Transversally Elliptic Operators

Yiannis Loizides, Essay for Index theory, Fall 2013

Atiyah and Singer were able to extend many of their results on elliptic operators on compact G-manifolds to a larger class known as the transversally elliptic operators. These are operators which are elliptic in directions perpendicular to the orbits of the group action. In particular, they were able to define the index of such an operator as a distribution on the group G, and show that many of the properties of the index for elliptic operators carried over with only slight modification. The majority of this essay will focus on properties of these operators and their indices (sketching proofs of only a selection of them), following closely Atiyah's 1971 lecture notes [2]. We'll then briefly describe two more recent developments: (1) a general cohomological formula for the index developed by Berline and Vergne [3], (2) an application by Paradan of the theory to give a new proof of the "quantization commutes with reduction" theorem [4].

1 Definition of the analytic index

Throughout, G will denote a compact Lie group with Lie algebra $\mathfrak{g}$, acting on a compact manifold X. We identify TX and T^*X using a G-invariant Riemannian metric on X. For each point $p \in X$ the action gives a map $\mathfrak{g} \to T_pX$ whose image we denote $\mathfrak{g}_X(p)$. Let $\pi : T^*X \to X$ be the projection (we'll also sometimes use π_{TX}).

Definition 1. Let

$$T^*_G X = \bigcup_{p \in X} \{\alpha \in T^*_p X : \alpha(\mathfrak{g}_X(p)) = 0\}.$$

We call a G-invariant pseudo-differential operator $P : \mathcal{D}(E) \to \mathcal{E}(F)$ *transversally elliptic* if its symbol $\sigma(P) \in \mathrm{Hom}(\pi^*E, \pi^*F)$ is invertible on $T^*_G X - 0$. Here $0 \simeq X$ is the zero section of T^*X. Note that since $\sigma(P^*) = \sigma(P)^*$, P^* is also transversally elliptic.

We would like to define the index of such an operator as

$$\mathrm{ind}(P) = \chi_{\ker(P)} - \chi_{\ker(P^*)}, \tag{1}$$

where χ_V is meant to denote the character of the representation V. But this expression requires further explanation because transversally elliptic operators need not be Fredholm (when extended to bounded operators between appropriate Sobolev spaces). For example, if $X = G$ with the left action, then $T^*_G G = 0$, and so the zero operator $E \to F$ is transversally elliptic but never Fredholm. Also G need not act by trace-class operators (for example the identity $e \in G$). But it is possible that smearing the group elements with a smooth function will yield a trace-class operator, or equivalently that the representation of the smooth group algebra will be trace-class. Let (ρ, V) denote a (possibly infinite-dimensional) representation of G. For $\phi \in \mathcal{D}(G)$ define

$$\rho(\phi) = \int_G \phi(g)\rho(g)dg.$$

The composition

$$\phi \in \mathcal{D}(G) \mapsto \mathrm{Tr}(\rho(\phi))$$

would then define a distribution on G which we denote $\chi_V \in \mathcal{D}'(G)$. This gives meaning to (1) provided we can show that for P transversally elliptic, the representations $\ker(P)$ and $\ker(P^*)$ of $\mathcal{D}(G)$ are trace-class.

Theorem 1. *Let $P : \mathcal{D}(E) \to \mathcal{E}(F)$ be a G-invariant transversally elliptic operator on X, and let ρ denote the representation of G on* $\ker(P)$*. Then the composition*

$$\phi \in \mathcal{D}(G) \mapsto \mathrm{Tr}(\rho(\phi))$$

defines a distribution on G.

Proof. The G-action gives the extra structure needed to make up for the fact that P is not elliptic. The basic idea is to build an elliptic operator out of P together with the G action, to which we apply the powerful theorems on elliptic operators to get facts about P. Use a hermitian metric and compatible connection on F to define a Laplace operator on sections of F ("Bochner Laplacian") which has spectrum contained in $[0, \infty)$. If we add the identity operator we get an operator L acting on sections of F with spectrum contained in $[1, \infty)$. Using the functional calculus we have L^s for any $s \in \mathbb{R}$, which has order $2s$ and trivial kernel. One way to think of L is as a bounded operator between appropriate Sobolev spaces, and in fact we could compose with the explicit isomorphism between two Sobolev spaces discussed in class to get a bounded operator on a single Sobolev space. Choosing an appropriate s, L^sP will have order 2 and the same kernel as P. This means that without loss of generality we can restrict to the case where P is second-order (recall that the order is well-defined also for pseudo-differential operators, it referring to the exponent in the estimates that the total symbol satisfies).

Let $Y_1, ..., Y_n$ denote an orthonormal basis for $\mathfrak{g}$, and use the same notation for the corresponding left-invariant vector fields on G. Let $\tilde{Y}_1, ..., \tilde{Y}_n$ denote the corresponding first-order differential operators (Lie derivatives) on E. Let

$$\Delta_E = 1 - \sum_i \tilde{Y}_i^2.$$

The $-\tilde{Y}_i^2$ are positive operators, and so Δ_E will have spectrum contained in $[1, \infty)$. It is constructed precisely so that its symbol is injective along the G-orbit directions.

Now define

$$A = (P, \Delta_E) : \mathcal{D}(E) \to \mathcal{D}(F) \oplus \mathcal{D}(E).$$

We have

$$\sigma(A) = (\sigma(P), \sigma(\Delta_E)).$$

which gives an injective homomorphism $\sigma(A)(\xi) \in \mathrm{Hom}(E, F \oplus E)$ for each $\xi \in T^*X - 0$ since $\sigma(P)(\xi)$ is injective for $\xi \in T^*_G X - 0$ while $\sigma(\Delta_E)(\xi)$ is injective for ξ pointing along an orbit direction (we're using a G-invariant Riemannian metric to identify TX and T^*X). It follows that $\sigma(A)^*\sigma(A)$ is an isomorphism on $T^*X - 0$, showing that A^*A is elliptic.

