

THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE

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These notes are meant to be a brief, self-contained introduction to the Atiyah-Hirzebruch spectral sequence (AHSS). The main references I consulted were *General cohomology theory and K-theory* by Hilton, Hatcher's notes on spectral sequences, some lectures notes on algebraic topology by Alex Kupers, and some lectures notes by Dan Ramras (the latter three are available online). Hatcher's notes do not cover the AHSS, although they do cover other related examples.

So that we do not need to worry about any convergence questions, throughout X will be a finite CW complex. Let $h(-)$ denote a reduced generalized cohomology theory (we review the definition below). The AHSS is a tool that relates $h(X)$ to the ordinary cohomology of X .

1. SPECTRAL SEQUENCES AND EXACT COUPLES

A *spectral sequence* (SS) is a sequence (E_n, d_n) of abelian groups equipped with differentials ($d_n^2 = 0$) such that $E_{n+1} = \ker(d_n)/\text{im}(d_n) = H(E_n, d_n)$ (here $H(E_n, d_n)$ means the homology/cohomology of E_n with respect to the differential d_n). Often the groups E_n are also bigraded, meaning

$$E_n = \bigoplus_{p,q \in \mathbb{Z}} E_n^{p,q}.$$

One imagines the groups $E_n^{p,q}$ arranged on a 2-dimensional grid with p, q being the x, y -coordinates, and for this reason the E_n are called the *pages* of the SS.

Here is an algebraic machine that produces a spectral sequence. Suppose we have two abelian groups $D = D_1$ and $E = E_1$, and an exact triangle

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ \gamma \swarrow & & \searrow \beta \\ E & & \end{array}$$

This means that the image of each map is the kernel of the next map. This is called an *exact couple*. Define a differential $d = d_1 = \beta\gamma$ (not $\gamma\beta = 0!$), and (E_1, d_1) is the first page of the SS.

Proposition 1.1. *Let $d = \beta\gamma$. The groups*

$$E' = H(E, d), \quad D' = \text{im}(\alpha)$$

and maps

$$\alpha' = \alpha, \quad \beta' = \beta \circ \alpha^{-1}, \quad \gamma' = \gamma$$

form a new exact couple, called the derived couple of $(E, D, \alpha, \beta, \gamma)$.

Proof. There are several small things to check. Note that β' is well-defined, because if $x, y \in \alpha^{-1}(z)$ then $x - y \in \ker(\alpha) = \text{im}(\gamma)$, say $x = y + \gamma(e)$, in which case

$$\beta(x) = \beta(y) + \beta(\gamma(e)) = \beta(y) + de$$

so we see that $[\beta(x)] = [\beta(y)]$ in homology $E' = H(E, d)$. To see that γ' is also well-defined, note that if $e \in \ker(\beta\gamma)$ then $\gamma(e) \in \ker(\beta) = \text{im}(\alpha)$, so that $\gamma(e)$ indeed lies in $\alpha(D)$. Moreover if $e \in \text{im}(\beta\gamma)$ then $\gamma(e) = 0$ since $\gamma\beta = 0$. So we've shown that the maps are well-defined. The fact that the composition of any two in a row is 0 is immediate. The other checks are similar. \square

Iterating this construction we get a sequence of exact couples $(E_n, D_n, \alpha_n, \beta_n, \gamma_n)$ where (recursively):

$$\begin{aligned} D_n &= D'_{n-1} = \alpha^{n-1}D, & E_n &= E'_{n-1} = H(E_{n-1}, d_{n-1}) \\ \alpha_n &= \alpha, & \beta_n &= \beta\alpha^{1-n}, & \gamma_n &= \gamma, & d_n &= \beta_n\gamma_n. \end{aligned}$$

And so we obtain a SS:

$$(E_1, d_1), \quad (E_2, d_2), \quad (E_3, d_3), \quad \dots$$

Proposition 1.2.

$$E_{n+1} = H(E_n, d_n) \simeq \frac{\gamma^{-1}(\alpha^n D)}{\beta(\ker \alpha^n)}$$

