Geometric K-Homology and the Freed-Hopkins-Teleman Theorem

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References

Based on:

- L. *Geometric K-homology and the Freed-Hopkins-Teleman theorem*. arxiv:1804.05213 [L]
- L., E. Meinrenken, Y. Song. *Spinor modules for Hamiltonian loop group spaces*. arxiv:1706.07493 [LMS]
- L., Y. Song. *Quantization of Hamiltonian loop group spaces*. arxiv:1804.00110 [LS]

Plan

- Short introduction to the Freed-Hopkins-Teleman theorem.
- Describe a map $\mathrm{K}_0^G(G, \mathcal{A}^{(k+\mathrm{h}^\vee)}) \to R_k(LG)$.
- More explicit description for D-cycles.
- Application: two approaches to 'quantization' for Hamiltonian LG-spaces give the same result.

Loop group

G compact simple Lie grp, $\pi_0(G) = \pi_1(G) = 0$
$T \subset G$ max. torus
$LG = \{\gamma\colon S^1 \to G\}$ (fixed Sobolev class $> \frac{1}{2}$)

- $\Omega G \subset LG$ based loop group ($\gamma(0) = e$)
- $G \subset LG$ constant loops
- $\Pi = \ker(\exp_T) \subset \Omega G$; $\qquad \gamma_\eta(s) = \exp(s\eta)$, $\eta \in \Pi$

$$1 \to S^1_{\text{cent}} \to \widehat{LG} \to LG \to 1, \qquad \widehat{LG} \circlearrowleft S^1_{\text{rot}}$$

$$c(\xi_1, \xi_2) = \int_{S^1} B\big(\xi_1(s), \xi_2'(s)\big)\, ds, \qquad \xi_1, \xi_2 \in L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$$

B = basic inner product on $\mathfrak{g}$

Verlinde ring

Representation of $S^1_{\text{rot}} \ltimes \widehat{LG}$ such that S^1_{cent} acts with weight k and S^1_{rot} weights bounded below by 0 is called a **level k positive energy representation**.

Irreducible level k P.E.R. $V_\lambda \stackrel{1:1}{\longleftrightarrow} \lambda \in \Pi^*_k = \Pi^* \cap k\mathfrak{a}$

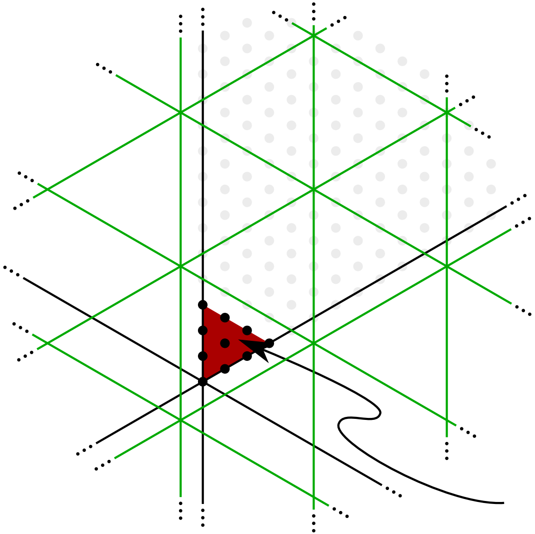
($\mathfrak{a}$ = fundamental Weyl alcove)

Verlinde ring $R_k(LG) = \mathbb{Z}[\Pi^*_k] \simeq R(G)/I_k(G)$

$$R_k(LG) \simeq R^{-\infty}(T)^{W_{\text{aff}}-\text{anti},\, k+\text{h}^\vee}$$

$$\left(\text{compare } R(G) \simeq R(T)^{W-\text{anti}}\right)$$

W_{aff} alternating formal characters, $G = SU(3)$, $k = 3$



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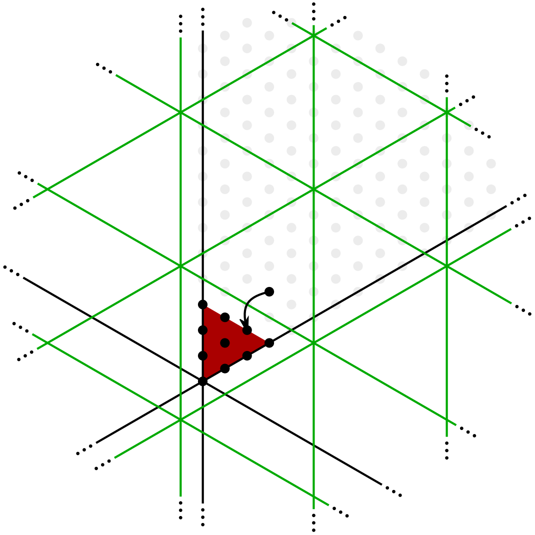

