac{2\pi}{i}X)}$$

Expanding in powers of k we have

$$\operatorname{Ch}^{s}(\mathcal{L}_{k,\Delta,C},\frac{2\pi}{i}X)^{1/2} = e^{2\pi i \langle k\phi + \varphi + \frac{1}{2}s_{\Delta}^{+},X \rangle} u_{s}^{k} \sum_{j=0}^{\dim(C)/2} \omega_{j}k^{j}$$

where u_s is the phase factor for the action of s on L (a locally constant function on C_0^s), and ω_j is a (possibly inhomogeneous) form which does not depend on k. Let

$$\overline{\varphi} = \varphi - \mathbf{h}^{\vee}\phi + \frac{1}{2}s_{\Delta}^{+}$$

and note this is invariant under translations by elements of Λ_{Δ} . Define

$$\nu_{s,j}(X) = \chi_s(X) \frac{\hat{\mathcal{A}}(C^s, \frac{2\pi}{i}X)e^{2\pi i\langle\overline{\varphi}, X\rangle}\omega_j \mathrm{Ch}^s(\mathcal{E}, \frac{2\pi}{i}X)}{\mathcal{D}_{\mathbb{R}}^s(\nu_{C^s, C}, \frac{2\pi}{i}X)},$$

a differential form, smooth and compactly supported in X, which is independent of k. (The reason for replacing $S\nu_C$ with $S^n\nu_C$ was in order that $\nu_{s,j}(X)$ is smooth in X. This allows us to apply the stationary phase principle below.)

The sum over $\zeta \in \Lambda_{\Delta}$ and integral over t can be exchanged, by definition of the distribution $Q_{\Delta,C,s,n}$. Then

$$m_{\Delta,C,n}(\lambda,k) = \sum_{s,j} k^j \sum_{\zeta \in \Lambda_\Delta} s^{\ell B^{\flat}(\zeta) - \lambda} \int_{\mathcal{C}_0^s} u_s^k \int_{\mathfrak{t}} dX e^{2\pi i \ell \langle \phi + B^{\flat}(\zeta) - \lambda/\ell, X \rangle} \nu_{s,j}(X).$$
(4.72)

Let S be the support of $\operatorname{Ch}(b)$ in $\mathfrak{g}/\mathfrak{t}$. Then $S' = \phi_{\mathfrak{g}/\mathfrak{t}}^{-1}(S) \cap C$ is compact, and its image $\Phi_a(S') \subset T$ is a compact set containing $\Phi_a(X \cap C) = \Phi(C) \cap T$. Since $\exp(\beta) \notin \Phi(C)$, by taking the support of $\operatorname{Ch}(b)$ sufficiently small, we can ensure that $\exp(\beta) \notin \Phi_a(S')$. Taking γ generic and sufficiently small, one can ensure γ_{Δ} and β are arbitrarily close, and in particular we can ensure that $\exp(\gamma_{\Delta}) \notin \Phi_a(S')$ either. Pulling back to C, it follows that the pull-back of $\operatorname{Ch}(b)$ vanishes on $\phi^{-1}(\mathfrak{b}') \cap C$, where \mathfrak{b}' is an open neighbourhood of γ_{Δ} in \mathfrak{t}^* . Let \mathfrak{b} be a smaller open neighbourhood, with closure contained in \mathfrak{b}' . The distance between points of \mathfrak{b} and points outside \mathfrak{b}' is bounded below by some $\epsilon > 0$.

Suppose $\lambda \in \ell \mathfrak{b} \Rightarrow \lambda/\ell \in \mathfrak{b}$. Since $\nu_{s,j}$ vanishes on $\phi^{-1}(\mathfrak{b}') \cap \mathcal{C}^s$, on the support of $\nu_{s,j}$ we have a lower bound $|\phi - \lambda/\ell| > \epsilon > 0$. Choose a closed fundamental domain Δ_0 for the action of Λ_{Δ} on Δ that contains β in its interior, and choose $\mathcal{C}_0 = \phi^{-1}(\Delta_0)$. Thus on the support of $\nu_{s,j}$

$$|\phi + B^{\flat}(\zeta) - \lambda/\ell| > \epsilon, \qquad \zeta \in \Lambda_{\Delta}.$$

For $\zeta \in \Lambda_{\Delta}$ outside of some closed ball D and for $x \in \mathcal{C}_0^s$ we have a stronger bound:

$$|\phi(x) + B^{\flat}(\zeta) - \lambda/\ell| > a|\zeta|,$$

where 0 < a < 1 is a constant. Then (4.72) can be split into two parts: a finite sum

$$m_{\Delta,C,n}^{(1)}(\lambda,k) = \sum_{s,j} k^j \sum_{\zeta \in \Lambda_\Delta \cap D} s^{\ell B^{\flat}(\zeta) - \lambda} \int_{\mathcal{C}_0^s} u_s^k \int_{\mathfrak{t}} dX e^{2\pi i \ell \langle \phi + B^{\flat}(\zeta) - \lambda/\ell, X \rangle} \nu_{s,j}(X), \tag{4.73}$$

and an infinite tail

$$m_{\Delta,C,n}^{(2)}(\lambda,k) = \sum_{s,j} k^j \sum_{\zeta \in \Lambda_\Delta \setminus \Lambda_\Delta \cap D} s^{\ell B^{\flat}(\zeta) - \lambda} \int_{\mathcal{C}_0^s} u_s^k \int_{\mathfrak{t}} dX e^{2\pi i \ell \langle \phi + B^{\flat}(\zeta) - \lambda/\ell, X \rangle} \nu_{s,j}(X).$$
(4.74)

In equation (4.73) we use the weak lower bound $|\phi + B^{\flat}(\zeta) - \lambda/\ell| > \epsilon$ on the support of $\nu_{s,j}$. At a fixed point $x \in C_0^s$, the principal of stationary phase gives a bound for the integral over t of the form $C_N(x)(\ell\epsilon)^{-N}$, where $N \in \mathbb{N}$ can be chosen, and $C_N(x)$ is a constant depending on x and N. As the support of the integrand restricted to the fundamental domain \mathcal{C}_0^s is compact, the constant can be taken to be *uniform* in x, and hence we obtain a bound of the form

$$|m_{\Delta,C,n}^{(1)}(\lambda,k)| \le \frac{c_N}{\ell^{N-\dim(C)/2}}$$

Taking $N > \dim(C)/2$ shows that $m_{\Delta,C,n}^{(1)}$ decays as $\ell \to \infty$.

In equation (4.74) we use the stronger lower bound $|\phi + B^{\flat}(\zeta) - \lambda/\ell| > a|\zeta|$ on the support of $\nu_{s,j}$. Arguing similar to the last paragraph, we obtain a bound

$$|m_{\Delta,C,n}^{(2)}(\lambda,k)| \leq \frac{c_N a^{-N}}{\ell^{N-\dim(C)/2}} \sum_{\zeta \in \Lambda_\Delta \setminus \Lambda_\Delta \cap D} \frac{1}{|\zeta|^N} \leq \frac{c'_N}{\ell^{N-\dim(C)/2}}.$$

This shows that $m_{\Delta,C,n}^{(2)}$ also decays as $\ell \to \infty$.

To summarize, in this section we have shown:

Theorem 4.4.12. The multiplicity function $m(\lambda, k)$ admits a decomposition

$$m = \sum_{\Delta,C} m_{\Delta,C}^{\rm qpol}$$

where $C \subset M^{T_{\Delta}}$. The contribution $m_{\Delta,C}^{\text{qpol}}$ vanishes unless $\mathcal{M}^{\beta} \cap \Psi^{-1}(\beta) \neq \emptyset$, where $\beta = \text{pr}_{\Delta}(0)$. Furthermore $m_{\Delta,C}^{\text{qpol}}$ is obtained by taking 'quasi-polynomial germs along Δ ' of a multiplicity function $m_{\Delta,C} = \mathcal{F}(Q_{\Delta,C})$, where

$$Q_{\Delta,C}(t,k) = \sum_{F \subset C^t} \int_F \frac{\hat{A}(F) \mathrm{Ch}(\mathcal{L}_{k,\Delta,C},t)^{1/2} \mathrm{Ch}(\wedge_{\mathbb{C}} \mathfrak{n}_- \otimes S\nu_C(\gamma_{\Delta}^+),t)}{\mathcal{D}_{\mathbb{R}}(\tilde{\nu}_{F,C},t)} \delta_{T_{\ell}T_{\Delta}}(t).$$

 $Q_{\Delta,C}(-,k) = q_{\Delta,C}(-,k)\delta_{T_{\ell}T_{\Delta}}$ is a distribution supported on $T_{\ell}T_{\Delta} \subset T$; the generalized function $q_{\Delta,C}(-,k)$ on $T_{\ell}T_{\Delta}$ takes the form of fixed-point contributions for a $T_{\ell}T_{\Delta}$ -equivariant spin-c structure $S_{k,\Delta,C}$ on C, tensored with the infinite-dimensional \mathbb{Z}_2 -graded bundle $\wedge_{\mathbb{C}} \mathfrak{n}_{-} \otimes S\nu_{C}(\gamma_{\Delta}^{+})$.