Proof. We prove this by induction. The base case is $n = 1$. By exactness

$$\ker(\beta\gamma) = \gamma^{-1}(\beta^{-1}(0)) = \gamma^{-1}(\alpha D), \quad \text{im}(\beta\gamma) = \beta(\ker(\alpha)),$$

so

$$E_2 = \frac{\ker(\beta\gamma)}{\text{im}(\beta\gamma)} = \frac{\gamma^{-1}(\alpha D)}{\beta(\ker \alpha)}.$$

Now suppose

$$E_n = \frac{\gamma^{-1}(\alpha^{n-1}D)}{\beta(\ker \alpha^{n-1})}.$$

Then

$$E_{n+1} = \frac{\ker(d_n)}{\text{im}(d_n)} = \frac{\ker(\beta_n\gamma_n)}{\text{im}(\beta_n\gamma_n)}.$$

We have

$$\ker(\beta_n\gamma_n) = \gamma_n^{-1}(\beta_n^{-1}(0)) = \gamma_n^{-1}(\text{im}(\alpha_n)).$$

But $\alpha_n: \alpha^{n-1}D \rightarrow \alpha^{n-1}D$, so $\text{im}(\alpha_n) = \alpha^n D$ and since $\gamma_n = \gamma$ we get

$$\ker(\beta_n\gamma_n) = \frac{\gamma^{-1}(\alpha^n D)}{\beta(\ker \alpha^{n-1})} \subset E_n.$$

(Here we use the notation $G'/H \subset G/H$ to denote the image of a subset $G' \subset G$ under the quotient map $G \rightarrow G/H$.) Now $\alpha_n: \alpha^{n-1}D \rightarrow \alpha^{n-1}D$ and $\beta_n = \beta\alpha^{1-n}$. Note that $\alpha^{1-n}(\ker \alpha_n)$ is exactly the set of elements of D that are killed by α^n . Consequently

$$\text{im}(\beta_n\gamma_n) = \beta_n(\ker \alpha_n) = \frac{\beta(\ker \alpha^n D)}{\beta(\ker \alpha^{n-1})} \subset E_n.$$

Putting this together with $\ker(\beta_n\gamma_n)$ above, we get the result. \square

2. GENERALIZED COHOMOLOGY THEORIES

A *reduced generalized cohomology theory* is a sequence of contravariant functors h^n from pointed spaces to abelian groups, together with a natural isomorphisms $h^n \circ \Sigma \simeq h^{n-1}$, where Σ is the suspension functor, satisfying:

- Homotopy invariance.
- Wedge axiom: $h \circ \vee_i = \Pi_i \circ h$ or

$$h\left(\bigvee X_i\right) = \prod_i h(X_i)$$

and similarly for morphisms.

- Exact sequences: let $f: A \rightarrow X$ and let C_f be the mapping cone of f so that there is an inclusion $X \hookrightarrow C_f$. Then for each n

$$h^n(C_f) \rightarrow h^n(X) \rightarrow h^n(A)$$

is exact in the middle.

The above are axioms for the reduced theory. If we start with an unreduced theory \underline{h} , the corresponding reduced theory is the kernel of the pullback to the basepoint

$$h^n(X) = \ker(\underline{h}^n(X) \rightarrow \underline{h}^n(\text{pt})).$$

It is convenient to define

$$h^n := h^n(S^0) = \underline{h}^n(\text{pt}).$$

Example 2.1. Reduced singular cohomology $h^n(X) = \tilde{H}^n(X; G)$ with coefficients in an abelian group G , and reduced K-theory $h^n(X) = \tilde{K}^n(X)$

We can iterate the last axiom to get a longer exact sequence

$$h^n(C_j) \rightarrow h^n(C_i) \rightarrow h^n(C_f) \rightarrow h^n(X) \rightarrow h^n(A)$$

where C_i (resp. C_j) is the mapping cone of the inclusion $i: Y \rightarrow C_f$ (resp. $j: C_f \hookrightarrow C_i$), and so on. Note that $C_i \approx \Sigma A$ (resp. $C_j \approx \Sigma X$), and using the natural isomorphism $h^n \circ \Sigma \simeq h^{n-1}$ we get

$$h^{n-1}(X) \rightarrow h^{n-1}(A) \rightarrow h^n(C_f) \rightarrow h^n(X) \rightarrow h^n(A).$$

