SUBELLIPTIC ESTIMATES & HYPOELLIPTIC LAPLACIAN FOR T^*S^1

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Contents

These are some expository notes (prepared for a seminar at Penn State) on subelliptic estimates for Bismut's hypoelliptic Laplacian in the simplest possible case, T^*S^1, where the operator more or less reduces to the 'Fokker-Planck operator'.

A reference is the book by Helffer-Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Chapter 5.

1. Hypoelliptic Laplacian on T^*S^1

Let $T^*S^1 = S^1 \times \mathbb{R}$ with coordinates $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $y \in \mathbb{R}$. The hypoelliptic Laplacian (or Fokker-Planck operator) is

$$L = -\partial_y^2 + y^2 - 1 + X, \qquad X = y\partial_x.$$

The operator X is (formally) skew-adjoint, so L is neither self-adjoint nor skew-adjoint. Let

$$a = \partial_x, \qquad b = \partial_y + y.$$

Then

$$L = b^*b + X, \qquad X = b^*a - a^*b.$$

The operators a, a^* commute with each other and with b, b^*. Moreover

$$[b, b^*] = 1, \qquad [b, X] = a.$$

2. The Sobolev scale

Introduce the operator

$$\Delta = 1 + a^*a + b^*b = -\partial_x^2 - \partial_y^2 + y^2.$$

It is an essentially self-adjoint, positive unbounded operator in the Hilbert space $L^2(T^*S^1)$ (initially with domain consisting of Schwartz functions, but then pass to its closed extension to obtain a self-adjoint operator).

Definition 2.1. For $s \geq 0$ let

$$H^s := \operatorname{dom}(\Delta^{s/2}) \subset H = L^2(T^*S^1).$$

For $s < 0$ set $H^s = (H^{-s})^*$. Let H^∞ be the intersection of all the H^s, and $H^{-\infty}$ the union of all the H^s. Then H^∞ coincides with the Schwartz functions and $H^{-\infty}$ with the tempered distributions. Define an inner product on H^s by

$$(u, v)_{H^s} = (\Delta^{s/2}u, \Delta^{s/2}v)_{L^2}, \qquad \|u\|_s = \|\Delta^{s/2}u\|.$$

The spaces H^s are Hilbert spaces and moreover for every $r \in \mathbb{R}$

$$\Delta^{s/2} \colon H^{r+s} \to H^r$$

is a unitary isomorphism.

Example 2.2. $H^1 = W^1 \cap \operatorname{dom}(y)$, where W^1 is the usual Sobolev space, and y is viewed as a multiplication operator in $L^2(T^*S^1)$. This follows from

$$\|\Delta^{1/2}u\|^2 = (\Delta u, u) = ((-\partial_x^2 - \partial_y^2)u, u) + (y^2 u, u) = \|(-\partial_x^2 - \partial_y^2)^{1/2}u\|^2 + \|yu\|^2$$

which implies $\operatorname{dom}(\Delta^{1/2}) = \operatorname{dom}((-\partial_x^2 - \partial_y^2)^{1/2}) \cap \operatorname{dom}(y)$.

Remark 2.3. For $s > t$ the embedding $H^s \hookrightarrow H^t$ is compact, and is trace class when $s \geq t + 4$. (In another talk we saw that Δ^{-2} is trace class.)

Definition 2.4. If an operator $T \colon H^\infty \to H^\infty$ extends to a bounded operator

$$T \colon H^{r+s} \to H^s$$

for all s, then we say that T *has (analytic) order* r. In particular $\Delta^{r/2}$ has order r. This defines a filtration on the algebra of operators $H^\infty \to H^\infty$.

Lemma 2.5. *The operators a, b, a^*, b^* all have order* 1. *The operators X and L have order* 2.

This requires some justification, at least for b. Note that

$$\|bu\|^2 = (b^*bu, u) \leq (\Delta u, u) = \|\Delta^{1/2}u\|^2 = \|u\|_1^2$$

which shows that b, viewed as an operator $H^1 \to H$ is bounded. To see that b viewed as an operator $H^{s+1} \to H^s$ is bounded, one can use the iterated commutators formula for the operator $\Delta^{s/2}$ that we saw in another talk.