This allows us to define subspaces (eigenspaces) of $\mathcal{D}(E)$

$$\ker(P)_\lambda = \{u \in \mathcal{D}(E) : Pu = 0, \Delta_E u = \lambda u\}$$

$$(A^*A)_\lambda = \{u \in \mathcal{D}(E) : A^*Au = \lambda^2 u\}.$$

Since A^*A is elliptic, $(A^*A)_\lambda$ is finite dimensional for all λ, and consists of C^∞-sections. Since $\ker(P)_\lambda \subset (A^*A)_\lambda$, we have that the $\ker(P)_\lambda$ are finite dimensional and consist of C^∞-sections. Let

$$b_\lambda := \dim((A^*A)_\lambda) \geq a_\lambda := \dim(\ker(P)_\lambda).$$

Since A^*A has spectrum contained in $[1, \infty)$ we can form $(A^*A)^{-s}$ for any $s > 0$. Because A^*A has order 4, $(A^*A)^{-s}$ has order $-4s$. Now fix s such that

$4s > \dim(X)$. Then $(A^*A)^{-s}$ maps delta distributions to continuous sections by Sobolev embedding. This means that $(A^*A)^{-s}$ must have a continuous Schwartz kernel, $K_s(x, y)$. In particular

$$\mathrm{Tr}(A^*A)^{-s} = \int_X \mathrm{Tr}(K_s(x, x))dx < \infty$$

On the other hand

$$\mathrm{Tr}(A^*A)^{-s} = \sum_\lambda b_\lambda \lambda^{-2s},$$

(in our notation the eigenvalues of A^*A are the λ^2) which shows that the sum converges (absolutely). Let f_λ denote the character of the finite dimensional representation $\ker(P)_\lambda$. Since G is compact, we can assume the representation is unitary, so that for each $g \in G$, $f_\lambda(g)$ is the trace of a unitary matrix. Since a unitary matrix has all its eigenvalues on the unit circle, its trace is less than $\dim(\ker(P)_\lambda) = a_\lambda \leq b_\lambda$. Consequently the absolute convergence of the series above implies that

$$h := \sum_\lambda f_\lambda \lambda^{-2s}$$

converges in sup-norm, showing that h is a continuous function on G.

Let ρ_λ denote the representation of G on $\ker(P)_\lambda$, so $f_\lambda = \mathrm{Tr} \circ \rho_\lambda$. Let e_j denote an orthonormal basis for $\ker(P)_\lambda$. We have

$$\begin{aligned} Y_k f_\lambda(g) &= \frac{d}{dt}\bigg|_0 f_\lambda(g \cdot \exp(tY_k)) \\ &= \sum_j \frac{d}{dt}\bigg|_0 \langle e_j, g\exp(tY_k)e_j\rangle \\ &= \sum_j \langle e_j, g(\tilde{Y}_k e_j)\rangle. \end{aligned}$$

Consequently with $\Delta := 1 - \sum_i Y_i^2$ (an operator of order two on G) we have

$$\begin{aligned} \Delta f_\lambda(g) &= \sum_j \langle e_j, g\Delta_E e_j\rangle \\ &= \sum_j \langle e_j, \lambda g e_j\rangle = \lambda f_\lambda(g). \end{aligned}$$

This gives that

$$H_{-4s}(G) \ni \Delta^{2s} h = \sum_\lambda \Delta^{2s} \frac{f_\lambda}{\lambda^{2s}} = \sum_\lambda \lambda^{2s} \frac{f_\lambda}{\lambda^{2s}} = \sum_\lambda f_\lambda = \chi_{\ker(P)},$$

where the equalities are of distributions (which makes commuting Δ^{2s} past the infinite sum automatic; alternatively think of Δ^{2s} as a continuous linear mapping between Sobolev spaces). This shows that $\chi_{\ker(P)}$ defines a distribution in $H_{-t}(G)$ when $t > \dim(X)$. $\square$

Since P commutes with the G-action, it intertwines Δ_E with Δ_F, and so in particular it intertwines the λ-eigenspaces $\mathcal{D}(E)_\lambda \subset \mathcal{D}(E)$ and $\mathcal{D}(F)_\lambda \subset \mathcal{D}(F)$. Consequently by restriction we get linear maps

$$P_\lambda : \mathcal{D}(E)_\lambda \to \mathcal{D}(F)_\lambda.$$

We can use a trick similar to the one used in the proof to get a parametrix for P_λ. The symbol of the operator

$$(P, \Delta_E - \lambda) : \mathcal{D}(E) \to \mathcal{D}(F) \oplus \mathcal{D}(E)$$

is injective for each $\xi \in T^*X - 0$. It thus has a left parametrix T, that is, $T \circ (P, \Delta_E - \lambda) - 1$ has a C^∞ Schwarz kernel, and in particular is compact. The restriction to $\mathcal{D}(F)_\lambda$, denoted T_λ, is a left parametrix for P_λ. Similarly P_λ^* has a left parametrix S_λ. Since $S_\lambda P_\lambda^* - 1$ has C^∞ Schwarz kernel, taking adjoints shows that $P_\lambda S_\lambda^* - 1$ has a C^∞ Schwarz kernel, or in other words S_λ^* is a right parametrix for P_λ.