4.4.6 Example: S^4 .

Our first example is the 4-sphere, a quasi-Hamiltonian SU(2)-space, c.f. [42] for details on its construction. See [35] for the analogous discussion for Duistermaat-Heckman distributions.

The 4-sphere is prequantizable at any level $k \in \mathbb{N}$. We identify $T \subset SU(2)$ with the circle S^1 , and write the character corresponding to $m \in \mathbb{Z}$ as t^m . The fixed-point formula is discussed in detail in [42], where it is shown that

$$Q(S^4)(t,k) = \frac{1 - t^{\ell}}{(1 - t)(1 - t^{-1})}, \qquad \ell = k + \mathbf{h}^{\vee} = k + 2.$$

Identify $\mathbf{t} = \mathbf{t}^* = \mathbb{R}$ with pairing given by multiplication, and $\Lambda = \mathbb{Z}$. The basic inner product satisfies B(1,1) = 2. Then $\Xi = B^{\sharp}(\Lambda^*) = \frac{1}{2}\mathbb{Z}$, hence $T_{\ell} = \ell^{-1}\Xi/\Lambda \simeq \mathbb{Z}_{2\ell} = \{t \in U(1) | t^{2\ell} = 1\}$. The multiplicity function is

$$m(\lambda,k) = \frac{1}{2\ell} \nabla_{\alpha} \sum_{t \in T_{\ell}}^{\prime} \frac{t^{-\lambda} - t^{\ell-\lambda}}{(1-t)(1-t^{-1})},$$

where $\alpha = -2$ is the negative root of SU(2). Using the notation of examples 1 and 3, this can be written

$$m(\lambda,k) = \frac{1}{2\ell} \nabla_{\alpha} \Big(V_2(\lambda,\ell) - V_2(\lambda-\ell,\ell) \Big).$$

Example 3 gave the decomposition formula for V_2 expanded around a center $\gamma \in (0, 2)$, which gives

$$\frac{1}{2\ell}\nabla_{\alpha}V_2(\lambda,\ell) = -\frac{\lambda-1}{\ell} + 1 + \sum_{n>0}\theta_+(\lambda-2n\ell) + \sum_{n\leq 0}\theta_-(\lambda-2n\ell),$$

where

$$\theta_+(\lambda) = x_+(\lambda+2) - x_+(\lambda), \qquad \theta_-(\lambda) = x_-(\lambda+2) - x_-(\lambda),$$

 $(x_{\pm} \text{ are defined in example 3})$. There is a similar expansion for the other term, except that now one expands first around a center $\gamma \in (-2, 0)$ to compensate for the shift by ℓ (see Section 4.2.4). Taking the difference, one obtains the norm-square localization formula:

$$m(\lambda,k) = 1 + \sum_{n>0} \theta_+ \left(\lambda - 2n\ell\right) + \sum_{n \le 0} \theta_- \left(\lambda - 2n\ell\right)$$
$$- \sum_{n \ge 0} \theta_+ \left(\lambda - (2n+1)\ell\right) - \sum_{n < 0} \theta_- \left(\lambda - (2n+1)\ell\right).$$

(One can check this agrees with Remark 4.3.8.)

4.4.7 Example: fusion of two conjugacy classes of SU(2).

We use notation as in the example above for G = SU(2). Note that $\ell = k + h^{\vee} = k + 2$. Let $C = G \cdot h$ be the conjugacy class of the element $h = \exp(B^{\sharp}(\rho)/2)$, $\rho = \frac{1}{2}\alpha$, where α is the positive root. C is a quasi-Hamiltonian SU(2) space with moment map the inclusion. Let $M = C \otimes C$ be the *fusion product* (c.f. [2]); as a G-space $M = C \times C$ with the diagonal action, and the moment map is the product of the two moment maps. It is known that M is prequantizable at level k if and only if k is even (c.f. [44]).

Remark 4.4.13. Similar to the previous example (S^4) , this is an example of a multiplicity free q-

Hamiltonian SU(2)-space with surjective moment map. Apparently there is only one other such example besides S^4 , $C \otimes C$ ([32]).

There are four fixed-points for the action of a regular element $t \in T$: (h,h), (h,h^{-1}) , (h^{-1},h) , (h^{-1},h) , (h^{-1},h) . The contribution from $F_1 = (h,h)$ is

$$m_{F_1}(\lambda, k) = \frac{1}{2\ell} \sum_{t \in T_\ell}' \frac{t^{k\rho - \lambda}}{1 - t^{-\alpha}},$$

(we have used the factor of $(1 - t^{-\alpha})$ in the formula for m to reduce the degree of the denominator). The denominator factors

$$1 - t^{-\alpha} = (1 - t^{-\rho})(1 + t^{-\rho}).$$

Using notation as in the previous example, $\rho \in \Lambda^*$ is identified with the generator $1 \in \mathbb{Z}$, and so m_{F_1} is $((2\ell)^{-1} \text{ times})$ a translate of a Verlinde series for the list $\boldsymbol{\alpha} = \{-1, a\}, a(x) = -x + \frac{1}{2}$ that was considered in 1 and 3. The contributions from the other three fixed points are similar. Adding them together, we find the norm-square localization formula for $m = \sum m_{F_i}$

$$m(\lambda,k) = \left(1 - 2\sum_{n\geq 0} 1_{[2n\ell+2+k,\infty)} + 2\sum_{n<0} 1_{(-\infty,2n\ell+k]} - 2\sum_{n\leq 0} 1_{(-\infty,2n\ell-2]} + 2\sum_{n>0} 1_{[2n\ell,\infty)}\right) \delta_{2\mathbb{Z}}(\lambda).$$

4.4.8 Example: a multiplicity-free q-Hamiltonian SU(3)-space.

There is an example due to C. Woodward of a multiplicity-free q-Hamiltonian G = SU(3)-space M with moment polytope an equilateral triangle inscribed in the alcove. We discussed this example already in Chapter 2, where we considered its Duistermaat-Heckman measure. Here we discuss this example in more detail and determine when M is prequantizable. To follow some of the discussion below, one needs to consult the image of the moment polytope in Chapter 2 (figure 2.3), from which we will read off several facts about this space. One way to construct this space is to begin with a similar Hamiltonian SU(3)-space M' described in [64], [65], and then construct M by a surgery. We outline this construction below.

Let G = SU(3) with maximal torus T. Choose simple roots α_1 , α_2 and let ρ denote the half-sum of the positive roots. Identify $\mathfrak{g} \simeq \mathfrak{g}^*$ using the basic inner product. The vertices of the alcove are the fundamental weights μ_1, μ_2 . The simply connected compact Lie group G_2 has SU(3) as a subgroup, with common maximal torus T. G_2 has 12 roots, the 6 long roots coincide with the roots of SU(3). The corresponding short coroots generate the (common) integral lattice. The root and weight lattices of G_2 coincide.

Let (M', ω') denote the G_2 -coadjoint orbit through the point $\frac{1}{2}\rho$, thus

$$M' = G_2/K, \qquad T \subset K \subset G_2.$$

The stabilizer K of $\frac{1}{2}\rho$ in G_2 is 4-dimensional, hence M' is 10-dimensional. As a Hamiltonian SU(3)-space, M' is multiplicity-free with trivial principal stabilizer. The moment map

$$\phi':M'\to\mathfrak{g}^*$$

is transverse to \mathfrak{t}^* , and its moment polytope Δ is an equilateral triangle inscribed in the fundamental

alcove, with vertices at $\frac{1}{2}\rho$, $\frac{1}{2}\mu_1$, $\frac{1}{2}\mu_2$ ([64],[65]).

Let $M_0 = M' \setminus (\phi')^{-1} (G \cdot \rho/2)$, and let ω'_0 , ϕ'_0 be the restrictions. Then $\phi'_0(M_0)$ is contained in the neighbourhood around $0 \in \mathfrak{g} \simeq \mathfrak{g}^*$ where the exponential map restricts to a diffeomorphism. Thus $(M_0, \phi'_0, \omega'_0)$ can be exponentiated (c.f. [2]), to a q-Hamiltonian SU(3)-space $(M_0, \Phi_0 = \exp(\phi'_0), \omega_0)$, where

$$\omega_0 = \omega'_0 + (\phi'_0)^* \varpi,$$

and ϖ is an explicit primitive of $\exp^* \eta$.

