2.3. Product and Quotient Rules; Higher-Order Derivatives

The Product Rule

If $f(x)$ and $g(x)$ are differentiable at x, then so is their product and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

or equivalently

$$(fg)' = fg' + gf'$$

Example

Differentiate $f(x) = (2x - 5)(1 - x)$.

The Product Rule

Example

Differentiate $f(x) = (x^3 - 2x^2 + 5)(\sqrt{x} + 2x)$.

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions, then so is the quotient $Q(x) = f(x)/g(x)$ and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g^2(x)}$$

or equivalently

$$\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

Example

Differentiate $y = \dfrac{1+x^2}{1-x^2}$.

The Quotient Rule

Example

Find all points on the graph of $f(x) = \frac{x^2 + x - 1}{x^2 - x + 1}$ where the tangent line is horizontal.

Product rule and Quotient Rule

Example

Differentiate $g(x) = \dfrac{(x^2 + x + 1)(4 - x)}{2x - 1}$.

The Second Derivative

The second derivative of a function is the derivative of its derivative. If $y = f(x)$, the second derivative is denoted by

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

The second derivative gives the rate of change of the rate of change of the original function.

Example

Find the second derivative of $f(x) = x^{10} - 4x^6 - 27x + 4$.

The Second Derivative

Example

Find the second derivative of $y = (x^2 - 2x)\left(x - \frac{1}{x}\right)$.

Higher-Order Derivatives

The *n*th Derivative

For any positive integer n, the nth derivative of a function is obtained from the function by differentiating successively n times. If the original function is $y = f(x)$, the nth derivative is denoted by

$$\frac{d^n y}{dx^n} \quad \text{or} \quad f^{(n)}(x)$$

Example

Find $f^{(4)}(x)$ if $f(x) = x^6 - 2x^5 + x^4 - 3x^3 + 5x - 6$.