Let

$$H_s(E)_\lambda = \ker((\Delta_E - \lambda) : H_s(E) \to H_{s-2}(E)).$$

Now extend P_λ as a map

$$P_\lambda : H_s(E)_\lambda \to H_{s-m}(F)_\lambda$$

(it intertwines Δ_E and Δ_F because it commutes with the G-action). The existence of the parametrices implies that, when projected down to the Calkin algebra, P_λ is invertible (in the Calkin algebra, the parametrices become a left and a right inverse, which must therefore coincide). Consequently P_λ is Fredholm for each λ. Its index is

$$\operatorname{ind}(P_\lambda) = \chi_{\ker(P_\lambda)} - \chi_{\ker(P_\lambda^*)}$$

Taking a homotopy of P through transversally elliptic operators leads to a homotopy of the operators P_λ. In particular $\mathrm{ind}(P_\lambda)$ does not change. Since

$$\mathrm{ind}(P) = \sum_\lambda \mathrm{ind}(P_\lambda)$$

(as an equality of distributions on G), we have that $\mathrm{ind}(P)$ depends only on the homotopy class of the symbol $\sigma(P)$ in the space of transversally elliptic symbols. Since the K-group $K_G(T^*_G X)$ may be defined in terms of homotopy classes of transversally elliptic symbols, this shows that we obtain a well-defined map

$$\mathrm{ind} : K_G(T^*_G X) \to \mathcal{D}'(G),$$

the analytic index map.

To summarize in different words, a transversally elliptic operator P gives rise to a collection of Fredholm operators P_λ, with P_λ defined on the (usually infinite dimensional) λ-eigenspace of the operator Δ_E. The kernel of P_λ is finite dimensional and does not depend on which Sobolev exponent s one takes to define the eigenspaces of Δ_E (its kernel consists of smooth sections). Consequently $\mathrm{ind}(P_\lambda)$ is a smooth function on G independent of s, and by the theorem $\mathrm{ind}(P) = \Sigma_\lambda \mathrm{ind}(P_\lambda)$ is a well-defined distribution on G independent of s. Note however that $\ker(P)$ *is* dependent on s in general. This is because it is the closure in H_s of $\Sigma_\lambda \ker(P_\lambda)$, which will be different for different s (since the topologies on H_s differ for different s).

2 Properties of the analytic index

We'll discuss a number of general properties of the index which are proved by Atiyah in lectures 3 and 4.

2.1 The multiplicative property

If H acts freely on X then we have an isomorphism

$$K_H(X) = K(X/H).$$

Applying this in the case of a $G \times H$-manifold X, on which H acts freely, we have

$$K_{G\times H}(T^*_{G\times H} X) = K_G((T^*_{G\times H} X)/H) = K_G(T^*_G(X/H)), \tag{2}$$

where in the second equality we use the natural identification of H-basic covectors on X with covectors on X/H. The isomorphism is induced by pullback by the map $\pi : X \to X/H$. We would like to relate the index maps for the two sides of (2). Note that the index map $\mathrm{ind}_{G\times H}$ yields a distribution on $G \times H$, whereas ind_G yields a distribution on G. Let V_α for $\alpha \in \hat{H}$ where $\hat{H}$ denotes the set of irreducible representations of H (indexed by dominant weights), and let χ_α denote the corresponding character. For each V_α we get a vector bundle $\underline{V}_\alpha := X \times_H V_\alpha$. Atiyah proves the following result:

Theorem 2. *Let $a \in K_G(T^*_G(X/H))$, then*

$$\mathrm{ind}_{G\times H}(\pi^* a) = \sum_{\alpha \in \hat{H}} \mathrm{ind}_G(a \otimes \underline{W}^*_\alpha) \cdot \chi_\alpha.$$

Here we use the same notation for the vector bundle $\underline{W}^*_\alpha$ and its pullback to $T^*_G(X/H)$. The equality is of distributions on $G \times H$. In particular this shows that the H-invariant part of $\mathrm{ind}_{G\times H}(\pi^* a)$ is $\mathrm{ind}_G(a)$.

We recall the definition of the symbol class $[E, F, \sigma] \in K(T^*_G X)$ (specializing to our situation). Let E, F be vector bundles over X and let $\sigma \in \mathrm{Hom}(\pi^* E, \pi^* F)$ be an isomorphism on $T^*_G X - 0$. Choose a vector bundle F' such that $F \oplus F' = X \times \mathbb{C}^n$. We use the map $\sigma \oplus 1$ to glue the bundle $\pi^* E \oplus \pi^* F'$ restricted to $T^*_G X \subset (T^*_G X)^+$ to the trivial bundle of rank n defined in a neighbourhood of the point at infinity in $(T^*_G X)^+$. This yields a vector bundle W. Then $[E, F, \sigma] := [W] - n$, where $n := [(T^*_G X)^+ \times \mathbb{C}^n]$.

Addition of symbol classes is straightforward. We have

$$[E_0, E_1, \sigma] + [F_0, F_1, \tau] = [E_0 \oplus F_0, E_1 \oplus F_1, \sigma \oplus \tau].$$

Products are slightly trickier. Introducing Hermitian metrics on E_i, F_i (so that σ^* is defined), a product of symbol classes is given by

$$[E_0, E_1, \sigma][F_0, F_1, \tau] = \big[(E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_1 \otimes F_0) \oplus (E_0 \otimes F_1), \theta\big]$$

where

$$\theta = \begin{pmatrix} \sigma \otimes 1 & -1 \otimes \tau^* \\ 1 \otimes \tau & \sigma^* \otimes 1 \end{pmatrix}.$$

That this is equivalent to the product induced from tensor products of vector bundles requires some explanation.

The method given by Atiyah [1] is indirect. For each pair $Y \subset X$ of compact Hausdorff spaces, Atiyah defines the set $L(X,Y)$ which consists of finite-length chain complexes of vector bundles on X which are exact on Y, modulo the equivalence relation given by addition of elementary sequences: those of the form $\cdots \to 0 \to E_0 \to E_0 \to 0 \to \cdots$ where the map $E_0 \to E_0$ is the identity. He then shows that there is a unique natural transformation of functors $\chi_{X,Y} : L(X,Y) \to K(X,Y)$ satisfying

$$\chi_{X,\emptyset}(E := 0 \to E_n \to \cdots \to E_0 \to 0) = \sum_{i=0}^{n} (-1)^i [E_i].$$

This natural transformation is called the *Euler characteristic*. Given a complex E in $L(X,Y)$,

$$E := 0 \to E_n \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} E_0 \to 0$$

it is always possible to find a complex of length two in the same equivalence class. It can be constructed by putting Hermitian metrics on the bundles E_i and then considering the complex

$$0 \to F_1 \xrightarrow{f} F_0 \to 0$$

where

$$F_0 = \bigoplus E_{2i} \qquad F_1 = \bigoplus E_{2i-1}$$

and

$$f = \sum f_{2i-1} + \sum f_{2i}^*.$$

This is analogous to the trick used for example when one wants to think of the de Rham differential d as an odd operator on a $\mathbb{Z}/2$-graded vector space. Then one applies the "difference bundle contruction" (as seen in class) to obtain the class $\chi(E)$ in $K(X,Y)$.