Let $c = \exp(\mu_1) \in SU(3)$ denote a non-trivial element of the center $Z(SU(3)) \simeq \mathbb{Z}_3$. Since c is central, $(M_1 := M_0, \Phi_1 := c\Phi_0, \omega_1 := \omega_0)$ and $(M_2 := M_0, \Phi_2 := c^2\Phi_0, \omega_2 := \omega_0)$ are again q-Hamiltonian SU(3)-spaces. Our claim is that these three q-Hamiltonian spaces glue together:

Proposition 4.4.14 (Woodward). There is a multiplicity-free, transverse q-Hamiltonian SU(3)-space (M, ω, Φ) with G-equivariant open embeddings

$$\iota_i: M_i \hookrightarrow M, \qquad i = 0, 1, 2$$

such that $\iota_i^* \omega = \omega_i, \ \iota_i^* \Phi = \Phi_i$. Moreover

$$M = \iota_0(M_0) \cup \iota_1(M_1) = \iota_0(M_0) \cup \iota_2(M_2) = \iota_0(M_0) \cup G\Phi^{-1}(\rho/2).$$

Proof. We begin by describing a G-equivariant identification of a certain open set in M_0 with open sets in M_i . Gluing the M_i together on these open sets then produces the smooth G-manifold M, and the embeddings ι_i .

Let $\mathring{\mathfrak{a}}$ denote the interior of the fundamental alcove. Let σ_i be the open face opposite μ_i , i = 1, 2. Set

$$\tau_i = \mathring{\mathfrak{a}} \cup \sigma_i,$$

and

$$U_i = (c^i \Phi)^{-1} (G \exp(\tau_i)) \subset M_i$$

These are open dense G-invariant submanifolds, and are multiplicity-free q-Hamiltonian⁵ G-spaces, with trivial generic stabilizer and moment images (in \mathfrak{a})

$$\tau_i \cap \Delta$$
.

A multiplicity-free q-Hamiltonian (or Hamiltonian) G-space is uniquely determined by its generic stabilizer (up to conjugacy) and its moment image (c.f. [64], [32]). The subset $\tau_i \cap \Delta \subset \mathfrak{a}$ is the same as the moment image for $\Phi^{-1}(G \exp(\tau_i)) \subset M_0$. Thus there exist G-equivariant diffeomorphisms

$$\iota_i: U_i \to \Phi^{-1}(G\exp(\tau_i)) \subset M_0$$

intertwining $c^i \Phi$ with Φ and also intertwining the q-Hamiltonian 2-forms. Define M to be the manifold obtained by gluing M_0 to M_1 using ι_1 to identify U_1 with the open dense subset $\Phi^{-1}(G \exp(\tau_1))$ of M_0 . The 2-form ω and moment map Φ are defined on M by gluing the 2-forms and moment maps for M_0 ,

⁵We can equally consider them as *Hamiltonian G*-spaces, since their moment images are contained in the neighbourhood of $e \in G$ where the exponential map is a diffeomorphism.

 M_1 , using the fact that they are intertwined by ι_1 . Arguing as above, uniqueness for multiplicity-free G-spaces implies that ι_2 extends to a map $\iota_2 : M_2 \hookrightarrow M$ as well, having the properties stated in the proposition.

- Remark 4.4.15. 1. As $M_1 = M_2 = M_0$, a somewhat confusing point is that the maps ι_i are not the identity maps. Rather the ι_i take advantage of the extra symmetry of the alcove and the fact that M_0 is multiplicity-free, to identify an open subset of M_0 with a different open subset of M_0 .
 - 2. We can describe the gluing more simply over a slightly smaller open subset. Let

$$U'_i = (c^i \Phi)^{-1}(\exp(\mathfrak{a})) \subset M_i.$$

These are smooth T-invariant submanifolds. Since M_0 is multiplicity-free, there is a simple model for these spaces: they are the unique multiplicity free T-spaces with trivial generic stabilizer and with moment map image $\mathring{\mathfrak{a}} \cap \Delta$ (e.g. they are each diffeomorphic to \mathbb{CP}^2 with 3 points removed). Thus there are T-equivariant symplectomorphisms

$$\iota_i: U_i \to U_0', \qquad i = 1, 2$$

intertwining the moment maps $\log(c^i \Phi)$ and $\log(\Phi)$. Exponentiating, these maps intertwine $c^i \Phi$ with Φ . Taking the flow-out under the action of G, we obtain G-equivariant diffeomorphisms

$$\iota_i: GU_i' = G \times_T U_i' \to G \times_T U_0' = GU_0'$$

intertwining $c^i \Phi$ with Φ . The set GU'_i is open and dense in U_i and one must then argue that the map ι_i extends smoothly—this follows for example from the uniqueness theorem for multiplicity-free spaces referred to above.

Since M' is a coadjoint orbit, M' is pre-quantizable at level k if and only if $\frac{k}{2}\rho$ lies in the weight lattice of G_2 , if and only if $k \in 2\mathbb{Z}$ (the weight lattices of SU(3) and G_2 coincide).

Proposition 4.4.16. The q-Hamiltonian SU(3)-space (M, ω, Φ) is prequantizable at level k if and only if $k \in 2\mathbb{Z}$.

Proof. First we explain why M is not prequantizable for $k \notin 2\mathbb{Z}$. The point $\exp(\rho/4)$ is contained in the image of the fixed-point subset $F := M^H$, $H = \exp(\mathbb{R}\rho)$. Let $q : \mathfrak{t}^* \to \mathfrak{t}^*/\operatorname{ann}(\mathbb{R}\rho) = \mathfrak{h}^*$. If Mis prequantizable at level k then $q(k\rho/4)$ must be in the weight lattice for H, because it is the weight of the H action on the fibre $L|_m$ of the level k prequantum line bundle on the Hamiltonian LG-space $\Psi : \mathcal{M} \to L\mathfrak{g}^*$, for some point $m \in \Psi^{-1}(\rho/4)$. The integral lattice of H is $\mathbb{R}\rho \cap \Lambda$, where Λ is the integral lattice of $T \subset SU(3)$. Thus the weight lattice is $(\mathbb{R}\rho \cap \Lambda)^* = \Lambda^*/\operatorname{ann}(\mathbb{R}\rho)$. It follows that $q(k\rho/4)$ is in the H weight lattice if and only if $k \in 2\mathbb{Z}$.

We recall the prequantization criterion from [44]: a q-Hamiltonian G-space (M, ω, Φ) is prequantizable if and only if the class $[(\omega, \eta)]$ in the relative cohomology group $H^3(\Phi, \mathbb{R})$ is integral. This means one must check that for each 2-cycle $\Sigma \subset M$, and 3-chain B in G with $\partial B = \Phi_* \Sigma$ the quantity

$$\int_{\Sigma} \omega - \int_{B} \eta \in \mathbb{Z}.$$
(4.75)



Figure 4.1: The left image shows a single contribution to the norm-square localization formula for a multiplicity-free q-Hamiltonian SU(3)-space (at level k = 2). The right image shows the sum of the first 6 contributions.

If the cycle Σ is contained entirely in a cross-section M_i (for i = 0, 1 or 2) then, using the primitive ϖ for η , the condition reduces to the ordinary prequantization criterion for a Hamiltonian *G*-space. By construction, each M_i is an open subset of the Hamiltonian *G*-space M', and M' is prequantizable at level $k \in 2\mathbb{Z}$. Thus (4.75) holds when Σ is contained entirely in one of the M_i 's.

Let $U = M_0$ and let $V = \Phi^{-1}(G \exp(V_0))$ where V_0 is a small neighbourhood of $\rho/2$ inside the fundamental alcove. V is an open subset of $M_1 \cap M_2$. To complete the proof, it suffices to show that $H_2(M)$ is generated by the image of $H_2(U)$, $H_2(V)$ under the push-forward maps. Using the Mayer-Vietoris sequence, it suffices to show that $H_1(U \cap V) = 0$. The moment image of $U \cap V$ is contained in the interior $\mathring{\mathfrak{a}}$ of the fundamental alcove, thus $U \cap V \sim G \times_T S$ where S is a multiplicity-free Hamiltonian T-space. Topologically $S \simeq S^3 \times \mathbb{R}$ (c.f. [64]), hence

$$U \cap V \sim G \times_T S^3.$$

As G is simply connected, $H_G^i(N) = H^i(N)$ for i = 1, 2 and any G-space N. Therefore

$$H^1(U \cap V) = H^1_G(G \times_T S^3) = 0$$

while $H^2(U \cap V) = \mathbb{Z}^2$ has no torsion. Thus $H_1(U \cap V) = 0$.

We now describe the fixed-point contributions and the norm-square localization formula for M. There are 3 fixed-point sets F_0, F_1, F_2 , each of which is topologically a 2-torus fixed by a 1-dimensional subtorus of T. Let m_i , i = 0, 1, 2 be the contribution to the multiplicity from F_i . It is enough to work out m_0 for $F := F_0$, since m_1, m_2 will then be determined using the (shifted) anti-symmetry under the Weyl group:

$$m_i(\lambda, \ell) = (-1)^{|w_i|} m_0(w_i(\lambda + \rho) - \rho, \ell), \qquad i = 1, 2$$

where w_i denotes the reflection corresponding to the simple root α_i of SU(3). Note that $\rho = \alpha_1 + \alpha_2$,

 $B(\rho,\rho) = 2$ and $B(\rho,\alpha_i) = 1$. In figure 2.3, the image of F in T corresponds (under the exponential map) to the disjoint union $S \sqcup (-S)$, where S is the line segment joining $\frac{1}{2}\alpha_2$ to $\frac{1}{2}\alpha_1$, with α_1 , α_2 the simple roots.

