Quantization of Hamiltonian LG-spaces

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Collaborators

The Hamiltonian $[Q, R] = 0$ Theorem

- G compact Lie group, Lie algebra $\mathfrak{g}$
- $\phi : M \to \mathfrak{g}^*$ compact Hamiltonian G-space
- $L \to M$ G-equiv prequantum line bundle
- D_L Dolbeault-Dirac operator for compatible almost $\mathbb{C}$ structure, twisted by L
- $Q(M) := \mathrm{index}(D_L) \in R(G)$

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Theorem (Meinrenken) "Guillemin-Sternberg principle"

Assume 0 is a regular value. Then

$$Q(M)^G = Q(M//G) \qquad \left(\text{where } M//G = \phi^{-1}(0)/G\right)$$

"Witten deformation" approach (Tian-Zhang)

v_M Hamiltonian vector field of $\frac{1}{2}||\phi||^2$

$$D_t = D - itc(v_M), \qquad t \in \mathbb{R}$$

Then

$$D_t^2 = D^2 - it[D, c(v_M)] + t^2||v_M||^2$$

$\rightarrow$ Cross term bounded on $L^2(M, S)^G$.

Large t, elements in $\ker(D_t^2)^G$ concentrate near

$$\phi^{-1}(0) \subset \{v_M = 0\} = \mathrm{Crit}(||\phi||^2)$$

Abelianize the problem

If $\phi : M \to \mathfrak{g}^*$ **transverse** to $\mathfrak{t}^*$ $\Rightarrow$

$$X := \phi^{-1}(\mathfrak{t}^*), \qquad S_{TX} := \mathrm{Hom}_{\mathrm{Cliff}(\mathfrak{g}/\mathfrak{t})}(S_{\mathfrak{g}/\mathfrak{t}}, S_{TM})$$

$$Q(X) := \mathrm{index}(D_X) \in R(T).$$

$$\Rightarrow \qquad Q(M)(t) = \frac{Q(X)(t)}{\prod_{\alpha>0}(1 - t^{-\alpha})}, \qquad t \in T$$

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Not transverse? $\to$ use $\mathfrak{t}^* \subset U \subset \mathfrak{g}^*$

$$X = \phi^{-1}(U), \qquad Q(X) := \langle D_X, \phi^*_{\mathfrak{g}/\mathfrak{t}} b \rangle$$

with $b \in K^0_T(\mathfrak{g}/\mathfrak{t})$ Bott element

The loop group

Definition

The loop group LG is the set of maps $S^1 \to G$ of fixed Sobolev class $s > 1/2$. It is a (Banach) Lie group.

Useful subgroups

- $G \subset LG$ constant loops
- $LT \subset LG$ abelian subgroup
- $\Omega G \subset LG$ based loop group ($\gamma(0) = e$)
- $\Lambda = \ker(\exp : \mathfrak{t} \to T) \subset \Omega G$ exponential loops

$$\lambda \in \Lambda, \qquad \gamma_\lambda(t) = \exp(t\lambda)$$

Central extensions

We will assume G compact, simple, $\pi_0(G) = \pi_1(G) = 0$.

Central extensions

$$1 \to U(1) \to \widehat{LG} \to LG \to 1$$

classified by an integer called the **level**.

Lie algebra cocycle is:

$$c(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{S^1} B(\xi_1(t), \xi_2'(t))dt, \qquad \xi_1, \xi_2 \in L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$$

B is the (basic) inner product on $\mathfrak{g}$

$L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$, LG acts by gauge transformations:

$$g \cdot \xi = \mathrm{Ad}_g \xi - dgg^{-1}.$$

Definition

A *Hamiltonian LG-space* $(\mathcal{M}, \omega, \Psi)$ consists of a weakly symplectic (Banach) LG-manifold, equipped with a proper moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$.

Quotient by $\Omega G \subset LG$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Psi} & L\mathfrak{g}^* \\ \downarrow /\Omega G & & \downarrow /\Omega G \\ M & \xrightarrow{\Phi} & G \end{array}$$

M is a **quasi-Hamiltonian** G-space (Alekseev-Malkin-Meinrenken). Need not have a G-equivariant spin-c structure.

