3.4. Optimization

Absolute Maxima and Minima of a function

Let f be a function defined on an interval I containing the number c. Then

- $f(c)$ is the absolute maximum of f on I if $f(c) \geq f(x)$ for all x in I.
- $f(c)$ is the absolute minimum of f on I if $f(c) \leq f(x)$ for all x in I.

Collectively, absolute maxima and minima are called absolute extrema.

Absolute Extrema on a Closed interval

The Extreme Value Property

A function $f(x)$ that is continuous on the closed interval $a \leq x \leq b$ attains its absolute extrema on the interval either at an endpoint of the interval (a or b) or at a critical number c such that $a < c < b$.

How to Find the Absolute Extrema of a Continuous Function f on $a \leq x \leq b$

Step 1. Find all critical numbers of f in $a < x < b$.

Step 2. Compute $f(x)$ at the critical numbers found in step 1 and at the endpoints $x = a$ and $x = b$.

Step 3. The largest and smallest values found in step 2 are, respectively, the absolute maximum and absolute minimum values of $f(x)$ on $a \leq x \leq b$.

Absolute Extrema on a Closed interval

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(x) = x^3 + 3x^2 + 1; \quad -3 \le x \le 2.$$

Absolute Extrema on a Closed interval

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(t) = \frac{t^2}{t-1}; \quad -2 \le t \le 1.$$

Absolute Extrema on a general interval

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(u) = u + \frac{16}{u}; \quad u > 0.$$

Absolute Extrema on a general interval

The Second Derivative Test for Absolute Extrema

Suppose that $f(x)$ is continuous on I where $x = c$ is the only critical number and that $f'(c) = 0$. Then

- if $f''(x) > 0$, the absolute minimum of $f(x)$ on I is $f(c)$.
- if $f''(x) < 0$, the absolute maximum of $f(x)$ on I is $f(c)$.

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(u) = u + \frac{16}{u}; \quad u > 0.$$