Lemma 2.6. *The operators $[\Delta^{s/2}, X]$, $[\Delta^{s/2}, L]$ have order s.*

This is one less than the 'naive' order, which would be $s + 2 - 1 = s + 1$. One expects this because

$$[\Delta, X] = b^*a + ba^*$$

has order 2 instead of 3. Again the general case can be handled using the iterated commutators formula.

Consideration of orders of operators will be useful in proving the subelliptic estimate.

3. Subelliptic estimate

The main goal is:

Theorem 3.1 (Subelliptic estimate)**.** *For all s there is a constant $C > 0$ with*

$$\|u\|^2_{s+\frac{1}{4}} \leq C(\|Lu\|^2_s + \|u\|^2_s)$$

for all $u \in H_\infty$.

Remark 3.2. It is enough to check this for $s = 0$, since then

$$\begin{aligned}\|u\|^2_{s+\frac{1}{4}} &= \|\Delta^{s/2}u\|^2_{\frac{1}{4}} \\ &\leq C(\|L\Delta^{s/2}u\|^2 + \|\Delta^{s/2}u\|^2) \\ &\leq C(\|Lu\|^2_s + \|u\|^2_s + \|[L,\Delta^{s/2}]u\|^2)\end{aligned}$$

and $[L, \Delta^{s/2}]$ has order s so the commutator term is bounded by $C'\|u\|^2_s$ for some constant C'.

We will try to prove an inequality of the form

$$\|\Delta^{\frac{q}{2}}u\|^2 \leq C(\|Lu\|^2 + \|u\|^2) \tag{1}$$

with $q > 0$, and see in the course of the proof why $q \leq \frac{1}{4}$ works. Throughout C will denote a positive constant (not depending on u) that can change from line to line. We use some basic facts over and over again:

(a) Let $\Re$ denote the $\mathbb{R}$ part of a complex number. The bilinear form $\Re(\cdot,\cdot)$ is symmetric and satisfies the Cauchy-Schwartz inequality

$$|\Re(u, v)| \leq \|u\|\|v\|.$$

The symmetry of $\Re(\cdot,\cdot)$ is useful because it implies that

$$\Re(Xu, u) = 0$$

for any $u \in H^\infty$ (X is formally skew-adjoint). Thus one useful trick will be to replace b^*b with $L = b^*b + X$ in certain expressions.

(b) One has the inequality $2\|u\|\|v\| \leq \|u\|^2 + \|v\|^2$. We will often simply drop the factor '2', since don't attempt to optimize our constants.

We start by proving similar estimates for a couple simpler operators.

Lemma 3.3. *One has the inequalities*

$$\|bu\|^2 \leq \|Lu\|^2 + \|u\|^2.$$

Proof. As X is formally skew-adjoint, $\Re(Xu, u) = 0$. Thus

$$\|bu\|^2 = \Re(b^*bu, u) = \Re(Lu, u) \leq \|Lu\|^2 + \|u\|^2. \tag{2}$$

□

Lemma 3.4. *For $q \leq \frac{1}{2}$ one has*

$$|(Lu, \Delta^{q-1}a^*bu)| \leq C(\|Lu\|^2 + \|u\|^2).$$

Proof. For $q \leq \frac{1}{2}$ the operator $\Delta^{q-1}a^*$ has order 0 so is bounded. Hence, by the Cauchy-Schwartz inequality

$$|(Lu, \Delta^{q-1}a^*bu)| \leq \|Lu\|\|\Delta^{q-1}a^*\|\|bu\| \leq C\|Lu\|\|bu\| \leq C(\|Lu\|^2 + \|bu\|^2),$$

and Lemma 3.3 gives the result. □

Lemma 3.5. *For $q \leq \frac{1}{2}$ one has*

$$\|b^*a\Delta^{q-1}u\|^2 \leq C(\|Lu\|^2 + \|u\|^2).$$

Proof. Since $(1 + b^*b)$ commutes with a and with Δ,

$$b^*a\Delta^{q-1} = b^*(1 + b^*b)^{-1/2}a\Delta^{q-1}(1 + b^*b)^{1/2}.$$

For $q \leq \frac{1}{2}$ the operator $a\Delta^{q-1}$ is bounded,hence

$$\|b^*a\Delta^{q-1}u\| \leq \|b^*(1 + b^*b)^{-1/2}\|\|a\Delta^{q-1}\|\|(1 + b^*b)^{1/2}u\| = C\|(1 + b^*b)^{1/2}u\|.$$