As Φ is transverse to $T, X = \Phi^{-1}(T)$ is a smooth 4-dimensional (degenerate) q-Hamiltonian T-space, and the normal bundle to F splits as

$$\nu(F,M) = \nu(F,X) \oplus \mathfrak{g}/\mathfrak{t}.$$

The factor in the denominator of the fixed-point formula coming from $\mathfrak{g}/\mathfrak{t}$ cancels with the product $\prod_{\alpha<0}(1-t^{\alpha})$. The stabilizer of F is the 1-dimensional subtorus

$$T_F = \exp(\mathbb{R}\rho) \subset T$$

and it acts with weight -1 on $\nu(F, X)$ (this can be read off from the corresponding Hamiltonian space). This normal bundle is necessarily trivial (its *T*-equivariant curvature for any choice of connection must vanish in cohomology, since the *T*-action on *F* has no fixed-points).

The order of the group $T_{\ell} = \ell^{-1}B^{\sharp}(\Lambda^*)/\Lambda$ is $3\ell^2$. Since F is a torus, $\hat{A}(F) = 1$. The group T/T_F acts freely on F, and F is a multiplicity free q-Hamiltonian T/T_F -space. In particular the pullback of the 2-form ω_F is symplectic. The weight lattice of T/T_F is isomorphic to $\Lambda^* \cap \operatorname{ann}(\mathfrak{t}_F)$, and so the symplectic volume of F can be read-off from the moment image: it is 3, since the moment image of F has length 3 times the \mathbb{Z} -basis element $\mu_1 - \mu_2$ of $\Lambda^* \cap \operatorname{ann}(\mathfrak{t}_F)$ (see figure 2.3).

On the covering space $\hat{F} = \mathfrak{t} \times_T F$ (fibre product), the symplectic form has a *T*-equivariant extension $\omega_F - \phi_F$, where $\phi_F : \hat{F} \to \mathfrak{t}$ satisfies

$$\phi_F(\lambda \cdot x) = \phi_F(x) + B^{\flat}(\lambda), \qquad \lambda \in \Lambda.$$
(4.76)

Recall that the determinant line bundle $\mathcal{L}_k|_F$ is $T_{2\ell}$ -equivariant and can be obtained as a quotient by Λ of a *T*-equivariant line bundle $\hat{\mathcal{L}}_k \to \hat{F}$, where the *T* and Λ actions on $\hat{\mathcal{L}}_k$ satisfy (c.f. (4.38))

$$t\lambda t^{-1}\lambda^{-1} = t^{2\ell B^{\flat}(\lambda)}.$$

Choosing a connection equivariant for the actions of T, Λ on $\hat{\mathcal{L}}_k$, the corresponding moment map μ satisfies

$$\mu(\lambda \cdot x) = \mu(x) + 2\ell B^{\flat}(\lambda), \qquad \lambda \in \Lambda$$

Comparing to (4.76) implies that $c_1(\mathcal{L}_k) = 2\ell[\omega_F]$ in cohomology. Hence

$$\int_F \operatorname{Ch}(\mathcal{L}_k)^{1/2} = 3\ell, \qquad \ell = k + \mathbf{h}^{\vee} = k + 3.$$

From the remarks above, the fixed-point formula for the multiplicity m_0 is

$$m_0(\lambda, k) = \frac{1}{\ell} \sum_{t \in T_\ell \cap T_F}' \frac{t^{k \frac{\rho}{4} - \lambda}}{1 - t}.$$
(4.77)

The group $T_{\ell} \cap T_F \simeq \mathbb{Z}_{\ell}$, so that the sum in equation (4.77) is essentially a 1-dimensional Verlinde series, of the kind considered in examples 1 and 3—more precisely it is the pullback of the 1-dimensional example,

under the projection $\mathfrak{t}^* \to \mathfrak{t}^*/\operatorname{ann}(\mathfrak{t}_F)$. Identifying $\mathfrak{t}^*/\operatorname{ann}(\mathfrak{t}_F)$ with \mathbb{R} such that $\mathbb{Z} \simeq \Lambda^*/\operatorname{ann}(\mathfrak{t}_F)$, the projection map is $\lambda \mapsto B(\lambda, \rho)$. Thus in the notation of examples 1 and 3

$$m_0(\lambda, k) = \frac{1}{\ell} V_1 \Big(B(\lambda, \rho) - \frac{k}{2}, \ell \Big).$$

In order to compensate for the shift by $\frac{k}{2}$, we apply the decomposition formula for V_1 about a center $\gamma \in (-1,0)$ (see Section 4.2.4). The formula given in example 3 (substituting $B(\lambda,\rho) - \frac{k}{2} = B(\lambda+\rho,\rho) - 2 - \frac{k}{2}$ in place of λ) gives

$$m_0(\lambda, k) = \frac{-B(\lambda + \rho, \rho)}{\ell} + \left[\sum_{n \ge 0} \mathbb{1}_{[2+k/2+n\ell,\infty)} - \sum_{n < 0} \mathbb{1}_{(-\infty,1+k/2+n\ell)}\right] \Big(B(\lambda + \rho, \rho)\Big).$$

(In this expression, the sums over $n \ge 0$ and n < 0 in the square bracket define a step function on \mathbb{R} , which is then being applied to $B(\lambda + \rho, \rho)$.)

The full multiplicity is

$$m(\lambda, k) = m_0(\lambda, k) - m_0(w_1(\lambda + \rho) - \rho, k) - m_0(w_2(\lambda + \rho) - \rho, k).$$

The part which is polynomial in λ is

$$\frac{1}{\ell}B(-\xi+w_1\xi+w_2\xi,\rho),\qquad \xi=\lambda+\rho,$$

and this vanishes identically (for any ξ). This is an example of the 'cancellations' observed in Section 4.4.5. Let $w_0 = e \in W$. Thus the norm-square localization formula in this case can be written

$$m(\lambda,k) = \sum_{i=0,1,2} (-1)^{|w_i|} \left[\sum_{n \ge 0} \mathbb{1}_{[2+k/2+n\ell,\infty)} - \sum_{n < 0} \mathbb{1}_{(-\infty,1+k/2+n\ell]} \right] \Big(B(w_i(\lambda+\rho),\rho) \Big).$$

See Figure 4.1.

4.5 The function d and proof of quasi-polynomial behaviour.

In this section we study the support of the terms $m_{\Delta,C}^{\text{qpol}}$ in the norm-square localization formula (Theorem 4.4.12), leading to a proof that m(0,k) is quasi-polynomial for all $k \ge 1$. As explained in the introduction, this result is closely related to the [Q, R] = 0 Theorem for Hamiltonian LG-spaces ([4]). The last step in proving the [Q, R] = 0 Theorem (not carried out here) would be to apply the 'delocalized' index formulas of Section 4.4.4 and a stationary phase argument to deduce that m(0, k) equals the quantization of the reduced space $Q(\mathcal{M}//LG)$ for large k (similar to [37] in case 0 is a regular value).

Throughout this section we use the basic inner product to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. For $\Delta \neq \mathfrak{t}^*$, the contribution $m^{\text{qpol}}_{\Delta,C}(-,k)$ is supported in a half-space of the form

$$\langle -,\beta\rangle \ge d,\tag{4.78}$$

where $\beta = \operatorname{pr}_{\Delta}(0)$ is the nearest point in Δ to the origin, and d is a lower bound depending on Δ , k and certain ' ρ -shifts'. We show:

Proposition 4.5.1. For $k \ge 1$ and Δ such that $0 \notin \Delta$, d > 0.

As a corollary, the corresponding terms $m_{\Delta,C}^{\text{qpol}}$ do not contribute to the multiplicity m(0,k), and we obtain our main result:

Corollary 4.5.2. The multiplicity m(0,k) of the basic level k representation in Q(M,k) is a quasipolynomial function of k.

Proof. Proposition 4.5.1 implies that

$$m(0,k) = \sum_{\{\Delta,C|0\in\Delta\}} m^{\text{qpol}}_{\Delta,C}(0,k), \qquad \forall k \ge 1.$$

But if $0 \in \Delta$, then $|T_{\ell}| m_{\Delta,C}^{\text{qpol}}(0,k)$ is a quasi-polynomial function of k. Thus $|T_{\ell}| m(0,k)$ is a quasi-polynomial function of k. But m(0,k) is an integer for $k \ge 1$, and so $|T_{\ell}| = |T_1|\ell^{\dim(T)}$ must divide this quasi-polynomial, implying that m(0,k) is itself quasi-polynomial for $k \ge 1$.

A bound d can be obtained from Lemma 4.4.4. Proposition 4.5.1 will follow from a lower bound on d obtained below in Proposition 4.5.5, together with a Lie algebraic inequality, Theorem 4.5.9.