The key fact is that χ intertwines tensor products of chain complexes with the product in $K(X,Y)$ (induced from tensor products of vector bundles). This is clear for $Y = \emptyset$ from the formula for the natural transformation χ. It is less clear for general Y, but follows from the naturality of χ together with the properties of products in $L(X,Y)$ and $K(X,Y)$. In the case of symbols we take $X = D_G M$ the (closed) disc "bundle" in $T_G^* M$ (so the discs will vary in dimension if the group action fails to be locally free) and $Y = S_G M$ the sphere bundle in $T^*_G M$. Even though the "fibres" now vary dimension, the construction works the same because it is performed fibre-wise. It would be nice to have a more direct argument that the formula for the product of symbol classes above is correct, but it is not obvious to me how to do this.

With this we can state the multiplicative property of the index.

Theorem 3. *Let X be a compact G-space and Y a compact $G \times H$-space. External tensor product induces a multiplication*

$$K_G(T^*_G X) \otimes K_{G\times H}(T^*_H Y) \to K_{G\times H}(T^*_{G\times H}(X \times Y)).$$

By the discussion above, this can be computed on symbol classes using the formula for θ given above. Moreover

$$\mathrm{ind}^{X\times Y}_{G\times H}(ab) = \mathrm{ind}^X_G(a) \cdot \mathrm{ind}^Y_{G\times H}(b).$$

The proof involves a short calculation using transversally elliptic operators representing the classes a, b, together with the concrete formula for products of symbol classes discussed above. A remark that Atiyah makes following the proof helps to explain why one needs for example $b \in K_{G\times H}(T^*_H Y)$. If B is a transversally elliptic operator with symbol b, then $b \in K_{G\times H}(T^*_H Y)$ implies that

$$\mathrm{ind}(B) = \sum_{\alpha,\beta} C_{\alpha\beta}\chi_\alpha \phi_\beta$$

where χ_α, ϕ_β are the characters of the irreducible representations of G, H respectively, and the $C_{\alpha\beta}$ are coefficients. Since $b \in K_{G\times H}(T^*_H Y)$ and not $K_{G\times H}(T^*_{G\times H} Y)$, for each β only finitely many of the $C_{\alpha\beta}$ are nonzero. This ensures that $\mathrm{ind}(B)$ can be multiplied by a distribution on G (in general this is not possible).

The multiplicative property turns out to be extremely useful. For example, below we'll see examples where a useful map between two different K-groups can be constructed by setting one of the arguments in the product equal to a fixed element. This includes the induction and restriction maps. Another application is to fibre bundles which are not products. One can always write a vector bundle E as an associated bundle $P \times_{O(n)} \mathbb{R}^n$, and it is sometimes possible to deduce information by applying the multiplicative property to $P \times \mathbb{R}^n$.

2.2 Excision and wrong-way maps

The next result Atiyah discusses is excision, which allows the index map to be extended to non-compact spaces.

Theorem 4. *Let $j : U \to X$ be an open G-equivariant embedding, where X is compact. Then there is an induced map*

$$j_* : K_G(T^*_G U) \to K_G(T^*_G X).$$

Moreover the composition $\mathrm{ind} \circ j_*$ *is independent of the choice of embedding.*

As in the elliptic case, there is a "wrong-way" map $j_!$ associated to any G-equivariant embedding $j : X \to Y$. Let $\pi : N \to X$ be the normal bundle of X in Y, then $TN \simeq \pi^*(N \oplus N)$ is a complex vector bundle over TX so we have the symbol on N defining the Bott element

$$\mathrm{Cliff} : (T\pi)^* \wedge^{ev} (TN^{1,0}) \to (T\pi)^* \wedge^{odd} (TN^{1,0}).$$

(By "symbol on N" we mean a map of bundles over $TN \simeq T^*N$.) Let $a \in K_G(T^*_G X)$. We can represent a by a transversally elliptic symbol $\sigma \in \mathrm{Hom}(\pi^*_{TX} E_0, \pi^*_{TX} E_1)$ where E_0, E_1 are vector bundles over X. Pull σ, $\pi^*_{TX} E_0$, $\pi^*_{TX} E_1$ back by $T\pi$ to obtain a symbol $(T\pi)^*\sigma$ on N. We can then multiply this with the symbol Cliff (as described earlier). This gives a symbol $\mathrm{Cliff} \cdot (T\pi)^*\sigma$ on N which is transversally elliptic and hence defines a class in $T^*_G N$. Composing with the map j_* induced by the open inclusion $T^*_G N \to T^*_G Y$ we get an element $j_!(a) \in K_G(T^*_G Y)$.

Similar to the elliptic case, we have the following theorem.

Theorem 5. *Let X be compact and let $j : X \to Y$ be a G-equivariant embedding. Then* $\mathrm{ind}_Y \circ j_! = \mathrm{ind}_X$.