The sequences for different n fit together into a long exact sequence. In particular for $f: A \rightarrow X$ an inclusion we define

$$h^n(X, A) := h^n(C_f)$$

and obtain the long exact sequence for pairs (X, A)

$$\cdots \rightarrow h^{n-1}(A) \rightarrow h^n(X, A) \rightarrow h^n(X) \rightarrow h^n(A) \rightarrow \cdots$$

For the pairs $\iota: A \hookrightarrow X$ we will be dealing with (CW complexes and inclusions of subcomplexes), the mapping cone $C_\iota \approx X/A$ the quotient space with the quotient topology, so we can also write this as

$$\cdots \rightarrow h^{n-1}(A) \rightarrow h^n(X/A) \rightarrow h^n(X) \rightarrow h^n(A) \rightarrow \cdots$$

The easy maps in this sequence are induced by $A \rightarrow X$, $X \rightarrow X/A$. The harder connecting map $h^{n-1}(A) \rightarrow h^n(X/A)$ is obtained by composing the isomorphism $h^{n-1}(A) \simeq h^n(\Sigma A)$ with the map induced by the composition $X/A \approx \text{Cone}(\iota) \hookrightarrow \text{Cone}(X \hookrightarrow \text{Cone}(\iota)) \approx \Sigma A$.

2.1. Maps between spheres of the same dimension. Recall that a map between spheres of the same dimension is determined up to homotopy by its degree. Here we note a couple consequences of this that will be useful later on.

Remark 2.2. Let I, J be finite index sets. Consider a continuous basepoint-preserving map

$$f: \vee_{i \in I} S_i^p \rightarrow \vee_{j \in J} S_j^p$$

between finite wedge sums of spheres. By the wedge axiom we get a map

$$f^*: \prod_{j \in J} h^n(S_j^p) \rightarrow \prod_{i \in I} h^n(S_i^p).$$

Such a map is determined by its components. The (j_0, i_0) component is the composition

$$h^n(S_{j_0}^p) \hookrightarrow \prod_{j \in J} h^n(S_j^p) \rightarrow \prod_{i \in I} h^n(S_i^p) \twoheadrightarrow h^n(S_{i_0}^p).$$

The first map is induced from the collapse map $\vee_j S_j^p \rightarrow S_{j_0}^p$ that is the identity on $S_{j_0}^p$ and collapses all other spheres to a point. The second map is induced from the inclusion $S_{i_0}^p \hookrightarrow \vee_i S_i^p$.

Lemma 2.3. *Let $f: S^p \rightarrow S^p$ be a continuous map, $p \geq 1$. Then for any reduced cohomology theory h , the induced map*

$$f^*: h^q(S^p) \rightarrow h^q(S^p)$$

is multiplication by $\text{degree}(f)$.

Proof. If the map has degree 0 or 1 the result is clear by homotopy invariance. Now suppose f has degree $m > 0$. Then up to homotopy f can be written as a composition

$$S^p \xrightarrow{f_1} \vee_{n=1}^m S^p \xrightarrow{f_2} S^p$$

where f_1 has the property that its composition with the collapse map to each of the wedge summands has degree 1, and f_2 is the “folding” map that acts as the identity on each of the wedge summands. It follows from the wedge axiom and Remark 2.2 that f_1^* is the map $(x_1, \dots, x_m) \mapsto x_1 + \dots + x_m$ and f_2^* is the map $x \mapsto (x, \dots, x)$. Thus $f^*(x) = mx$. A similar trick handles the case $m < 0$. \square

3. THE SPECTRAL SEQUENCE

Let X be a finite CW complex, and let X_p be the p -skeleton (for $p < 0$ set X_p to be the basepoint). For each p , there is a long exact sequence associated to the inclusion $X_{p-1} \hookrightarrow X_p$, and the AHSS will be a powerful tool for extracting information from all of these exact sequences.