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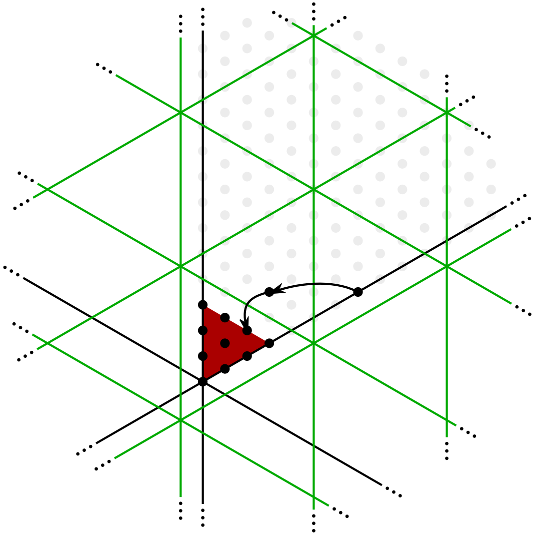

Twisted K-homology

$\mathcal{A}$ = bundle over X with fibres $\mathbb{K}(H)$
Dixmier-Douady class $\mathrm{DD}(\mathcal{A}) \in H^3_G(X,\mathbb{Z}) \times H^1_G(X,\mathbb{Z}_2)$

Analytic definition: $\mathrm{K}^G_0(X,\mathcal{A}) = \mathrm{KK}_G(C_0(\mathcal{A}),\mathbb{C})$ is the $\mathcal{A}$-twisted K-homology of X (even degree)

Cycles: (H,ρ,F), $H = \mathbb{Z}_2$-graded Hilbert space

$$\rho\colon C_0(\mathcal{A}) \to \mathbb{B}(H)$$

F 'abstract 0^{th} order odd elliptic operator': $a \in C_0(\mathcal{A})$

$$[F,\rho(a)], \quad \rho(a)(F-F^*), \quad \rho(a)(1-F^2) \quad \in \mathbb{K}(H)$$

Example: M even-dim, Riem. $\mathcal{A} = \mathrm{Cliff}(TM)$
'Fundamental class' $[M]$ with $F = \chi(d+d^*)$ (bounded version of de Rham-Dirac operator)

Geometric 'D-cycles'

D-cycle $(M, E, \Phi, \mathcal{S})$ for $\mathrm{K}_0(X, \mathcal{A})$ is

- M compact even-dim. Riem. manifold
- E Hermitian vector bundle on M
- $\Phi\colon M \to X$ continuous
- Morita bimodule $\Phi^*\mathcal{A} \circlearrowleft \mathcal{S} \circlearrowright \mathrm{Cliff}(TM)$

$$\Rightarrow \quad \Phi^*\mathrm{DD}(\mathcal{A}) = \mathrm{DD}(\mathrm{Cliff}(TM))$$

$$(\Phi, \mathcal{S})_*([E] \cap [M]) \in \mathrm{K}_0^G(X, \mathcal{A})$$

Freed-Hopkins-Teleman Theorem

$X = G \circlearrowleft G$ (conjugation) $H^3_G(G, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1 = 0 \Rightarrow \mathrm{DD}(\mathcal{A}) \in \mathbb{Z}$
<u>Notation</u>: $\mathcal{A}^{(\ell)}$ for $\mathrm{DD}(\mathcal{A}^{(\ell)}) = \ell$

Theorem (Freed, Hopkins, Teleman)

Let $k > 0$, and $\mathrm{h}^\vee$ = the dual Coxeter number. Then

$$R_k(LG) \simeq \mathrm{K}_0^G(G, \mathcal{A}^{(k+\mathrm{h}^\vee)}).$$

Given P.E.R., F-H-T construct a cycle for twisted K-theory, inducing isomorphism above.