Fix Δ, C with $0 \notin \Delta$, and let $\beta \neq 0$ be the nearest point of Δ to the origin. $m_{\Delta,C}$ is supported in a half-space of the form

$$\langle -,\beta \rangle \ge d, \qquad \beta = \operatorname{pr}_{\Delta}(0).$$
 (4.79)

The multiplicity function $m_{\Delta,C}^{\text{qpol}}$ is obtained by taking the 'quasi-polynomial germ along Δ ' of $m_{\Delta,C}$, and thus has support contained in the same half-space. The bound d (depending on k, Δ) is the minimum value for the pairing of β with the weights of T_{Δ} appearing in the inverse Fourier transform $Q_{\Delta,C}$ (formula of Lemma 4.4.4). Taking the perturbation γ small enough that γ_{Δ}^+ is sufficiently near β , the weights of the action of T_{Δ} on $S\nu_C(\gamma_{\Delta}^+)$ will pair non-negatively with β . Thus from the formula in Lemma 4.4.4 we read off the lower bound:

$$d := k \langle \phi_{\Delta}, \beta \rangle + \langle \varphi_{\Delta}, \beta \rangle + \frac{1}{2} \langle s_{\Delta}^{+}, \beta \rangle - \sum_{\alpha > 0, \langle \alpha, \beta \rangle > 0} \langle \alpha, \beta \rangle.$$
(4.80)

(Below we recall the definitions of ϕ_{Δ} , φ_{Δ} , s_{Δ}^{+} from Section 4.4.)

Remark 4.5.3. One can independently guess this bound using, for example, the general norm-square localization formulas of [53] applied to the spin-c structure on the covering space \mathcal{N} and the Kirwan vector field for the moment map ϕ . (This is not quite rigorous as \mathcal{N} is non-compact, and we have not obtained our formula by deforming the symbol of a Dirac operator.)

Each term in (4.80) other than the last is invariant under the Weyl group, while the last term is minimized for $\beta \in \mathfrak{t}_+$. So from now on we restrict to the 'worst' case when $\beta \in \mathfrak{t}_+$. The last term then becomes $2\langle \rho, \beta \rangle$.

The set of *affine roots* of \mathfrak{g} is

$$\mathcal{R}_{\text{aff}} = \{(\alpha, 0) | \alpha \in \mathcal{R}\} \cup \{(\alpha, n) | \alpha \in \mathcal{R} \cup \{0\}, 0 \neq n \in \mathbb{Z}\}.$$

(This is the set of roots of the affine Kac-Moody algebra corresponding to \mathfrak{g} .) The elements of $\mathcal{R}_{\rm aff}$ can

be viewed as *affine* linear functions on \mathfrak{t} , by associating to (α, n) the affine linear function

$$\langle \alpha, \xi \rangle + n, \qquad \xi \in \mathfrak{t}.$$

The fundamental alcove \mathfrak{a} determines a subset of *positive affine roots* $\mathcal{R}_{aff,+}$: those $(\alpha, n) \in \mathcal{R}_{aff}$ for which the corresponding affine linear function is positive on the interior \mathfrak{a} . Equivalently

$$\mathcal{R}_{\mathrm{aff},+} = \{(\alpha, n) | \alpha \in \mathcal{R} \cup \{0\}, n > 0\} \cup \{(\alpha, 0) | \alpha \in \mathcal{R}_+\}.$$

The Stiefel diagram of \mathfrak{g} is the affine hyperplane arrangement $\{H_{\alpha,n} | (\alpha, n) \in \mathcal{R}_{aff}\}$ in \mathfrak{t} where

$$H_{\alpha,n} = \{\xi \in \mathfrak{t} | \langle \alpha, \xi \rangle + n = 0\}$$

is the zero set of the corresponding affine linear function. The affine Weyl group $W_{\text{aff}} \simeq W \ltimes \Lambda$ is generated by reflections in the hyperplanes of the Stiefel diagram, and acts simply transitively on the open chambers. The fundamental alcove is the unique closed chamber $\mathfrak{a} \subset \mathfrak{t}_+$ containing the origin. The *vertices* of the Stiefel diagram are the points κ such that

$$\cap \{H_{\alpha,n} | \kappa \in H_{\alpha,n}\} = \{\kappa\}.$$

Below we use the following easy result, which is Lemma 3.1 from [47]:

Proposition 4.5.4. Let σ be an open face of the fundamental alcove, and let G_{σ} be the stabilizer of $\exp(\zeta)$ for any $\zeta \in \sigma$ (it does not depend on the choice of ζ). Let $\mathcal{R}'_{\sigma,+} \subset \mathcal{R}_{\sigma}$ be the set of roots which are positive on the cone $\mathbb{R} \cdot (\mathring{\mathfrak{a}} - \zeta)$.⁶ Let ρ'_{σ} be the corresponding ρ -element for G_{σ} . Then the difference

$$\rho - \rho'_{\sigma}$$

is the orthogonal projection of ρ onto the dilated face $h^{\vee}\sigma$.

In particular, if $\sigma = \{\nu\}$ is a vertex of \mathfrak{a} then

$$\rho - \rho'_{\nu} = \mathbf{h}^{\vee} \nu. \tag{4.81}$$

Proposition 4.5.5. Let $0 \neq \beta \in \mathfrak{t}_+$ be the nearest point of Δ to the origin, and assume $m_{\Delta,C}^{\text{qpol}}$ is non-zero. Then there is a lower bound for d of the form

$$d > \mathbf{h}^{\vee} \|\kappa\|^2 - \langle \rho + \rho_{\kappa}, \kappa \rangle. \tag{4.82}$$

where κ is a vertex in the closure of the open face of the Stiefel diagram containing β , $G_{\exp(\kappa)}$ is the stabilizer of $\exp(\kappa)$, and ρ_{κ} is the half-sum of the positive roots $\mathcal{R}_{\kappa,+} \subset \mathcal{R}_+$ for the group $G_{\exp(\kappa)}$.

Remark 4.5.6. This lower bound is 'universal' in the sense that it contains only Lie-algebraic data, with no reference to the space \mathcal{M} .

Proof. The fibre product $C \times_T \mathfrak{t}$ is a submanifold of \mathcal{N} . Since C is fixed by T_{Δ} , the image $\phi(C \times_T \mathfrak{t})$ is contained in a disjoint union of translates of $\operatorname{ann}(\mathfrak{t}_{\Delta})$ (one translate for each element of the quotient

 $^{^{6}}$ This is the same set of positive roots which appeared in Section 4.3.3. See also Definition 4.5.11 and Remark 4.5.12 below, for the relation to the affine Kac-Moody algebra associated to \mathfrak{g} .

 $\Lambda/\operatorname{ann}(\mathfrak{t}_{\Delta})$). The affine subspace Δ is one of these translates. As ϕ , φ are abstract moment maps, the compositions

$$\operatorname{pr}_{\mathfrak{t}^*_{\Lambda}} \circ \phi, \qquad \operatorname{pr}_{\mathfrak{t}^*_{\Lambda}} \circ \varphi$$

are constant along the part of the fibre product $C \times_T \mathfrak{t}$ lying over Δ . These constant values are ϕ_{Δ} , φ_{Δ} respectively. In particular they can be described in terms of lifts of any point $x \in C$.

By Theorem 4.4.9, we can assume $\exp(\beta) \in \Phi(C)$. Choose $x \in C \cap \Phi^{-1}(\exp(\beta))$ and a lift $\hat{x} \in \pi^{-1}(x)$. We will describe $\phi_{\Delta}, \varphi_{\Delta}, s_{\Delta}^{+}$ in terms of the action of T_{Δ} on $\pi^* L|_{\hat{x}}, \pi^* S_0|_{\hat{x}}, T_x N$ respectively.

Remark 4.5.7. Theorem 4.4.9 that we have just invoked, and which is intimately tied with norm-square localization, plays an important role. The existence of a point $x \in C \cap \Phi^{-1}(\exp(\beta))$ leads to improved bounds for the terms involving φ_{Δ} and s_{Δ}^+ .