The long exact sequence for the inclusion $X_{p-1} \hookrightarrow X_p$ looks like

$$\cdots \rightarrow h^{p+q-1}(X_{p-1}) \xrightarrow{\beta} h^{p+q}(X_p/X_{p-1}) \xrightarrow{\gamma} h^{p+q}(X_p) \xrightarrow{\alpha} h^{p+q}(X_{p-1}) \rightarrow \cdots$$

We've used a shifted indexing $n = p + q$, $q \in \mathbb{Z}$, because it will be more convenient later on. Note also that X_p/X_{p-1} is a wedge of p -spheres, and so by the wedge and suspension axioms

$$h^{p+q}(X_p/X_{p-1}) = h^{p+q}(\vee_{I_p} S^p) = \prod_{I_p} h^q,$$

where I_p is the set of p -cells.

We now define an exact couple:

$$D_1 = D = \bigoplus_{p,q \in \mathbb{Z}} D^{p,q}, \quad E_1 = E = \bigoplus_{p,q \in \mathbb{Z}} E^{p,q}$$

where

$$D_1^{p,q} = D^{p,q} = h^{p+q}(X_p), \quad E_1^{p,q} = E^{p,q} = h^{p+q}(X_p/X_{p-1}) \simeq \prod_{I_p} h^q.$$

The maps α, β, γ are as indicated in the sequences above. Our exact couple looks like

$$\begin{array}{ccc} D & \xrightarrow{(-1,1)} & D \\ \nwarrow \gamma & \alpha & \nearrow \beta \\ (0,0) & & (1,0) \\ & \searrow & \swarrow \\ & E & \end{array}$$

The groups are bigraded and it is convenient to visualize them as lying on a 2-dimensional grid, with the group $D^{p,q}$ (resp. $E^{p,q}$) sitting at the point (p, q) . The numbers in the diagram above indicate the bidegrees of the maps.

Definition 3.1. The AHSS for the h -cohomology of X is the SS obtained from the above exact couple.

Note that E_1 (and consequently all subsequent pages) is supported in the right half plane ($p \geq 0$)—one says that the AHSS is a “right half plane SS”.

3.1. The E_2 page. We can compute the E_2 page of the AHSS for $h^\bullet(X)$ in terms of the groups h^n and the ordinary cohomology of X .

Theorem 3.2. *The E_2 page of the AHSS is $E_2^{p,q} = \tilde{H}^p(X; h^q)$.¹*

Proof. We prove this by arguing that $E_1^{p,q} = H^p(X_p, X_{p-1}; h^q)$ and that $d_1: E_1^{p,q} \rightarrow E_1^{p+1,q}$ is the coboundary map in cellular cohomology with coefficients $G = h^q$; the result then follows from the isomorphism between cellular and singular cohomology, see the appendix for a brief reminder on cellular cohomology.

We claim first that $E_1^{p,q} = H^{p+q}(X_p, X_{p-1}; h^q)$. Indeed $E_1^{p,q} = h^{p+q}(X_p/X_{p-1})$ by definition. But $X_p/X_{p-1} = \vee_{I_p} S^p$ so

$$\begin{aligned} E_1^{p,q} &= h^{p+q}(\vee_{I_p} S^p) \\ &= \prod_{I_p} h^q \\ &= \tilde{H}^{p+q}(\vee_{I_p} S^p; h^q) \\ &= \tilde{H}^{p+q}(X_p/X_{p-1}; h^q) \\ &= H^{p+q}(X_p, X_{p-1}; h^q). \end{aligned}$$

We next determine the differential. Using the wedge axiom, the map

$$d_1: h^{p+q-1}(X_p/X_{p-1}) \rightarrow h^{p+q}(X_{p+1}/X_p)$$

¹The same statement works if we replace h, \tilde{H} with their unreduced versions \underline{h}, H .

gives a map between finite products

$$\prod_{\sigma \in I_p} h^{p+q-1}(S_\sigma^p) \rightarrow \prod_{\tau \in I_{p+1}} h^{p+q}(S_\tau^{p+1})$$

and the latter is determined by each of its components. Fix $\sigma \in I_p$ and $\tau \in I_{p+1}$. We will look at the (σ, τ) component.