Squaring $\|(1 + b^*b)^{1/2}u\|$ we find

$$\|(1 + b^*b)^{1/2}u\|^2 = \Re((1 + b^*b)u, u) = \|u\|^2 + \Re(Lu, u) \leq \|u\|^2 + \|Lu\|^2 + \|u\|^2.$$

□

Lemma 3.6. *For $q \leq \frac{1}{2}$ one has*

$$|(Lu, b^*a\Delta^{q-1}u)| \leq C(\|Lu\|^2 + \|u\|^2).$$

Proof. Applying the Cauchy-Schwartz inequality to the left hand side,

$$|(Lu, b^*a\Delta^{q-1}u)| \leq \|Lu\|\|b^*a\Delta^{q-1}u\| \leq \|Lu\|^2 + \|b^*a\Delta^{q-1}u\|^2.$$

Lemma 3.5 gives the result. □

Lemma 3.7. *For $q \leq \frac{1}{4}$ we have a bound*

$$\|b\Delta^{q-1}a^*bu\|^2 \leq C(\|Lu\|^2 + \|u\|^2).$$

Proof. We have

$$\begin{aligned}
\|b\Delta^{q-1}a^*bu\|^2 &= \Re(b^*b\Delta^{q-1}a^*bu, \Delta^{q-1}a^*bu) \\
&= \Re(L\Delta^{q-1}a^*bu, \Delta^{q-1}a^*bu) \\
&= \Re(\Delta^{q-1}a^*bLu, \Delta^{q-1}a^*bu) + \Re([L, \Delta^{q-1}a^*b]u, \Delta^{q-1}a^*bu)
\end{aligned}$$

using $\Re(Xv, v) = 0$ with $v = \Delta^{q-1}a^*b$ in the second line, and then commuting L past $\Delta^{q-1}a^*b$ in the third line. The first term becomes

$$\Re(a\Delta^{2q-2}a^*bLu, bu).$$

For $q \leq \frac{1}{4}$, Δ^{2q-2} has order ≤ -3 and so $a\Delta^{2q-2}a^*b$ is bounded. So this is bounded by

$$C\|Lu\|\|bu\| \leq C(\|Lu\|^2 + \|u\|^2).$$

For the second term make the replacement $L = b^*b + X$, and use

$$[b^*b + X, \Delta^{q-1}a^*b] = -\Delta^{q-1}a^*b - [X, \Delta^{q-1}]a^*b - \Delta^{q-1}a^*a,$$

where we also used $[X,b]=-a$ to simplify the third term. Thus

$$\Re([L,\Delta^{q-1}a^*b]u,\Delta^{q-1}a^*bu)=-\|\Delta^{q-1}a^*bu\|^2-\Re([X,\Delta^{q-1}]a^*b,\Delta^{q-1}a^*bu)-\Re(\Delta^{q-1}a^*au,\Delta^{q-1}a^*bu).$$

We can drop the first term which is always negative. Write the second term as

$$\Re(a\Delta^{q-1}[\Delta^{q-1},X]a^*bu,bu)$$

and note that for $q\leq\frac{1}{4}$ the operator $a\Delta^{q-1}[\Delta^{q-1},X]a^*$ has order 0, so is bounded—thus this term is bounded by $C\|bu\|^2\leq C(\|Lu\|^2+\|u\|^2)$. Write the third term as

$$\Re(a\Delta^{2q-2}a^*au,bu)$$

and note $a\Delta^{2q-2}a^*a$ has order ≤ 0, so is bounded—hence this term is bounded by $C\|u\|\|bu\|\leq C(\|u\|^2+\|bu\|^2)\leq C(\|Lu\|^2+2\|u\|^2)$. □

Proof of Theorem 3.1: Note

$$\|\Delta^{\frac{q}{2}}u\|^2=(\Delta^q u,u)=(\Delta^{q-1}(1+a^*a+b^*b)u,u). \tag{3}$$