We'll describe some ideas from the proof since it is an example of the "associated bundle trick" mentioned above. First by excision and the tubular neighbourhood theorem, it is enough to consider $Y = N$ where N is a G-equivariant vector bundle over X. Now write $N = P \times_{O(n)} \mathbb{R}^n$ as an associated bundle of a G-equivariant principal $O(n)$-bundle, with G acting trivially on $\mathbb{R}^n$. Since the $O(n)$ action on P is free, the pullback maps

$$K_G(T_G X) \to K_{G \times O(n)}(T_{G \times O(n)} P)$$
$$K_G(T_G N) \to K_{G \times O(n)}(T_{G \times O(n)}(P \times \mathbb{R}^n))$$

are isomorphisms (again we're using the identification of $O(n)$-basic covectors on P with covectors on $P/O(n) = X$). We have a multiplication

$$K_{G\times O(n)}(T_{G\times O(n)}P) \otimes K_{G\times O(n)}(T\mathbb{R}^n) \to K_{G\times O(n)}(T_{G\times O(n)}(P \times \mathbb{R}^n)).$$

Under the isomorphisms above this becomes

$$K_G(T_G X) \otimes K_{G\times O(n)}(T\mathbb{R}^n) \to K_G(T_G N).$$

Taking the second argument in this product to be the Bott class in $K_{G\times O(n)}(T\mathbb{R}^n)$ realizes the Bott isomorphism $j_!$ as a special case of the product! The result then follows quickly from the multiplicative property of the index (using that the index of the Bott class is 1). (In fact the multiplicative property yields a slightly more general equation between distributions on $G \times O(n)$, and the desired result comes from taking the $O(n)$-invariant parts of the equation.)

2.3 Localization

Let $X^g = \{x \in X | gx = x\}$. Then we have the following result:

Theorem 6. (Localization in equivariant K-theory) *Let $\sigma \in K_G(T_G X)$ be a transversally elliptic symbol. Then*

$$\mathrm{supp}(\mathrm{ind}(\sigma)) \subset \bigcup_{X^g \neq \emptyset} g.$$

This is analogous to localization in equivariant cohomology (and is another instance of the analogy between integration for de Rham cohomology on the one hand, and the index map for K-theory on the other—and interestingly, more or less the same ideas can be used to prove both cases, c.f. Atiyah and Bott's paper on equivariant cohomology).

2.4 Induction and restriction

Another topic that Atiyah discusses is change of group. The first possibility is the inclusion of a subgroup $i : H \to G$. We have a product map

$$K_H(T_H X) \otimes K_{G\times H}(T_G G) \to K_{G\times H}(T_{G\times H}(G \times X))$$

where $G \times H$ acts on G via the action $(g,h) : g' \mapsto gg'h^{-1}$. Since the diagonal action of H on $G \times X$ is free, this becomes

$$K_H(T_H X) \otimes K_{G\times H}(T_G G) \to K_G(T_G(G \times_H X)).$$

Restricting the second argument to the trivial complex line bundle on $T_G G$ (equivalently: the symbol of the zero map $TG \times \mathbb{C} \to TG \times \{0\}$, which is G-transversally elliptic), yields an "induction map"

$$i_* : K_H(T_H X) \to K_G(T_G(G \times_H X)),$$

which is in fact an isomorphism. Equivalently, the map i_* comes from pulling back an H-tranversally elliptic symbol σ to $G \times X$ to obtain a $G \times H$-transversally elliptic symbol. Since the diagonal H action is free, the symbol descends to the quotient $G \times_H X$ and the result is $i_*\sigma$. We use the same symbol i_* to denote the induction map $\mathcal{D}'(H)^H \to \mathcal{D}'(G)^G$ (dual to the restriction map i^* on smooth functions $C^\infty(G)^G \to C^\infty(H)^H$).

Remark. The map i_* is not pushforward of distributions under inclusion, which would be dual to the map $C^\infty(G) \to C^\infty(H)$. The map $C^\infty(G)^G \to C^\infty(H)^H$ extends to a map $C^\infty(G) \to C^\infty(H)$, by precomposing with the averaging map (over conjugacy classes). This implies that i_* is the composition of pushforward from H to G, followed by averaging over G-orbits. (One can always average a distribution over G-orbits; it is just the dual map to averaging a smooth function over G-orbits.)

In this setting, Atiyah proves

$$i_* \circ \mathrm{ind}_H^X = \mathrm{ind}_G^{G\times_H X} \circ i_*.$$

The second possibility Atiyah discusses is restriction to a maximal torus $T \subset G$. The strategy is again to realize a map $K_G(T_G X) \to K_T(T_T X)$ as a special case of the product construction. If X is a G-manifold then $G \times_T X \simeq G/T \times X$ as G-spaces. We have a product map

$$K_G(T_G X) \otimes K_G(T(G/T)) \to K_G(T_G(G/T \times X)) = K_G(T_G(G \times_T X)).$$

To get a map that preserves the index we therefore need a symbol on G/T with index $1 \in R(G)$ (the representation ring of G). Now G/T is a complex manifold and in fact the $\bar{\partial}$-complex $(\wedge T^{0,1}(G/T))$ is known to have index $1 \in R(G)$ (corresponding to the one dimensional space of holomorphic functions on the compact manifold G/T, i.e. constants). Fixing this symbol as the second argument we obtain an index-preserving map $K_G(T_G X) \to K_G(T_G(G \times_T X))$. If we then compose this with the inverse of the induction isomorphism corresponding to $i : T \to G$ we obtain the restriction map

$$r : K_G(T_G X) \to K_T(T_T X).$$

And by the result on i_* described above we have

$$\mathrm{ind}_G = i_* \circ \mathrm{ind}_T \circ r,$$

which shows that the index of any G-transversally elliptic symbol is captured (in theory) by an appropriate "restriction" of it to a T-transversally elliptic symbol.

3 Example: circle action on a vector space

Above we summarized the first four (of ten!) of Atiyah's 1971 lectures. In lectures 5 and 6 he goes on to study in detail the case $X = V$ a vector space and $G = S^1$ the circle group. We will simply state part of a result which is the culmination of lectures 5 and 6.

Let V be a $2n$-dimensional real vector space, with a linear $H = S^1$ action fixing only the origin. Let A denote the vector field on V generated by the S^1 action (say A is the image of $i \in \mathbb{R}i = \mathrm{Lie}(S^1)$), so A vanishes only at the origin. Choose a complex structure on V so that the weights $m_1, ..., m_n > 0$. Let V_0 denote the tangent space to V at 0 (isomorphic to V itself of course).