Choose an element $w \in W$ (not unique in general) such that

$$\exp(\beta) \in w \exp(\sigma).$$

where σ is a face of the fundamental alcove \mathfrak{a} . Let $Y_{w\sigma}$ be the corresponding cross-section, which contains x. Recall the definitions of ϕ_{Δ} , φ_{Δ} , and s_{Δ}^+ :

- 1. By definition $\phi_{\Delta} = \operatorname{pr}_{\mathfrak{t}_{\Delta}^{*}}(\phi(\hat{x}) + \xi)$, where $\xi \in (\Delta \phi(\hat{x})) \cap B^{\flat}(\Lambda)$. Equivalently $\{\phi_{\Delta}\} = \Delta \cap \mathfrak{t}_{\Delta}$ (identifying $\mathfrak{t} \simeq \mathfrak{t}^{*}$). Thus $\langle \phi_{\Delta}, \beta \rangle = \|\beta\|^{2}$.
- 2. By definition $\varphi_{\Delta} = \operatorname{pr}_{\mathfrak{t}_{\Delta}^{*}}(\varphi(\hat{x}) + \mathbf{h}^{\vee}\xi)$, with ξ as in item 1. By Proposition 4.3.3, $\operatorname{pr}_{\mathfrak{t}_{\Delta}^{*}}(\varphi(\hat{x}))$ is the weight for the action of T_{Δ} on $S_{0}|_{\hat{x}}$. Choose the lift \hat{x} such that $\phi(\hat{x}) \in w \cdot \mathfrak{a}$. Then $\langle \varphi(\hat{x}), \beta \rangle$ is determined by equation (4.40):

$$\langle \varphi(\hat{x}), \beta \rangle = \frac{1}{2} \langle s_{w\sigma}, \beta \rangle + \langle w(\rho - \rho'_{\sigma}), \beta \rangle, \qquad (4.83)$$

where $s_{w\sigma}$ is the sum of the weights for the action of T_{Δ} on $T_x Y_{w\sigma}$, equipped with an almost complex structure which is compatible with the symplectic form on the cross-section. ρ'_{σ} is defined in Section 4.3.3 or Proposition 4.5.4.

3. The normal bundle ν_C to the component $C \subset N^{t_\Delta}$ has a complex structure such that the weights of the action of T_Δ pair positively with γ_Δ^+ (notation: $\nu_C(\gamma_\Delta^+)$). By definition s_Δ^+ is the sum of these γ_Δ^+ -polarized weights. The tangent space is a direct sum

$$T_x N = T_x Y_{w\sigma} \oplus \mathfrak{g}/\mathfrak{g}_{w\sigma}$$

and $\mathfrak{g}/\mathfrak{g}_{w\sigma}$ is identified with certain *G*-orbit directions. Thus $\frac{1}{2}\langle s_{\Delta}^+,\beta\rangle$ is a sum

$$\frac{1}{2}\langle s_{\Delta}^{+},\beta\rangle = \frac{1}{2}\langle s_{w\sigma}^{+},\beta\rangle + \langle \rho - \rho_{w\sigma},\beta\rangle$$

where $s_{w\sigma}^+$ is the sum of the polarized weights for the subspace $T_x Y_{w\sigma}$, and $\rho_{w\sigma}$ is the half-sum of the positive roots $\mathcal{R}_{w\sigma,+} \subset \mathcal{R}_+$ for the subgroup $G_{w\sigma}$. (We have used here the assumption $\beta \in \mathfrak{t}_+^*$.)

The combination

$$\frac{1}{2}\langle s_{w\sigma},\beta\rangle + \frac{1}{2}\langle s_{w\sigma}^+,\beta\rangle \ge 0,$$

since any weight appearing in $s_{w\sigma}$ having negative pairing with β will cancel with the corresponding weight appearing in $s_{w\sigma}^+$. Dropping this term and the term $k \|\beta\|^2$, we obtain the lower bound

$$d > \langle w(\rho - \rho'_{\sigma}) + \mathbf{h}^{\vee} \xi, \beta \rangle - \langle \rho + \rho_{w\sigma}, \beta \rangle.$$

$$(4.84)$$

Note that β is contained in the compact, convex set $\overline{w\sigma} + \xi$. The right-side of this inequality is a linear function of β , and thus its minimum value on this set must occur at a vertex $\kappa \in \overline{w\sigma} + \xi$. Hence

$$d > \langle w(\rho - \rho'_{\sigma}) + \mathbf{h}^{\vee} \xi, \kappa \rangle - \langle \rho + \rho_{w\sigma}, \kappa \rangle.$$

Then $w^{-1}(\kappa - \xi)$ is some vertex ν of \mathfrak{a} . Consider the sum

$$w\rho'_{\sigma} + \rho_{wc}$$

 $w\rho'_{\sigma}$ is one-half the sum of a (possibly different) set $\mathcal{R}'_{w\sigma,+}$ of positive roots for the same subgroup $G_{w\sigma}$. Those roots in the intersection $\mathcal{R}_{w\sigma,+} \cap \mathcal{R}'_{w\sigma,+}$ add, while the others cancel. Therefore the sum $w\rho'_{\sigma} + \rho_{w\sigma}$ is a sum of a set of roots in \mathcal{R}_+ .

Let ρ_{κ} be the half-sum of the positive roots $\mathcal{R}_{\kappa,+} \subset \mathcal{R}_+$ for the group $G_{\exp(\kappa)} = G_{w\nu}$. By a similar argument

$$w\rho'_{\nu} + \rho_{\kappa}$$

is a sum of a set of roots in \mathcal{R}_+ . Since $\mathcal{R}_{w\sigma,+} \subset \mathcal{R}_{\kappa,+}$ and $\mathcal{R}'_{\sigma} \subset \mathcal{R}'_{\nu}$, the same positive roots will appear as in the sum $w\rho'_{\sigma} + \rho_{w\sigma}$, and possibly some additional ones. Since $\kappa \in \mathfrak{t}_+$ it follows that

$$\langle w\rho'_{\nu} + \rho_{\kappa}, \kappa \rangle \ge \langle w\rho'_{\sigma} + \rho_{w\sigma}, \kappa \rangle.$$

Thus we have the lower bound

$$d > \langle w(\rho - \rho_{\nu}') + h^{\vee}\xi, \kappa \rangle - \langle \rho + \rho_{\kappa}, \kappa \rangle$$

By equation (4.81)

$$\rho - \rho'_{\nu} = \mathbf{h}^{\vee} \mathbf{\nu}$$

thus

$$w(\rho - \rho'_{\nu}) + \mathbf{h}^{\vee} \xi = \mathbf{h}^{\vee} \kappa$$

Remark 4.5.8. In case $\beta = \kappa$ is a vertex of the Stiefel diagram, one reaches the same expression as (4.82) already at equation (4.84). Equation (4.82) shows that the case $\beta = \kappa$ is a vertex is the sharpest.

Theorem 4.5.9. Let $\kappa \in \mathfrak{t}_+$ be a vertex of the Stiefel diagram, and ρ_{κ} the half-sum of the positive roots $\mathcal{R}_{\kappa,+}$ for the group $G_{\exp(\kappa)}$. Then

$$\mathbf{h}^{\vee} \|\kappa\|^2 - \langle \rho + \rho_{\kappa}, \kappa \rangle \ge 0. \tag{4.85}$$

Remark 4.5.10. 1. In the course of the proof we will see that equation (4.85) is an equality when κ is vertex of the fundamental alcove.

2. Equation (4.85) can be written

$$\|\kappa - \tau_{\kappa}\| \ge \|\tau_{\kappa}\|, \qquad \tau_{\kappa} = \frac{\rho + \rho_{\kappa}}{2h^{\vee}}.$$

In this form, it is somewhat analogous to the 'magic inequality' proved in [51], which bounds the distances of points in the faces of the positive Weyl chamber from the 'strictly regular' points in the interior.

3. The inequality becomes simpler for G = SU(n). In this case, the vertices exponentiate to central elements, thus $G_{\exp(\kappa)} = G$ and $\tau_{\kappa} = \rho/h^{\vee}$. Via the basic inner product, the point ρ/h^{\vee} is the barycenter of the fundamental alcove, which is equidistant from the vertices of the alcove.

We begin the proof with a definition and some lemmas.

Definition 4.5.11. Let κ be a vertex of the Stiefel diagram. The roots \mathcal{R}_{κ} of $G_{\exp(\kappa)}$ are exactly those $\alpha \in \mathcal{R}$ such that $\kappa \in H_{\alpha,n}$ for some affine root (α, n) . Therefore

$$\mathcal{R}_{\kappa,+}' := \{ \alpha \in \mathcal{R}_{\kappa} | \kappa \in H_{\alpha,n}, \ (\alpha, n) \in \mathcal{R}_{\mathrm{aff},+} \}$$

is a set of *positive roots* for $G_{\exp(\kappa)}$. Let $\mathfrak{t}'_{\kappa,+}$ denote the corresponding positive Weyl chamber (for $G_{\exp(\kappa)}$), and ρ'_{κ} the corresponding half-sum of positive roots.

Remark 4.5.12. When $\kappa = \nu$ is a vertex of the fundamental alcove, this coincides with the set of positive roots $\mathcal{R}'_{\nu,+}$ introduced earlier: the set of roots of $G_{\exp(\nu)}$ which are positive on the cone $\mathbb{R} \cdot (\mathring{\mathfrak{a}} - \nu)$.

The two sets of positive roots $\mathcal{R}_{\kappa,+} \subset \mathcal{R}_+$ and $\mathcal{R}'_{\kappa,+}$ do not agree. Instead we have the following.