Consider first $\Delta^{q-1}b^*bu$. If $q\leq\frac{1}{2}$, the operator $\Delta^{q-1}b^*$ has non-positive order, so is bounded. Thus

$$\|\Delta^{q-1}b^*bu\|\leq C\|bu\|$$

for some constant C, and use Lemma 3.3. This shows that the b^*b term in (3) is easily bounded by a term of the form (1), as long as $q\leq\frac{1}{2}$. It remains to bound

$$\Re(\Delta^{q-1}a^*au,u). \tag{4}$$

Since we know how to bound terms involving b's, it makes sense to begin by making the replacement

$$a=[b,X]$$

i.e.

$$\begin{aligned}\Delta^{q-1}a^*a&=\Delta^{q-1}a^*(bX-Xb)\\&=\Delta^{q-1}a^*bX-X\Delta^{q-1}a^*b-[\Delta^{q-1},X]a^*b.\end{aligned}$$

Substituting this in (4), using that X is formally skew-adjoint and the symmetry of the bilinear form $\Re(\cdot,\cdot)$ we have

$$\begin{aligned}\Re(\Delta^{q-1}a^*au,u)&=\Re(\Delta^{q-1}a^*bXu,u)-\Re(X\Delta^{q-1}a^*bu,u)-\Re([\Delta^{q-1}a^*,X]bu,u)\\&=\Re(Xu,(\Delta^{q-1}a^*b+b^*a\Delta^{q-1})u)-\Re([\Delta^{q-1}a^*,X]bu,u).\end{aligned}$$

In the second term, the commutator $[\Delta^{q-1},X]$ has order $2(q-1)$, so $[\Delta^{q-1},X]a^*$ has order $2q-1$. With $q\leq\frac{1}{2}$ this operator is bounded, hence the commutator term above is bounded by

$$C\|bu\|\|u\|\leq C(\|bu\|^2+\|u\|^2)\leq C(\|Lu\|^2+2\|u\|^2),$$

using equation (2). This leaves us with

$$\Re(Xu,(\Delta^{q-1}a^*b+b^*a\Delta^{q-1})u).$$

Since we know how to bound terms involving L's and b's, it makes sense to make the replacement $X=L-b^*b$. So we have a difference of two terms

$$\Re(Lu,(\Delta^{q-1}a^*b+b^*a\Delta^{q-1})u) \tag{5}$$

and

$$\Re(b^*bu, (\Delta^{q-1}a^*b + b^*a\Delta^{q-1})u). \tag{6}$$

Let's consider (6) first. For one of the parts

$$\Re(b^*bu, \Delta^{q-1}a^*bu) = \Re(bu, b\Delta^{q-1}a^*bu) \leq \|bu\|^2 + \|b\Delta^{q-1}a^*bu\|^2 \tag{7}$$

and now apply Lemmas 3.3, 3.7. For the other part move the b^*b over and note that

$$b^*(bb^*)a\Delta^{q-1} = b^*(b^*b+1)a\Delta^{q-1} = b^*a\Delta^{q-1}(b^*b+1)$$

where we used the fact that b^*b+1 commutes with Δ, a. Thus

$$\Re(b^*bu, b^*a\Delta^{q-1}u) = \Re(u, b^*a\Delta^{q-1}(b^*b+1)u) = \Re(b\Delta^{q-1}a^*bu, bu) + \Re(u, b^*a\Delta^{q-1}u).$$

The first term in this expression is bounded as in (7), while for the second part we use that $a\Delta^{q-1}$ is bounded to get an upper bound of the form $C\|bu\|\|u\| \leq C(\|bu\|^2 + \|u\|^2)$ and then use Lemma 3.3. This completes the discussion of (6).

For (5) we use Lemma 3.5 to bound the second term. For the first term use the Cauchy-Schwartz inequality

$$\Re(Lu, \Delta^{q-1}a^*bu) \leq \|Lu\|\|\Delta^{q-1}a^*bu\| \leq \|Lu\|^2 + \|\Delta^{q-1}a^*bu\|^2.$$

The operator $\Delta^{q-1}a^*$ is bounded, so one gets a bound for the second term in this expression of the form

$$C\|bu\|^2$$

and now use Lemma 3.3. □

4. M-ACCRETIVITY

The argument here is a simplification of that appearing in Chapter 5 of the book by Helffer-Nier.