Since $V_0 \simeq V$ is a complex vector space, we have a canonical generator for $K_H(V_0)$ namely the Bott class, which (to be consistent with Atiyah's notation) we denote $\bar{\partial} = j_!(1) = [\lambda_V]$ (here $j : \{0\} \to V$ is the inclusion of the origin). (Explicitly this is the symbol $\lambda_V = \mathrm{Cliff} : \wedge^{ev} V^{1,0} \to \wedge^{odd} V^{1,0}$, where these are though of as vector bundles over V.) We now use the vector field A to define two classes in $K_H(T_H V)$.

First extend the symbol λ_V trivially to TV (pull the bundles and the bundle map λ_V back by the map $TV \to V_0$ "translation to the origin"). So λ_V becomes a bundle map $\pi^*_{TV} \wedge^{ev} V^{1,0} \to \pi^*_{TV} \wedge^{odd} V^{1,0}$ which is invertible away from the zero section $V \subset TV$. The problem is that the characteristic set

V does not intersect $T_H V$ in a compact set, so λ_V is not H-transversally elliptic. To fix this we put

$$\lambda_V^{\pm}(x, \xi) := \lambda_V(x, \xi \pm A(x)).$$

These maps have characteristic set equal to the graph of A ($-A$ respectively), which intersects $T_H V$ in a single point: $(0, 0)$. Thus $[\lambda_V^{\pm}] =: \bar{\partial}^{\pm} \in K_H(T_H V)$ is a transversally elliptic symbol.

A central result of lectures 5 and 6 is the computation of the index of these classes. We introduce the following notation. Let $f(z)$ be a meromorphic function with poles only on the unit circle $S^1 \subset \mathbb{C}$. Then f has Taylor expansions around 0 and around ∞, i.e. $f(z) = \Sigma a_n z^n$ for $|z| < 1$ and $f(z) = \Sigma b_n z^{-n}$ for $|z| > 1$. We denote these two Taylor series by f^+ and f^- respectively. Each defines a distribution on S^1, defined either by integrating against the series termwise or equivalently one can deform S^1 to either a slightly smaller (for f^+) or slightly larger (for f^-) circle of radius $1 \pm \epsilon$, integrate against f and then let ϵ go to zero. As one sees from the description in terms of contours, the difference between these two distributions is a distribution which involves taking residues at the poles of f.

Theorem 7. *Let $R(H) = \mathbb{Z}[t, t^{-1}]$ be the representation ring of $H = S^1$. For the classes $\bar{\partial}^{\pm}$ defined above we have*

$$\mathrm{ind}_H(\bar{\partial}^{\pm}) = \left[\prod_i \frac{1}{1 - t^{-m_i}}\right]^{\pm}.$$

In the lectures, Atiyah in fact obtains a complete description of the image of the index map for the case of S^1 acting on a vector space V. The image forms an $R(H)$-submodule of the distributions on S^1. There is a torsion part which is generated by the image of the difference $\bar{\partial}^+ - \bar{\partial}^-$, and there is a free part which is generated by the image of $\bar{\partial}^+$ (or equivalently $\bar{\partial}^-$; this is a choice of splitting).

In the later lectures Atiyah goes on to study in detail the more general case of a torus acting on a vector space. As he points out this gives a pretty good handle on the index map, since the general case can in some sense be reduced to this case by the pushforward $j_!$ and induction maps.

4 Brief remarks on a cohomological formula

In the 1971 lecture notes, Atiyah obtains a cohomological formula for the index in a special case. About 25 years passed before a fully general cohomological formula was published by Berline and Vergne [3]. For completeness we'll quote their result, though we will not explain all the various parts which appear. There are some complications in the transversally elliptic case. Given a class $a \in K_G(T_G M)$ one must pick a sufficiently "nice" representative symbol σ, called *G-transversally good*. One needs conditions on the growth of σ along the fibres of TM, including directional information involving a cone of directions around $T_G M$ inside TM. The terms of the formula require some modification to terms which are more analytically manageable. Also the formula only expresses the germs of the index around points $g \in G$—it is then a theorem that there is a unique distribution on G corresponding to these germs.

Theorem 8. *Let σ be a G-transversally good symbol on M. Let $g \in G$ and $Y \in \mathfrak{g}_g$ (the infinitesimal stabilizer of g for the conjugation action) sufficiently close to zero. The germ of the analytic index at $g \in G$ is given by*

$$\mathrm{ind}(\sigma)(ge^Y) = (2\pi i)^{-\dim M^g} \int_{T^*M^g} ch_g(\mathbb{A}^\theta_\sigma)(Y) J(M^g)(Y)^{-1} D_g(\nu)(Y)^{-1}.$$

There is a one-one correspondence between distributions on a slice $U = \{ge^Y : Y \in \mathfrak{g}_g \text{ close to zero}\}$ for the G action on itself by conjugation, and G-invariant distributions on $G \cdot U$ (G acts by conjugation). This correspondence is understood implicitly in the formula. Referring to the formula: $D_g(\nu)(Y)$ is an equivariant class associated to the normal bundle ν of M^g in M similar to the Atiyah-Segal-Singer formula, J is roughly the inverse of the equivariant $\hat{A}$ class, $\mathbb{A}^\theta_\sigma$ is a "superconnection" (first-order odd linear differential operator on a graded vector bundle over TM^g with Leibniz rule) encoding the information in the symbol σ and modified by the canonical 1-form θ on $T^*M^g = T^*(M^g)$.

5 Quantization commutes with reduction

In this last section we'll summarize Paradan's paper [4], which gives a proof of the "quantization commutes with reduction" theorem. The proof is an application of the index theory for transversally elliptic operators discussed above.