Cycle for $\mathrm{K}_0^G(G, \mathcal{A}^{(k+\mathrm{h}^\vee)}) \longrightarrow ?? \in R_k(LG)$

Main result

Theorem 1 (L.)

Construction of a map

$$\mathscr{I}: K_0^G(G, \mathcal{A}^{(k+\mathrm{h}^\vee)}) \to R^{-\infty}(T)^{W_{\mathrm{aff}}-\mathrm{anti}, k+\mathrm{h}^\vee} \simeq R_k(LG)$$

inverse to F-H-T isomorphism.

Theorem 2 (L.)

For a D-cycle $x = (M, E, \Phi, \mathcal{S})$, $\mathscr{I}(x)$ given by T-equivariant L^2-index of 1^{st} order elliptic operator on Π-covering space of $\Phi^{-1}(U)$, $U \subset G$ a tubular neighborhood of $T = \mathfrak{t}/\Pi$.

$$PG = \{\gamma\colon \mathbb{R} \to G | \gamma(s)^{-1}\gamma(s+1) = \text{const}\}$$

$$LG \circlearrowright PG \longrightarrow G$$

V = level ℓ P.E.R. $\Rightarrow$

$$\mathcal{A} = PG \times_{LG} \mathbb{K}(V^*) \qquad \text{DD}(\mathcal{A}) = \ell$$

(soon $\ell = k + \mathrm{h}^\vee$)

A Dixmier-Douady bundle over T

$$T \times \Pi \hookrightarrow LG$$

pullback the $U(1)$ central extension $\widehat{LG}$

$$1 \to U(1) \to T \ltimes \widehat{\Pi} \to T \times \Pi \to 1$$

$$L^2(\widehat{\Pi})_{(-\ell)} \circlearrowleft T \ltimes \widehat{\Pi}$$

$$\mathcal{B}^{(\ell)} = \mathfrak{t} \times_{\Pi} \mathbb{K}(L^2(\widehat{\Pi})_{(-\ell)})$$

$$\mathcal{A}^{(\ell)}|_T \simeq_{\mathcal{E}} \mathcal{B}^{(\ell)}, \qquad \mathcal{E} = \mathfrak{t} \times_{\Pi} \left(V \otimes L^2(\widehat{\Pi})_{(-\ell)}\right)$$

Definition of $\mathscr{I}$

- $\mathrm{K}_0^G(G, \mathcal{A}^{(\ell)}) \to \mathrm{K}_0^T(U, \mathcal{A}^{(\ell)}|_U) \xrightarrow{\sim} \mathrm{K}_0^T(T, \mathcal{A}^{(\ell)}|_T)$
- Morita isomorphism $\mathcal{A}^{(\ell)}|_T \simeq_{\mathcal{E}} \mathcal{B}^{(\ell)}$
- $C_0(\mathcal{B}^{(\ell)})$ is a (twisted) crossed product algebra $\Pi \ltimes_{(\ell)} C_0(\mathfrak{t})$ $\rightsquigarrow$ element in K-homology group

$$\mathrm{K}_0^{T \ltimes \widehat{\Pi}}(\mathfrak{t}) = \mathrm{KK}^{T \ltimes \widehat{\Pi}}(C_0(\mathfrak{t}), \mathbb{C})$$

(also used a Green-Julg-type isomorphism here)

- Analytic assembly map ('integration over $\mathfrak{t}$')

$$\mathrm{K}_0^{T \ltimes \widehat{\Pi}}(\mathfrak{t}) \to \mathrm{K}_0(C^*(T \ltimes \widehat{\Pi}))$$

- Track Weyl symmetry $\Rightarrow$ range contained in subgroup isomorphic to $R^{-\infty}(T)^{W_{\mathrm{aff}}-\mathrm{anti},\ell}$.