Lemma 4.5.13. Let $\kappa \in \mathfrak{t}_+$ be a vertex of the Stiefel diagram. The two sets of positive roots $\mathcal{R}_{\kappa,+} \subset \mathcal{R}_+$ and $\mathcal{R}'_{\kappa,+}$ (Definition 4.5.11) are related as follows:

$$\mathcal{R}_{\kappa,+}' = \{ \alpha | -\alpha \in \mathcal{R}_{\kappa,+}, \, \langle \alpha, \kappa \rangle < 0 \} \cup \{ \alpha \in \mathcal{R}_{\kappa,+} | \langle \alpha, \kappa \rangle = 0 \}.$$

Proof. Let $(\alpha, n) \in \mathcal{R}_{aff,+}$ be a positive affine root $(n > 0 \text{ and } \alpha \in \mathcal{R} \cup \{0\} \text{ or } n = 0 \text{ and } \alpha \in \mathcal{R}_+)$ such that $\kappa \in H_{\alpha,n}$, that is

$$\langle \alpha, \kappa \rangle + n = 0.$$

Since $n \ge 0$, $\langle \alpha, \kappa \rangle \le 0$. If $\langle \alpha, \kappa \rangle = 0$ then n = 0 and hence $\alpha \in \mathcal{R}_+$. On the other hand if $\langle \alpha, \kappa \rangle < 0$ then $-\alpha \in \mathcal{R}_{\kappa,+}$, since $\kappa \in \mathfrak{t}_+$.

Let $H = G_{\kappa} \subset G_{\exp(\kappa)}$. Then a set of positive roots for H is

$$\mathcal{R}_{H,+} = \{ \alpha \in \mathcal{R}_{\kappa,+} | \langle \alpha, \kappa \rangle = 0 \}.$$

By definition

$$\langle \rho_H, \kappa \rangle = 0. \tag{4.86}$$

By the lemma above

$$\mathcal{R}'_{\kappa,+} = - ig(\mathcal{R}_{\kappa,+} \setminus \mathcal{R}_{H,+} ig) \cup \mathcal{R}_{H,+}$$

hence

$$\rho_{\kappa}' = -\rho_{\kappa} + 2\rho_H \quad \Rightarrow \quad \rho_{\kappa}' + \rho_{\kappa} = 2\rho_H. \tag{4.87}$$

Lemma 4.5.14. There is a unique $w = (w_0, \xi) \in W_{\text{aff}} = W \ltimes \Lambda$ satisfying

- 1. $\kappa \in w\mathfrak{a}$ and
- 2. $w\mathfrak{a} \kappa \subset \mathfrak{t}'_{\kappa+}$.

Proof. Choose w_1 satisfying the first condition; this fixes w_1 up to an element in the stabilizer $(W_{\text{aff}})_{\kappa}$ of κ in the affine Weyl group. $(W_{\text{aff}})_{\kappa}$ is generated by reflections in hyperplanes $H_{\alpha,n}$ with $\kappa \in H_{\alpha,n}$, in particular $\alpha \in R_{\kappa}$ and $(W_{\text{aff}})_{\kappa} \simeq W_{G_{\exp(\kappa)}}$ the Weyl group of $G_{\exp(\kappa)}$. There is therefore a unique $w_2 \in (W_{\text{aff}})_{\kappa}$ such that $w = w_2 w_1$ satisfies $w\mathfrak{a} \subset \kappa + \mathfrak{t}'_{\kappa,+}$.

Lemma 4.5.15. Let $w = (w_0, \xi) \in W_{\text{aff}}$ be the element from Lemma 4.5.14. Then

$$\mathbf{h}^{\vee}\kappa = w\rho - \rho'_{\kappa}.$$

Proof. There is a unique vertex ν of \mathfrak{a} such that $w\nu = \kappa$. By equation (4.81),

$$\rho - \rho'_{\nu} = \mathbf{h}^{\vee} \mathbf{\nu}$$

where ρ'_{ν} is the half sum of the positive roots for $G_{\exp(\nu)}$ corresponding to the $G_{\exp(\nu)}$ -Weyl chamber $\mathbb{R}_{>0} \cdot (\mathfrak{a} - \nu)$. Hence

$$\mathbf{h}^{\vee}\kappa = \mathbf{h}^{\vee}w\nu = w\rho - w_0\rho'_{\nu}.$$

Now $w_0 \rho'_{\nu}$ is the half sum of the positive roots for $G_{\exp(w_0\nu)} = G_{\exp(\kappa)}$ corresponding to the $G_{\exp(\kappa)}$ -Weyl chamber

$$\mathbb{R}_{>0}\cdot(w_0\mathfrak{a}-w_0\nu).$$

But by Lemma 4.5.14

$$w_0\mathfrak{a} - w_0\nu = w\mathfrak{a} - \kappa \subset \mathfrak{t}'_{\kappa,+},$$

hence $w_0 \rho'_{\nu} = \rho'_{\kappa}$.

Corollary 4.5.16. Let w be the element from Lemma 4.5.14. Then $w\mathfrak{a} \subset \mathfrak{t}_+$.

Proof. For each $\alpha \in \mathcal{R}_+$, we must show that $\alpha|_{w\mathfrak{a}} \geq 0$. If $\langle \alpha, \kappa \rangle > 0$ then this is clear, since $\kappa \in w\mathfrak{a}$ and $\alpha|_{w\mathfrak{a}}$ does not change sign. So assume $\langle \alpha, \kappa \rangle = 0 \Rightarrow \alpha \in \mathcal{R}_{H,+}$. Since $\mathcal{R}_{H,+} \subset \mathcal{R}'_{\kappa,+}$, we have $\langle \alpha, \rho'_{\kappa} \rangle > 0$. On the other hand, by Lemma 4.5.15

$$\langle \alpha, w\rho \rangle = \mathbf{h}^{\vee} \langle \alpha, \kappa \rangle + \langle \alpha, \rho_{\kappa}' \rangle = \langle \alpha, \rho_{\kappa}' \rangle > 0.$$

Recall that $\frac{1}{\mathbf{h}^{\vee}}\rho \in \mathring{\mathfrak{a}}$, so $\frac{1}{\mathbf{h}^{\vee}}\langle \alpha, w\rho \rangle > 0$ implies $\alpha|_{w\mathfrak{a}} \ge 0$.

Combining Lemma 4.5.15 with equation (4.87) gives

$$\mathbf{h}^{\vee}\kappa - (\rho + \rho_{\kappa}) = w\rho - \rho_{\kappa}' - \rho - \rho_{\kappa} = w\rho - \rho - 2\rho_{H}.$$
(4.88)

Recall that for an element $w \in W$, the difference $\rho - w\rho$ is a sum of positive roots, more precisely

$$\rho - w\rho = \sum_{\alpha \in \mathcal{H}_w} \alpha$$

where \mathcal{H}_w is the set of positive roots such that a line segment connecting any point in the interior \mathring{a} to a point in $w\mathring{a}$ crosses the hyperplane H_{α} . An analogous result holds for the affine Kac-Moody algebra associated to \mathfrak{g} (c.f. [29]):

$$\rho - w\rho = \sum_{(\alpha,n)\in\mathcal{H}_w} \alpha,\tag{4.89}$$

where $w \in W_{\text{aff}}$ and \mathcal{H}_w is the set of positive affine roots $(\alpha, n) \in \mathcal{R}_{\text{aff},+}$ such that a line segment connecting \mathfrak{a} to $w\mathfrak{a}$ crosses $H_{\alpha,n}$.

Proof of Theorem 4.5.9. Combining equations (4.88) and (4.89),

$$\mathbf{h}^{\vee}\kappa - (\rho + \rho_{\kappa}) = -2\rho_H - \sum_{(\alpha,n)\in\mathcal{H}_w} \alpha.$$

Taking the inner product with κ and using equation (4.86) gives

$$\mathbf{h}^{\vee} \|\kappa\|^2 - \langle \rho + \rho_{\kappa}, \kappa \rangle = -\sum_{(\alpha, n) \in \mathcal{H}_w} \langle \alpha, \kappa \rangle.$$
(4.90)

By Corollary 4.5.16, a line segment from \mathring{a} to $w\mathring{a}$ is contained entirely in \mathring{t}_+ . Let $(\alpha, n) \in \mathcal{R}_{aff,+}$, then

$$H_{\alpha,n} \cap \mathring{\mathfrak{t}}_+ \neq \emptyset \quad \Leftrightarrow \quad n > 0 \text{ and } \alpha \in \mathcal{R}_-.$$

Hence the affine roots $(\alpha, n) \in \mathcal{H}_w$ have $\alpha \in \mathcal{R}_- \Rightarrow \langle \alpha, \kappa \rangle \leq 0$. The result follows from (4.90).

Remark 4.5.17. Note that for κ a vertex of the fundamental alcove, $\mathcal{H}_w = \emptyset$, and equation (4.90) shows that the inequality in Theorem 4.5.9 becomes an equality.