Let (M, ω) be a compact symplectic manifold with moment map $\phi : M \to \mathfrak{g}^*$. Identify $\mathfrak{g} = \mathfrak{g}^*$ using a G-invariant inner product. Choose a G-invariant compatible almost complex structure J. For any G-equivariant hermitian vector bundle $E \to M$ equipped with a compatible connection, let $\mathcal{D}_E$ denote the corresponding Dirac-Dolbeault operator $\Gamma(\wedge^{ev} T^{0,1} M \otimes E) \to \Gamma(\wedge^{odd} T^{0,1} M \otimes E)$. The *Riemann-Roch character* is an $R(G)$-module homomorphism $RR(M, \cdot) : K_G(M) \to R(G)$ defined by

$$RR(M, E) = \mathrm{ind}_G(\mathcal{D}_E).$$

The Spin_c quantization of M is obtained by applying this map to a prequantum line bundle $L \to M$. Since the symbol of the Dirac-Dolbeault operator is the Bott class (tensored with E), we can equivalently describe the map RR as the composition

$$K_G(M) \xrightarrow{\lambda_M} K_G(TM) \xrightarrow{\mathrm{ind}_G} R(G).$$

(Here λ_M is the Bott isomorphism.)

The basic strategy is to construct a homotopy of $\lambda_M(E) =: \lambda^E$ in the space of transversally elliptic symbols (even though $\lambda_M(E)$ is initially *elliptic*), using the fact that this preserves the index. Let $\hat{\phi}(m)$ denote the vector in $T_m M$ which is the image of $\phi(m) \in \mathfrak{g}$ under the map $\mathfrak{g} \to T_m M$ (in other words, $\hat{\phi}$ is the vector field generated by the Hamiltonian $\frac{1}{2}||\phi||^2$). For $s \in [0, 1]$ we define symbols

$$\lambda^E(s)(x, \xi) = \lambda^E(x, \xi - s\hat{\phi}(x)).$$

Put $\lambda^E_1 = \lambda^E(1)$. In particular this shows that $[\lambda^E] = [\lambda^E_1] \in K_G(TM)$. Since the characteristic set $\mathrm{char}(\lambda^E)$ (i.e. the subset of TM where λ^E is not invertible) of λ^E was the zero section $M \subset TM$, the characteristic set of λ^E_1 is the graph Γ_ϕ of $\hat{\phi}$. Notice that

$$\Gamma_\phi \cap T_G M = \{m \in M : \hat{\phi}(m) = 0\} = \mathrm{Crit}(||\phi||^2) =: C.$$

Now the critical values of $||\phi||^2$ can be indexed by a finite discrete subset $B \subset \mathfrak{t}_+$ containing 0. Let C^β for $\beta \in B$ be the union of connected components of C corresponding to β. For each C^β choose a G-invariant open neighbourhood $U^\beta \supset C^\beta$ such that $\overline{U^\alpha} \cap \overline{U^\beta} = \emptyset$ for $\alpha \neq \beta$. Since $\Gamma_\phi \cap U^\beta \cap T_G M = C^\beta$ is compact, the restriction $\lambda^E_\beta := \lambda^E_1|_{U^\beta}$ defines a G-transversally elliptic symbol in $K_G(T_G U^\beta)$. Let $i^\beta : U^\beta \to M$ be the inclusion.

Theorem 9. *Using the notation introduced above, we have the following equality in $K_G(T_G M)$:*

$$\lambda^E = \sum_{\beta \in B} i_*^\beta \lambda_\beta^E.$$

Proof. (Sketch) We would like to apply excision, but for this we need a transversally elliptic symbol in the same class (of $K_G(T_G M)$) as λ_1^E but having characteristic set contained in $TM|_U$ where U is the union of the U^β (at the moment we have only that $\text{char}(\lambda_1^E) \cap T_G M \subset U$). The problem is that $\text{char}(\lambda_1^E) \cap U$ is not a compact set, and so we want to deform the symbol so that it becomes compact. Let $\chi_\beta : M \to [0,1]$ be a smooth bump function supported in U^β and non-zero on C^β, and put $\chi = \Sigma \chi_\beta$. Consider the homotopy

$$\lambda_t^E(x, \xi) = \lambda^E(x, (t + (1-t)\chi(x))\xi - \hat{\phi}(x)).$$

(Clearly λ_1^E here agrees with our earlier definition.) One can check that this is G-transversally elliptic for each t, and hence λ_1^E defines the same class as λ_0^E in $K_G(T_G M)$. Consider

$$\lambda_0^E(x, \xi) = \lambda^E(x, \chi(x)\xi - \hat{\phi}(x)).$$

This has characteristic set $\{(x, \xi) : \chi(x)\xi = \hat{\phi}(x)\}$. If $x \notin U$ then $\chi(x) = 0 \Rightarrow \hat{\phi}(x) = 0$ which implies that $x \in C \subset U$, a contradiction. Hence $\text{char}(\lambda_0^E) \subset TM|_U$, so we can now apply excision and homotopy invariance:

$$\lambda^E = \lambda_1^E = \lambda_0^E = \sum_\beta i_*^\beta (\lambda_0^E|_{U^\beta}) \in K_G(T_G M).$$

A similar homotopy shows $\lambda_0^E|_{U^\beta} = \lambda_\beta^E$ in $K_G(T_G U^\beta)$, and the result follows. □

This theorem gives the decomposition

$$RR(M, E) = \sum_\beta \text{ind}_{U^\beta}(\lambda_\beta^E) =: \sum_\beta RR_\beta(M, E).$$

Information about $RR(M, E)$ can be obtained from studying the pieces $RR_\beta(M, E)$. By considering local normal forms near $\phi^{-1}(0)$ Paradan proves

Theorem 10. *Suppose 0 is a regular value of ϕ, and let L be a prequantum line bundle. Let $M_{red} = \phi^{-1}(0)/G$. Then*

$$RR_0(M, L)^G = RR(M_{red}, L_{red}).$$

This is at least plausible given that RR_0 is localized in a small neighbourhood of $\phi^{-1}(0)$. (Paradan has more general results covering vector bundles E satisfying an appropriate condition, and also results covering the singular case.)