$\mathscr{I}$ for D-cycles

$(M, E, \Phi, \mathcal{S}) \rightsquigarrow$ analytic cycle (H, ρ, F) (push-forward to G)

$H = L^2$-sections of Cliff(TM)-mod (infinite dim)

Morita morphism $\mathcal{A}^{(\ell)}|_T \simeq_{\mathcal{E}} \mathcal{B}^{(\ell)}$
$\rightsquigarrow$ Hilbert bundle fibres $L^2(\widehat{\Pi})_{(-\ell)}$ ($\otimes$ finite dim)
$\rightsquigarrow$ Π-covering space

Analytic assembly $\rightsquigarrow$ T-equivariant L^2-index

Hamiltonian LG-spaces

$L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$ connections on $S^1 \times G$

$$g \cdot \xi = \mathrm{Ad}_g \xi - dgg^{-1}, \qquad g \in LG$$

same as coadjoint action on $L\mathfrak{g}^* \times \{1\} \subset \widehat{L\mathfrak{g}}^*$

Definition

A *Hamiltonian LG-space* $(\mathcal{M}, \omega_{\mathcal{M}}, \Psi_{\mathcal{M}})$ consists of a LG-manifold $\mathcal{M}$, symplectic form $\omega_{\mathcal{M}}$, proper moment map

$$\Phi_{\mathcal{M}} \colon \mathcal{M} \to L\mathfrak{g}^*.$$

Quantization, index pairing approach

- $\mathcal{X} = \Phi_{\mathcal{M}}^{-1}(\mathfrak{t})$, $\mathfrak{t} = \mathrm{Lie}(T)$, is maybe singular
- $\exists$ finite dim smooth spin-c 'thickening' $\mathcal{Y}$ [LMS]
- twist by level k prequantum line bundle L, take 'index pairing' [LS]:

$$[\mathrm{PD}(\mathcal{X})] \cap [\mathrm{D}^L] \in R^{-\infty}(T)^{W_{\mathrm{aff}}-\mathrm{anti},k+\mathrm{h}^\vee} \simeq R_k(LG)$$

$\star$ Can do Witten deformation!

Quantization, twisted K-homology approach

Quotient by $\Omega G \subset LG$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Phi_{\mathcal{M}}} & L\mathfrak{g}^* \\ \big\downarrow /\Omega G & & \big\downarrow /\Omega G \\ M & \xrightarrow{\Phi} & G \end{array}$$

M is a **quasi-Hamiltonian** G-space [Alekseev-Malkin-Meinrenken]

(Ham LG-spaces) $\overset{1:1}{\longleftrightarrow}$ (q-Ham G-spaces)

- M need not have a spin-c structure
- M gives rise to a canonical level $\mathsf{h}^\vee$ D-cycle $(M, \mathbb{C}, \Phi, \mathcal{S})$ for $\mathsf{K}_0^G(G, \mathcal{A}^{(\mathsf{h}^\vee)})$ [Alekseev-Meinrenken]
- Level k prequantum $L \to \mathcal{M} \rightsquigarrow$ Morita morphism

$$\underline{\mathbb{C}} \simeq_{\mathcal{L}} \Phi^* \mathcal{A}^{(k)}$$

- Define 'quantization' of M as class of $(M, \mathbb{C}, \Phi, \mathcal{S} \otimes \mathcal{L})$ in $\mathsf{K}_0^G(G, \mathcal{A}^{(k+\mathsf{h}^\vee)})$ [Meinrenken]

Compatibility of the approaches

Let $x = (\Phi, \mathcal{S} \otimes \mathcal{L})_*[M]$ be the pushforward of the D-cycle

$$(M, \mathbb{C}, \Phi, \mathcal{S} \otimes \mathcal{L})$$

Theorem 1 $\Rightarrow$ under the isomorphism

$$R^{-\infty}(T)^{W_{\mathrm{aff}}-\mathrm{anti},k+\mathrm{h}^\vee} \simeq R_k(LG)$$

$\mathscr{I}(x)$ is the image of x under the F-H-T isomorphism.

Theorem 2 $\Rightarrow$

$$\mathscr{I}(x) = [\mathrm{PD}(\mathcal{X})] \cap [\mathrm{D}^L] \in R^{-\infty}(T)^{W_{\mathrm{aff}}-\mathrm{anti},k+\mathrm{h}^\vee}$$

□

Thanks!