3.1. Increasing and Decreasing Functions; Relative Extrema

Increasing and Decreasing Functions

Let $f(x)$ be a function defined on the interval $a < x < b$, and let x_1 and x_2 be two numbers in the interval. Then

- $f(x)$ is increasing on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
- $f(x)$ is decreasing on the interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

Intervals of Increase and Decrease

Procedure for using the derivative to determine intervals of increase and decrease

Step 1. Find all values of x for which $f'(x) = 0$ or $f'(x)$ is not continuous, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval $a < x < b$ determined in Step 1 and evaluate $f'(c)$. Then

- If $f'(c) > 0$, $f(x)$ is increasing on $a < x < b$.
- If $f'(c) < 0$, $f(x)$ is decreasing on $a < x < b$.

Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$F(x) = \frac{x^2}{x-3}.$$

Relative Extrema

Definition

- The graph of the function $f(x)$ is said to have a relative maximum at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a < x < b$ containing c.
- Similarly, the graph has a relative minimum at $x = c$ if $f(c) \leq f(x)$ on such an interval.
- Collectively, the relative maxima and minima of f are called its relative extrema.

Definition

A number c in the domain of $f(x)$ is called a critical number if either $f'(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a critical point for $f(x)$.

Relative Extrema

Relative extrema can only occur at critical points.

The First Derivative Test for Relative Extrema

Let c be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is

- A relative maximum if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
- A relative minimum if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c.
- Not a relative extremum if $f'(x)$ has the same sign on both sides of c.

Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Relative Extrema

Example

Find all critical numbers of the function

$$F(x) = \frac{x^2}{x-3}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = x\sqrt{4 - x}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Sketching the graph

Prodecure for sketching the graph of a continuous function using the derivative

Step 1. Determine the domain of $f(x)$. Set up a number line restricted to include only those numbers in the domain.

Step 2. Find $f'(x)$ and mark each critical number on the restricted number line. Then analyze the sign of $f'(x)$ to determine the intervals of increase and decrease for $f(x)$.

Step 3. For each critical number c, find $f(c)$ and plot the critical point $P(c, f(c))$ on a plane. Plot intercepts and other key points that can be easily found.

Step 4. Sketch the graph of f as a smooth curve joining the critical points in such a way that it rises where $f'(x) > 0$, falls where $f'(x) < 0$, and has a horizontal tangent where $f'(x) = 0$.

Sketching the graph

Example

Use calculus to sketch the graph of

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

Sketching the graph

Example

Use calculus to sketch the graph of

$$F(x) = \frac{x^2}{x-3}.$$

Sketching the graph

Example

Use calculus to sketch the graph of

$$f(x) = \frac{x+1}{x^2+x+1}.$$