Given this theorem, in order to prove $[Q,R]=0$ it remains to show that $RR_\beta(M,L)^G = 0$ for each $\beta \in B - \{0\}$. If we restrict $RR_\beta(M,L)$ to the maximal torus T we get a sum of characters of T (the weight decomposition) which we can think of as a discrete measure on $\mathfrak{t}^*$ with mass at each lattice point corresponding to the multiplicity. This "quantum multiplicity function" determines the representation. It turns out that the contribution RR_β to the quantum multiplicity function $RR(M,L)$ is supported outside the ball of radius $||\beta||$ in $\mathfrak{t}^*$ and so the desired result follows.

How this comes about is interesting. One first reduces to the case where $\beta \in \mathfrak{g}$ is central. This is done using the general induction/restriction properties of the index discussed earlier (but still requires some work since those maps are still quite abstract). There are actually two choices. One could consider M as a Hamiltonian T-space (with moment map $\phi_T = pr_{\mathfrak{t}} \circ \phi$), and Paradan proves an induction formula:

Theorem 11. *Let W be the Weyl group and Hol_T^G the* holomorphic induction map $R(T) \to R(G)$. *Let $RR^T_{\beta'}(M,E)$ denote the contribution to $RR^T(M,E)$ corresponding to the critical point β' of the T-moment map ϕ_T. Then we have*

$$RR^G_\beta(M,E) = \sum_{\beta' \in W\cdot\beta} \mathrm{Hol}^G_T(RR^T_{\beta'}(M,E)).$$

(See the paper for further information, e.g. on the holomorphic induction map.) Alternatively one could pass to the symplectic cross-section Y_β which is a Hamiltonian G_β-space (and β is central in G_β by definition). Paradan also proves an induction formula in this case:

Theorem 12. *Let $RR^{G_\beta}_\beta(Y_\beta,E)$ denote the contribution to $RR^{G_\beta}(Y_\beta,E)$ corresponding to the critical point β of the restricted moment map $\phi|_{Y_\beta}$ for Y_β. Then we have*

$$RR^G_\beta(M,E) = \mathrm{Hol}^G_{G_\beta}(RR^{G_\beta}_\beta(Y_\beta,E)).$$

Either formula allows one to reduce to the situation in which β is central.

The case where β is central is handled by making heavy use of the properties discussed in the previous sections (especially the multiplicative property), as well as Atiyah's result for the case of S^1 acting on a vector space. What is perhaps remarkable is that so little is used beyond the general abstract properties of the index map and the single explicit calculation for S^1. The key result is:

Theorem 13. *Let $\beta \in \mathfrak{g}$ be central. We have the following equality in the completed character ring $\hat{R}(G)$*

$$RR_\beta(M, E) = (-1)^{r_\nu} \sum_{k \in \mathbb{N}} RR_\beta(M^\beta, E \otimes \det \nu^{+,\beta} \otimes S^k(\nu \otimes \mathbb{C})^{+,\beta}).$$

The notation requires some explanation: S^k denotes the k^{th} symmetric power, ν denotes the normal bundle to M^β in M and r_ν is its rank. Let H be the torus generated by β. An H-equivariant complex vector bundle $F \to M^\beta$ splits into a sum of H-weight bundles with weights $\alpha_1, ..., \alpha_n$. The vector bundle $F^{+,\beta}$ is the sum of those weight bundles having $\langle \alpha_i, \beta \rangle > 0$; in other words it is the *β-polarized* part of the vector bundle F. Note also that since ν is already a complex vector bundle, $\nu \otimes \mathbb{C} \simeq \nu \oplus \bar{\nu}$ which implies that $(\nu \otimes \mathbb{C})^{+,\beta}$ has the same dimension as ν (but the weights are polarized).

Similar to above, $RR_\beta(M^\beta, F)$ denotes the contribution to $RR(M^\beta, F)$ corresponding to β for the restricted moment map $\phi|_{M^\beta}$. To keep notation simple we take M^β to denote the components of M^β which intersect $\phi^{-1}(\beta)$ non-trivially (since these are the only parts that contribute to the formula). The image of this restricted moment map is contained in the affine subspace through β which is orthogonal to β in $\mathfrak{g} = \mathfrak{g}^*$ (β is central). Putting $E = L$ a prequantum line bundle, it follows from the Kostant formula that H has weight β on $L|_{M^\beta}$. The result now follows fairly quickly from the theorem. Since the weight of H on L is β and since all the other vector bundles appearing in the formula are β-polarized, all the H-weights α on the right hand side satisfy $\langle \alpha, \beta \rangle \geq ||\beta||^2$ (i.e. they lie in the half-space not containing the origin which is determined by the affine subspace $\langle \alpha, \beta \rangle = ||\beta||^2$ through β in $\mathfrak{g}$). One can show using the fact that H is central that this property is preserved on applying the index map (this is proved for example in the appendix of Paradan's paper), which proves the desired result.

Roughly, the proof of the theorem itself involves two clever uses of the multiplicative property. One works on the small neighbourhood U^β where $RR_\beta(M,E)$ is localized. The local normal form is the normal bundle $\nu \to M^\beta$ (perhaps restricted to a small neighbourhood of $\phi^{-1}(\beta) \cap M^\beta$ in M^β). The multiplicative property is used on the vector bundle ν (using the "associated bundle trick" described earlier) to break things up into a contribution from the base M^β and the typical fibre. One is lead to consider the action of a torus on a typical fibre, which is a vector space. The multiplicative property is used again in a very clever way to extract the desired information from the one explicit computation for S^1 done by Atiyah! This is only a rough outline of the story—there are many further important details spanning several pages. Hopefully it gives something of the flavour of Paradan's approach to the $[Q,R]=0$ theorem.

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