One can get the result using Lemma 2.3 and staring at the following diagram:

$$\begin{array}{ccccc}
 h^{p+q}(D_\tau / \partial D_\tau) & \xleftarrow{\sim} & h^{p+q-1}(\partial D_\tau) & & \\
 \phi_\tau^* \uparrow & & \phi_\tau^* \uparrow & & m_{\sigma\tau} \\
 h^{p+q}(X_{p+1} / X_p) & \xleftarrow{\beta} & h^{p+q-1}(X_p) & & h^{p+q-1}(D_\sigma / \partial D_\sigma) \\
 & \swarrow d_1 & \uparrow \gamma = \pi^* & \searrow c_\sigma^* & \\
 & & h^{p+q-1}(X_p / X_{p-1}) & &
 \end{array}$$

Here D_τ is the closed $p+1$ -dimensional disk and likewise D_σ is a closed p -dimensional disk. The map of pairs

$$\phi_\tau: (D_\tau, \partial D_\tau) \rightarrow (X_{p+1}, X_p)$$

(the “attaching map” of the $p+1$ -cell τ) gives the top left corner, which commutes by naturality of the long exact sequences for relative cohomology. The lower left triangle is the definition of d_1 . The map

$$c_\sigma: X_p / X_{p-1} \rightarrow D_\sigma / \partial D_\sigma$$

is the map that collapses the complement of the p -cell σ to a point, and c_σ^* is the induced map on cohomology. The map $m_{\sigma\tau}$ is defined to map the diagram commute. Since it is induced by the continuous map $c_\sigma \circ \pi \circ \phi_\tau$ between two p -spheres, Lemma 2.3 tells us that $m_{\sigma\tau}$ is multiplication by $\deg(c_\sigma \circ \pi \circ \phi_\tau)$.

From the wedge axiom (now for maps) and Remark 2.2, the composition $\phi_\tau^* \circ d_1 \circ c_\sigma^*$ going from the far right to the upper left is the (σ, τ) component of d_1 , and the commutativity of the diagram shows that it is given by multiplication by $m_{\sigma\tau}$.

So we’ve shown that the (σ, τ) component of d_1 is multiplication by $m_{\sigma\tau}$. In particular there is a matrix of integers $m_{\sigma\tau}$ that completely describes d_1 , and moreover it is the same matrix for *any cohomology theory*, because it is just the degree of some map between two spheres. In particular it is the same for h and for \tilde{H} (reduced singular cohomology). \square

Example 3.3. Let $X = \mathbb{C}P^n$. This admits a CW structure with one cell in each even dimension. Thus

$$E_2^{p,q} = \tilde{H}^p(\mathbb{C}P^n; h^q) = \begin{cases} h^q & 0 \neq p \text{ even} \\ 0 & p \text{ odd or } 0. \end{cases}$$

Suppose moreover that $h^q = 0$ for q odd (for example this happens when $h = \tilde{H}$ or $h = \tilde{K}$). Then since higher differentials have bidegree $(n, 1-n)$, they necessarily vanish, and so $E_2 = E_3 = E_4 = \dots$. One says that the SS “collapses” at the E_2 page.

3.2. What does it compute? Let's now start to unravel what the SS can tell us, *assuming we can calculate all the pages E_2, E_3, \dots* . All the higher pages are subquotients of the first page:

$$E_{n+1} = \frac{\gamma^{-1}(\alpha^n D)}{\beta(\ker \alpha^n)}, \quad d_{n+1} = \beta \alpha^{-n} \gamma.$$

Note that d_{n+1} has bidegree $(1, 0) - n(-1, 1) = (n+1, -n)$.

For a general right half plane SS, the question of what it computes can be difficult. We are assuming that X is a **finite** CW complex, which makes the difficult questions about convergence go away. In particular, a finite CW complex is finite dimensional: this means that at some finite stage N one has $X_N = X_{N+1} = X_{N+2} = \dots = X$. Thus for $p > N$, $X_p/X_{p-1} = \text{pt}$, and the reduced cohomology of a point is 0. So we get the following key simplification:

E_1 is supported in the vertical strip $0 \leq p \leq N$.