Quantization for q-Hamiltonian G-spaces

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Substitute: push-forward in **twisted K-homology** (Meinrenken). Moreover, there is a $[Q, R] = 0$ Theorem (AMW '01).

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Can we work on $\mathcal{M}$, obtain an LG-rep directly, and prove $[Q, R] = 0$ using "Witten deformation"? We do something intermediate.

Plan

- Construct spinor bundle $S_{T\mathcal{M}} \to \mathcal{M}$.
- Finite-dim (non-compact) $X \subset \mathcal{M}$, induced spinor bundle S.
- Prove Dirac operator D_X Fredholm on each T-isotypic component of $L^2(X, S)$

$$\Rightarrow \qquad Q(X) := \mathrm{index}(D_X) \in R^{-\infty}(T).$$

- Witten deformation $D_t = D - itc(v_X)$ to prove $[Q, R] = 0$.

Remark

Also prove compatibility with the earlier twisted K-homology definition, and a “norm-square localization” formula for $Q(X)$.

Completing the tangent bundle

Can complete fibres of tangent bundle

$$TM \to \mathbf{T}M$$

such that ω becomes **strongly symplectic**:

$$\omega_m^\flat : \mathbf{T}_m\mathcal{M} \xrightarrow{\sim} (\mathbf{T}_m\mathcal{M})^*$$

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Definition

A complex structure J on (V, ω) is **compatible** with ω if

$$g_J(-,-) = \omega(J-,-)$$

defines a (positive definite) inner product.

Almost $\mathbb{C}$ structure J and spinor bundle

Existence

$\mathcal{M}$ has finite-dimensional symplectic cross-sections Y_σ

$$\mathcal{M} = \bigcup_\sigma LG \cdot Y_\sigma, \qquad LG \cdot Y_\sigma \simeq LG \times_{(LG)_\sigma} Y_\sigma$$

Patch together almost complex structures on the $LG \cdot Y_\sigma$

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The spinor bundle:

$$S_{\mathbf{T}\mathcal{M}} = \overline{\wedge_J \mathbf{T}\mathcal{M}} \qquad \text{("Fock space")}$$

Polarization

(V, ω) finite dim $\Rightarrow$ any two compatible J_1, J_2 homotopic.

Definition

A **polarization** of a real Hilbert space V is an element $J \in \mathbb{B}(V)$ such that $J^2 = -1 \; mod \, \mathbb{B}_{HS}(V)$.

Polarization of $L\mathfrak{g}$ with

$$(+i) \text{ eigenspace } = \bigoplus_{n>0} \mathfrak{g}_{\mathbb{C}} z^n, \quad (-i) \text{ eigenspace } = \bigoplus_{n<0} \mathfrak{g}_{\mathbb{C}} z^n$$

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Lemma

Up to homotopy, there is a unique LG-invariant compatible almost complex structure J on $\mathbf{T}\mathcal{M}$ such that the action map $L\mathfrak{g} \to T_m\mathcal{M}$ is polarization-preserving.

Finite-dimensional $X \subset \mathcal{M}$

For simplicity, in this talk we assume Ψ transverse to $\mathfrak{t}^*$. Then

$$X := \Psi^{-1}(\mathfrak{t}^*) \subset \mathcal{M}$$

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More generally we have:

Lemma

There is a $T \times \Lambda$-invariant submanifold $\mathfrak{t}^* \subset U \subset L\mathfrak{g}^*$, $\dim(U) = \dim(\mathfrak{g})$, which is transverse to Ψ. Thus $X = \Psi^{-1}(U)$ is smooth.

Idea: $L\mathfrak{g}^* = P_e G \to G$ is a principal ΩG-bundle, and can choose an LG-invariant connection. The connection tells you in which directions to "thicken" $\mathfrak{t}^* \subset L\mathfrak{g}^*$.

Spin-c structure on X

$X = \Psi^{-1}(\mathfrak{t}^*)$ has normal bundle

$$\nu(X, \mathcal{M}) \simeq X \times (L\mathfrak{g}/\mathfrak{t}), \qquad \text{(via } \mathbf{T}\Psi\text{)}$$

$$S_{TX} := \mathrm{Hom}_{\mathrm{Cliff}(L\mathfrak{g}/\mathfrak{t})}(S_{L\mathfrak{g}/\mathfrak{t}}, S_{\mathbf{T}\mathcal{M}})$$

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Subtleties

- What about non-uniqueness of Clifford representations in infinite dimensions—is S_{TX} above correct (or even non-zero)?
- Equivariance under action of the lattice Λ?