Bibliography

- A. Alekseev, H. Bursztyn, and E. Meinrenken. Pure spinors on Lie groups. Astérisque, 327:131–199, 2009.
- [2] A. Alekseev, A. Malkin, and E. Meinrenken. Lie group valued moment maps. J. Differential Geom., 48:445–495, 1998.
- [3] A. Alekseev and E. Meinrenken. Dirac structures and Dixmier–Douady bundles. Int. Math. Res. Not., 2012:904–956, 2012.
- [4] A. Alekseev, E. Meinrenken, and C. Woodward. The Verlinde formulas as fixed point formulas. J.Symplectic Geom., 1:1–46, 2001.
- [5] A. Alekseev, E. Meinrenken, and C. Woodward. Duistermaat-Heckman measures and moduli spaces of flat bundles over surfaces. *Geom. and Func. Anal.*, 12:1–31, 2002.
- [6] M. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Phil. Trans. R. Soc. Lond.*, 308:523–615, 1983.
- [7] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23:1–28, 1984.
- [8] M. F. Atiyah and I. M. Singer. The index of elliptic operators: III. Ann. of Math., 87:546–604, 1968.
- [9] V. Baldoni, A. Boysal, and M. Vergne. Multiple Bernoulli series and volumes of moduli spaces of flat bundles over surfaces. J. Symb. Comp., 68:27–60, 2015.
- [10] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*. Springer-Verlag, 1992.
- [11] N. Berline and M. Vergne. Zéros d'un champs de vecteurs et classes caractéristiques équivariantes. Duke Math. J., 50:539–549, 1983.
- [12] N. Berline and M. Vergne. The equivariant index and Kirillov's character formula. Amer. J. of Math., 107(5):1159–1190, 1985.
- [13] R. Bott. The stable homotopy of the classical groups. Ann. of Math., 70:313–337, 1959.
- [14] R. Bott, T. Tolman, and J. Weitsman. Surjectivity for Hamiltonian loop group spaces. *Invent. Math.*, 155:225–251, 2004.
- [15] A. Boysal and M. Vergne. Multiple Bernoulli series, an Euler-MacLaurin formula, and wall crossings. Ann. de l'Ins. Fourier, 62:821–858, 2012.

- [16] M. Brion and M. Vergne. Arrangement of hyperplanes, I: rational functions and Jeffrey-Kirwan residue. Ann. Sci. Ecole Norm. Sup. (4), 32:715–741, 1999.
- [17] M. Brion and M. Vergne. Arrangement of hyperplanes, II: the Szenes formula and Eisenstein series. Duke Math J., 103:279–302, 2000.
- [18] A. Cannas da Silva and V. Guillemin. On the Kostant multiplicity formula for group actions with non-isolated fixed points. Adv. Math., 126:1–15, 1996.
- [19] E. Ehrhart. Polynômes arithmétiques et méthode des polyedres en combinatoire. Birkhauser, 1977.
- [20] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory II. J. of the Am. Math. Soc., 26:595–644, 2013.
- [21] V. Ginzburg, V. Guillemin, and Y. Karshon. Cobordism theory and localization formulas for Hamiltonian group actions. Int. Math. Res. Notices, 5:221–234, 1996.
- [22] V. Ginzburg, V. Guillemin, and Y. Karshon. Moment maps, Cobordisms, and Hamiltonian Group Actions. AMS, 2002.
- [23] V. Guillemin, E. Lerman, and S. Sternberg. On the Kostant multiplicity formula. J. Geom. Phys., 5:721–750, 1988.
- [24] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67(3):515–538, 1982.
- [25] V. Guillemin and S. Sternberg. Supersymmetry and Equivariant de Rham Theory. Springer, 1999.
- [26] M. Harada and Y. Karshon. Localization for equivariant cohomology with varying polarization. Communications in Analysis and Geometry, 20:869–947, 2012.
- [27] L. Jeffrey and F. Kirwan. Intersection theory on moduli spaces of holomorphic vector bundles of arbitrary rank on a Riemann surface. Annals of Math., 148:109–196, 1998.
- [28] L.C. Jeffrey and F.C. Kirwan. Localization for nonabelian group actions. *Topology*, 34:291–327, 1995.
- [29] V. Kac. Infinite-dimensional Lie algebras. Cam. Univ. Press, 1994.
- [30] G. Kasparov. Equivariant KK-theory and the Novikov conjecture. Invent. Math., 91:147–201, 1988.
- [31] F. Kirwan. Cohomology of quotients in symplectic and algebraic geometry. Princeton Univ. Press, 1984.
- [32] F. Knop. Multiplicity free quasi-Hamiltonian manifolds. arXiv:1612.03843, 2016.
- [33] S. Kumar and M. Vergne. Equivariant cohomology with generalized coefficients. Astérisque, 215:109–204, 1993.
- [34] H. B. Lawson and M-L. Michelsohn. Spin Geometry. Princeton Univ. Press, 1989.
- [35] Y. Loizides. Norm-square localization for Hamiltonian LG-spaces. J. of Geom. and Phys., 114:420– 449, 2017.

- [36] Y. Loizides, E. Meinrenken, and Y. Song. Spinor modules for Hamiltonian loop group spaces. *In preparation.*
- [37] E. Meinrenken. On Riemann-Roch formulas for multiplicities. J. Amer. Math. Soc., 9:373–390, 1996.
- [38] E. Meinrenken. Symplectic surgery and the Spin-c Dirac operator. Adv. in Math., 134:240–277, 1998.
- [39] E. Meinrenken. Witten's formulas for intersection pairings on moduli spaces of flat G-bundles. Advances in Mathematics, 197:140–197, 2005.
- [40] E. Meinrenken. Equivariant cohomology and the Cartan model. Encyclopedia of mathematical physics, pages 242–250, 2006.
- [41] E. Meinrenken. On the quantization of conjugacy classes. *LEns. Math.*, 55(1):33–75, 2009.
- [42] E. Meinrenken. Quantization of q-Hamiltonian SU(2)-spaces. In Geometric Aspects of Geometry and Mechanics (in honor of the 65th birthday of Hans Duistermaat). Birkhaeuser, 2011.
- [43] E. Meinrenken. Lectures on group-valued moment maps and verlinde formulas. Contemp. Math, 583:175–210, 2012.
- [44] E. Meinrenken. Twisted K-homology and group-valued moment maps. Int. Math. Res. Not., 2012:4563–4618, 2012.
- [45] E. Meinrenken and R. Sjamaar. Singular reduction and quantization. Topology, 38(4):699–762, 1999.
- [46] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. J. Diff. Geom., 50:417–469, 1998.
- [47] E. Meinrenken and C. Woodward. Canonical bundles for Hamiltonian loop group manifolds. Pacific Journal of Mathematics, 198(2):477–487, 2001.
- [48] P-E. Paradan. Formules de localisation en cohomologie équivariante. Compositio Math., 117:243– 293, 1999.
- [49] P-E. Paradan. The moment map and equivariant cohomology with generalized coefficients. *Topology*, 39:401–444, 2000.
- [50] P-E. Paradan. Localization of the Riemann-Roch character. J. Fun. Anal., 187:442–509, 2001.
- [51] P.-E. Paradan and M. Vergne. Admissible coadjoint orbits for compact Lie groups. arXiv:1512.02367.
- [52] P-E. Paradan and M. Vergne. Equivariant Dirac operators and differentiable geometric invariant theory. arXiv:1411.7772.
- [53] P-E. Paradan and Michele Vergne. Witten non abelian localization for equivariant K-theory, and the [Q, R] = 0 theorem. arXiv:1504.07502.

- [54] P. Prato and S. Wu. Duistermaat-Heckman measures in a non-compact setting. *Comp. Math*, pages 113–128, 1994.
- [55] A. Pressley and G.B. Segal. Loop groups. Clarendon Press, 1986.
- [56] J. Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. J. of the Aust. Math. Soc. (Ser. A), 47:368–381, 1989.
- [57] A. Szenes. The combinatorics of the Verlinde formula. In Vector bundles in algebraic geometry (Durham, 1993). London Math. Soc. Lec. Notes Ser., 208:241–253.
- [58] A. Szenes. Iterated residues and multiple Bernoulli polynomials. Int. Math. Res. Not., 18:937–956, 1998.
- [59] A. Szenes. Residue theorem for rational trigonometric sums and Verlinde's formula. Duke Math. J., 118:189–227, 2003.
- [60] A. Szenes and M. Vergne. [Q, R] = 0 and Kostant partition functions. arXiv:1006.4149.
- [61] M. Vergne. A note on the Jeffrey-Kirwan-Witten localisation formula. Topology, 35:243–266, 1996.
- [62] E. Verlinde. Fusion rules and modular transformations in 2D conformal field theory. Nuc. Phys. B, 300:360–376, 1988.
- [63] E. Witten. Two dimensional gauge theories revisited. J. Geom. Phys., 9:303–368, 1992.
- [64] C. Woodward. The classification of transversal multiplicity-free group actions. Ann. of Glob. Anal. and Geom., 14(1):3–42, 1996.
- [65] C. Woodward. Localization for the norm-square of the moment map and the two-dimensional Yang-Mills integral. J. Symplectic Geom., 3:17–53, 2005.