Because of this, d_{n+1} vanishes identically for $n \geq N$ (because either the domain or range or both is zero), hence $E_{N+1} = E_{N+2} = E_{N+3} = \dots$ doesn't change after we reach E_{N+1} . So let's define

$$E_\infty = E_{N+1} = E_{N+2} = \dots$$

to be this limiting page, so

$$E_\infty = \frac{\gamma^{-1}(\cap \alpha^n D)}{\beta(\cup \ker \alpha^n)}.$$

Theorem 3.4.

$$E_\infty^{p,q} \simeq \frac{\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_{p-1}))}{\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_p))}$$

Remark 3.5. Let's phrase this a bit differently. Define

$$F^m h^n(X) = \ker(h^n(X) \rightarrow h^n(X_m)).$$

This gives a descending *filtration* of $h^n(X)$:

$$h^n(X) = F^{-1}h^n(X) \supset F^0h^n(X) \supset F^1h^n(X) \supset \dots \supset F^{N-1}h^n(X) \supset F^N h^n(X) = 0.$$

Then the theorem says

$$E_\infty^{p,q} = \frac{F^{p-1}h^n(X)}{F^p h^n(X)}, \quad n = p + q.$$

In other words, what appears along the *diagonal* $p + q = n$ on the E_∞ page is the *associated graded* of the filtered abelian group $F^\bullet h^n(X)$. Thus E_∞ gives us $h^n(X)$ ‘up to extension problems’. The results of Theorems 3.4, 3.2 are usually summarized by saying: “there is a spectral sequence with $E_2^{p,q} = \tilde{H}^p(X; h^q)$ converging to $h^{p+q}(X)$ ”. The word “converging” here carries a fair bit of baggage: here it means the diagonals of the E_∞ page are the components of the associated graded of some implicit filtration on the groups $h^{p+q}(X)$.

Lemma 3.6.

$$E_\infty^{p,q} = \frac{\gamma^{-1}(\text{im}(h^{p+q}(X) \rightarrow h^{p+q}(X_p)))}{\ker(\gamma_{p,q}: h^{p+q}(X_p/X_{p-1}) \rightarrow h^{p+q}(X_p))}.$$

Proof. The bidegree of α^n is $(-n, n)$. Since D is supported in the right half-plane, the elements of $D_{p,q}$ are in the kernel of α^{p+1} . Thus

$$\cup \ker \alpha^n = D$$

and so the denominator of E_∞ is

$$\beta(D) = \text{im}(\beta) = \ker(\gamma).$$

For the numerator

$$(\cap \alpha^n D)^{p,q} = \lim_n \alpha^n (D^{p+n, q-n}) = \lim_n \alpha^n h^{p+q}(X_{p+n}) = \text{im}(h^{p+q}(X) \rightarrow h^{p+q}(X_p))$$

since for n sufficiently large $X_{p+n} = X$ by finite dimensionality. \square

Proof of 3.4. We use the description of E_∞ from the previous lemma. Let us define auxiliary groups

$$K^{p,q} = \text{im}(h^{p+q}(X) \rightarrow h^{p+q}(X_p)) \cap \text{im}(\gamma) = \text{im}(h^{p+q}(X) \rightarrow h^{p+q}(X_p)) \cap \ker(\alpha).$$

Trivially $\gamma^{-1}(S) = \gamma^{-1}(S \cap \text{im}(\gamma))$ for any subset S . So by the lemma

$$E_\infty^{p,q} = \frac{\gamma^{-1}(K^{p,q})}{(\ker \gamma)_{p,q}} \xrightarrow{\gamma} K^{p,q},$$

and by the first isomorphism theorem the map $E_\infty^{p,q} \rightarrow K^{p,q}$ induced by γ is an isomorphism, so

$$E_\infty^{p,q} \simeq K^{p,q}.$$

Now define a map

$$\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_{p-1})) \rightarrow K^{p,q} = \text{im}(h^{p+q}(X) \rightarrow h^{p+q}(X_p)) \cap \ker(\alpha)$$