We must show that for some $\lambda > 0$ the range of $L + \lambda$ is dense in H (we will see $\lambda = 2$ is large enough). Let

$$T = L + 2 = -\partial_y^2 + y^2 + 1 + X.$$

It is equivalent to show that if $f \in L^2(T^*S^1)$ and

$$(f, Tu) = 0, \quad \forall u \in C_c^\infty(T^*S^1) \quad \Rightarrow \quad f = 0$$

or, in other words, if $T^*f = 0$ (in the sense of distributions) then $f = 0$. There is a similar subelliptic estimate for L^*, and it implies that f must be *smooth* (e.g. one gets an inequality for the 1/4 Sobolev norm by a constant times the 0 Sobolev norm, so the 1/4 Sobolev norm must be finite, and so on). (We will see that, apart from the sub-elliptic estimate, the y^2 and X terms play no further role in proving m-accretivity.) We can assume f is real-valued (since $T^*\Re(f) = 0 = T^*\Im(f)$ so one could apply the argument below separately to the real and imaginary parts)—this is a small convenience because then the inner products below will all be symmetric.

Now the idea is to replace f with $\zeta_k^2 f$, where

$$\zeta_k(y) = \zeta(\tfrac{y}{k})$$

and ζ is a smooth compactly supported bump function in the y-direction (the non-compact direction). By dominated convergence $\|\zeta_k f\|^2 \to \|f\|^2$ as $k \to \infty$. Having the cutoff function will allow us to freely integrate by parts in the y-variable. At the end of the argument we will take $k \to \infty$ and obtain the result.

Since f is smooth and ζ_k has compact support, the equation $T^* f = 0$ implies in particular that

$$0 = (T^* f, \zeta_k^2 f) = (f, T(\zeta_k^2 f)) = (f, (-\partial_y^2 + 1 + y^2 + X)\zeta_k^2 f). \tag{8}$$

Using integration by parts, the $-\partial_y^2$ term equals[1]

$$(f, -\partial_y^2(\zeta_k f)) = \|\partial_y(\zeta_k f)\|^2 - \|(\partial_y \zeta_k) f\|^2.$$

Substituting this into (8), using $X\zeta_k = 0$, $(f, Xf) = 0$ we find

$$0 = \|\partial_y(\zeta_k f)\|^2 - \|(\partial_y \zeta_k) f\|^2 + (f, (1 + y^2)\zeta_k^2 f) \geq -\|(\partial_y \zeta_k) f\|^2 + (f, (1 + y^2)\zeta_k^2 f),$$

thus

$$\|\zeta_k f\|^2 + \|y\zeta_k f\|^2 \leq \|(\partial_y \zeta_k) f\|^2.$$

By the definition $\zeta_k(y) = \zeta(\frac{y}{k})$ we have

$$\|(\partial_y \zeta_k) f\|^2 \leq \|\partial_y \zeta_k\|^2 \|f\|^2 = \frac{C}{k^2}\|f\|^2$$

for some constant C. Thus, dropping the positive term $\|y\zeta_k f\|^2$ we have

$$\|\zeta_k f\|^2 \leq \frac{C}{k^2}\|f\|^2.$$

Taking $k \to \infty$ shows $\|\zeta_k f\|^2 \to 0$. Thus $f = 0 \in H$.

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[1]Integrating by parts once we have

$$\begin{aligned}(f, -\partial_y^2(\zeta_k^2 f)) &= (\partial_y f, (\partial_y \zeta_k)\zeta_k f + \zeta_k \partial_y(\zeta_k f)) \\ &= (\zeta_k \partial_y f, (\partial_y \zeta_k) f) + (\zeta_k \partial_y f, \partial_y(\zeta_k f)).\end{aligned}$$

Use $\zeta_k \partial_y f = \partial_y(\zeta_k f) - f\partial_y \zeta_k$ in the second term and note that

$$(\zeta_k \partial_y f, (\partial_y \zeta_k) f) - ((\partial_y \zeta_k) f, \partial_y(\zeta_k f)) = ((\partial_y \zeta_k) f, -f\partial_y \zeta_k)$$

(here we used the symmetry of $(\cdot, \cdot)$ for $\mathbb{R}$-valued arguments) to get

$$(f, -\partial_y^2(\zeta_k f)) = \|\partial_y(\zeta_k f)\|^2 - \|(\partial_y \zeta_k) f\|^2.$$