Non-uniqueness of Fock space representations

Theorem (Shale-Stinespring)

The Cliff(V)*-representations* $\overline{\wedge_J V}$ *and* $\overline{\wedge_{J'} V}$ *associated to two orthogonal complex structures* J, J' *are unitarily equivalent iff* $J - J'$ *is Hilbert-Schmidt. (Unique up to phase.)*

$$X \times L\mathfrak{g}/\mathfrak{t} \subset \mathbf{T}\mathcal{M}|_X, \qquad \text{(via } \mathbf{T}\Psi \text{)}$$

Using cross-sections

$$\mathcal{M} = LG \cdot \bigcup_\sigma Y_\sigma, \qquad LG \cdot Y_\sigma = LG \times_{(LG)_\sigma} Y_\sigma$$

one checks that almost complex structures indeed differ by Hilbert-Schmidt.

Restricted orthogonal group

Definition

Fix (V, J). The *restricted orthogonal group* $O_{\text{res}}(V)$ is the set of $a \in O(V)$ commuting with J modulo Hilbert-Schmidt operators.

$$\Rightarrow \qquad 1 \to U(1) \to \widehat{O}_{\text{res}}(V) \to O_{\text{res}}(V) \to 1$$

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$LT \circlearrowright L\mathfrak{g}/\mathfrak{t}$, and in fact

$$LT \to O_{\text{res}}(L\mathfrak{g}/\mathfrak{t})$$

$\Rightarrow S_{L\mathfrak{g}/\mathfrak{t}}$ is $\widehat{LT}^{(h^\vee)}$-equivariant. $T \times \Lambda \subset LT \Rightarrow T \ltimes \hat{\Lambda}^{(h^\vee)} \subset \widehat{LT}^{(h^\vee)}$

$$\hat{\lambda} t \hat{\lambda}^{-1} t^{-1} = t^{h^\vee B^\flat(\lambda)} \text{ in } T \ltimes \hat{\Lambda}^{(h^\vee)}$$

$h^\vee$ dual Coxeter number, B the basic inner product.

The index

$\Rightarrow S_{TX}$ is $T \ltimes \hat{\Lambda}^{(h^\vee)}$-equivariant. Twist by level k prequantum line bundle

$$S = S_{TX} \otimes L \qquad \left(T \ltimes \hat{\Lambda}^{(\ell)}\text{-equivariant, } \ell := k + h^\vee\right)$$

Let D be a Dirac operator for S, and $H = L^2(X, S)$.

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Let D be a Dirac operator for S, and $H = L^2(X, S)$.

Theorem

D is Fredholm on each isotypic component $H(\mu)$, $\mu \in \Lambda^*$.

Define

$$Q(X) = \text{index}(D) \in R^{-\infty}(T).$$

Note: $Q(X)$ in fact W_{aff} anti-symmetric. Dividing by Weyl-Kac denominator gives a character $Q(\mathcal{M})$ of a level k positive energy rep of LG.

Sketch of proof

Choose χ a normalizing function so that $F = \chi(D)$ is properly supported. Choose $f \in C_c^\infty(X)$, a T-invariant bump function such that $\{\lambda \cdot f | \lambda \in \Lambda\}$ is partition of unity. Set $A = f \cdot (F^2 - 1)$. Then

$$F^2 - 1 = \sum \hat{\lambda} A \hat{\lambda}^{-1}$$

essentially a direct sum of operators, and each term is compact.

As A is compact,

$$||A||_{H(\nu)} \xrightarrow{\nu \to \infty} 0.$$

But $\hat{\lambda}^{-1}$ maps $H(\mu)$ isometrically to $H(\mu + \ell B^\flat(\lambda))$, so

$$||\hat{\lambda} A \hat{\lambda}^{-1}||_{H(\mu)} = ||A||_{H(\mu + \ell B^\flat(\lambda))} \xrightarrow{\lambda \to \infty} 0.$$

$\Rightarrow$ sum converges in norm when restricted to $H(\mu)$.