that just takes an element of $h^{p+q}(X)$ to its pullback to $h^{p+q}(X_p)$. The elements in $K^{p,q}$ are elements of $h^{p+q}(X_p)$ that vanish when you pull back once more, i.e. apply $\alpha: h^{p+q}(X_p) \rightarrow h^{p+q}(X_{p-1})$, so the map above is surjective. Its kernel is $\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_p))$, and so by the first isomorphism theorem

$$E_\infty^{p,q} \simeq K^{p,q} \simeq \frac{\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_{p-1}))}{\ker(h^{p+q}(X) \rightarrow h^{p+q}(X_p))}.$$

\square

Here are some nice corollaries for X a finite CW complex (see Hilton's book for example):

- (a) If h^q is finitely generated for each q , then $h^q(X)$ is finitely generated.
- (b) If $f: X \rightarrow X'$ induces an isomorphism on singular cohomology, then f induces an isomorphism $h^\bullet(X') \rightarrow h^\bullet(X)$ for any generalized cohomology theory h .
- (c) For any h

$$h^q(X) \otimes \mathbb{Q} \simeq \bigoplus_{r+s=q} H^r(X, \text{pt}; \mathbb{Q}) \otimes h^s.$$

- (d) If $h^0 = G$ and $h^n = 0$ for $n \neq 0$ (i.e. if h satisfies the "dimension axiom") then $h^n(X) = H^n(X, \text{pt}; G)$ is necessarily singular cohomology with coefficients in G .

APPENDIX A. REVIEW OF CELLULAR COHOMOLOGY

Here is a very concise excerpt from Hatcher pp.202-203:

Cellular Cohomology. For a CW complex X this is defined via the cellular cochain complex formed by the horizontal sequence in the following diagram, where coefficients in a given group G are understood, and the cellular coboundary maps d_n are the compositions $\delta_n j_n$, making the triangles commute. Note that $d_n d_{n-1} = 0$ since $j_n \delta_{n-1} = 0$.

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \swarrow & & \searrow & \\
& & H^{n-1}(X^{n-1}) & & & & \\
j_{n-1} \nearrow & & \downarrow \delta_{n-1} & & & & \\
\cdots \longrightarrow H^{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{d_{n-1}} & H^n(X^n, X^{n-1}) & \xrightarrow{d_n} & H^{n+1}(X^{n+1}, X^n) & \longrightarrow \cdots & \\
& & \downarrow j_n & & \downarrow \delta_n & & \\
& & H^n(X^n) & & & & \\
& & \swarrow & & \searrow & & \\
H^n(X) \approx H^n(X^{n+1}) & & & & & & 0 \\
& & \swarrow & & & & \\
& & 0 & & & &
\end{array}$$

Theorem 3.5. $H^n(X; G) \approx \text{Ker } d_n / \text{Im } d_{n-1}$. Furthermore, the cellular cochain complex $\{H^n(X^n, X^{n-1}; G), d_n\}$ is isomorphic to the dual of the cellular chain complex, obtained by applying $\text{Hom}(-, G)$.

Proof: The universal coefficient theorem implies that $H^k(X^n, X^{n-1}; G) = 0$ for $k \neq n$. The long exact sequence of the pair (X^n, X^{n-1}) then gives isomorphisms $H^k(X^n; G) \approx H^k(X^{n-1}; G)$ for $k \neq n, n-1$. Hence by induction on n we obtain $H^k(X^n; G) = 0$ if $k > n$. Thus the diagonal sequences in the preceding diagram are exact. The universal coefficient theorem also gives $H^k(X, X^{n+1}; G) = 0$ for $k \leq n+1$, so $H^n(X; G) \approx H^n(X^{n+1}; G)$. The diagram then yields isomorphisms

$$H^n(X; G) \approx H^n(X^{n+1}; G) \approx \text{Ker } \delta_n \approx \text{Ker } d_n / \text{Im } d_{n-1} \approx \text{Ker } d_n / \text{Im } d_{n-1}$$