Witten deformation

Define

$$v := \Psi(1 + ||\Psi||^2)^{-1/2} : X \to \mathfrak{t}^* \simeq \mathfrak{t}, \qquad v_X \in \Gamma(TX)$$

$$D_t = D - itc(v_X), \qquad t \geq 0.$$

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$$D_t = D - itc(v_X), \qquad t \geq 0.$$

Then

$$\begin{aligned} D_t^2 &= D^2 + it[D, c(v_X)] + t^2 ||v_X||^2 \\ &= D^2 + t^2 ||v_X||^2 + 2t\langle \mu, v \rangle + 2itv^k \mathcal{L}_k^S + itB \end{aligned}$$

- μ is the *spin-c moment map*, $\mathcal{L}_\xi^{\det(S)} - \nabla_\xi^{\det(S)} = 2i\langle \mu, \xi \rangle$
- $v = \sum v^k e_k$, e_k an orthonormal basis of $\mathfrak{t}$
- B is a bounded operator

Localizing to Crit($||\Psi||^2$)

$$D_t^2 = D^2 + t^2||v_X||^2 + 2t\langle \mu, v\rangle + 2itv^k\mathcal{L}_k^S + itB$$

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Restricted to an isotypic component, $\mathcal{L}_k^S$ become **bounded** (as in Tian-Zhang). On other hand,

$$\langle\mu, v\rangle(x) \to \infty, \quad \text{as} \quad x \to \infty \text{ in } X.$$

$\Rightarrow D_t$ Fredholm on each isotypic component, and

$$\text{index}(D) = \text{index}(D_t), \qquad \forall t \geq 0$$

For large t, sections in kernel concentrate near

$$\{v_X = 0\} = \text{Crit}(||\Psi||^2) \cap X.$$

Which components of $\text{Crit}(||\Psi||^2)$ contribute?

$$\text{Crit}(||\Psi||^2) \cap X = \bigcup_{\beta \in \mathcal{B}} X^{\beta} \cap \Psi^{-1}(\beta)$$

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For large t, $C = X^\beta \cap \Psi^{-1}(\beta)$ does not contribute to $\mathrm{index}(D_t)^T$ if

$$d(\beta) = \langle \mu, \beta \rangle + \frac{1}{4}\mathrm{Tr}_{\nu(C,X)}(|\mathcal{L}_\beta|) - 2\langle \rho, \beta \rangle > 0$$

where $\langle \mu, \beta \rangle$ is constant on C, $\mathcal{L}_\beta \in \Gamma(\mathrm{End}(\nu(C, X)))$.
(Have returned to the general setting, with Ψ not necessarily transverse.)

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(Have returned to the general setting, with Ψ not necessarily transverse.)

Detailed behavior of $\mu \Rightarrow d(\beta)$ bounded below by expression only involving **Lie algebra data** (!)

An inequality

The Stiefel diagram is an affine hyperplane arrangement in $\mathfrak{t}$:

$$\langle \alpha, x \rangle + n = 0, \qquad \alpha \in \mathcal{R}, n \in \mathbb{Z}.$$

v a vertex and $K = G_{\exp(v)}$. Let ρ, ρ_K be half-sums of positive roots for G, K, and let

$$c_K = \frac{\rho + \rho_K}{2h^\vee}.$$

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Theorem: Identify $\mathfrak{t} \simeq \mathfrak{t}^*$ (basic inner product). Then

$$||v - c_K|| \geq ||c_K||.$$

Special case: $G = SU(N)$, then $c_K = c$ is the barycenter of the alcove.

Conclusion

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Remarks

- Non-transverse case: study instead index pairing between D and pull-back of Bott class on $\mathfrak{g}/\mathfrak{t}$. Introduces extra terms, argument mostly unchanged.
- More work gives a "norm-square localization formula" for $Q(X)$.
- Possible to interpret $(L^2(X, S), D)$ as defining a class in twisted K-homology $K_0^T(G, \mathcal{A}^{(\ell)})$—connects $Q(X)$ with earlier work on q-Hamiltonian spaces.