1.5. Limits

Behavior of $f(x)$ as x approaches c

Consider the behavior of $f(x) = \dfrac{x^2 - 3x + 2}{x - 1} = \dfrac{(x-1)(x-2)}{x-1}$ as x approaches 1.

x	0.8	0.9	0.99	1	1.01	1.1	1.2
$f(x)$	-1.2	-1.1	-1.01	undefined	-0.99	-0.9	-0.8

As x approaches 1, $f(x)$ approaches -1.

Definition

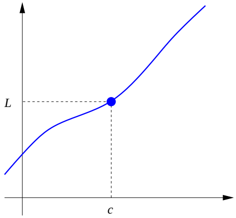
If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is the limit of $f(x)$ as x approaches c. The behavior is expressed by

$$\lim_{x \to c} f(x) = L$$

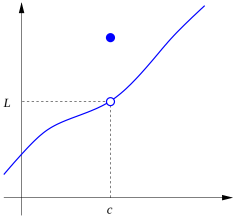
Functions for which the limit exists

It is important to remember that limits describes the behavior of a function near a particular point, not necessarily at the point itself.

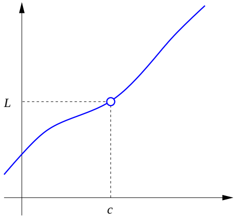
Three fuctions for which $\lim_{x \to c} f(x) = L$



$f(c) = L$



$f(c) \neq L$

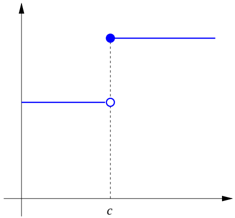


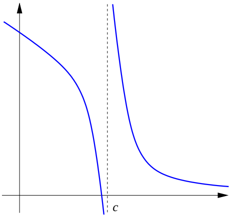
$f(c)$ does not exist

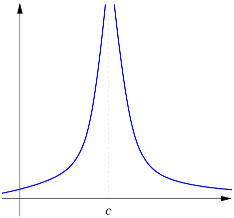
Functions for which the limit does not exist

It is possible that the limit $\lim_{x \to c} f(x)$ does not exist.

Example







Infinite limit

Properties of Limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

$$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

$$\lim_{x\to c}[kf(x)] = k \lim_{x\to c} f(x)$$

$$\lim_{x\to c}[f(x)g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)]$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{if } \lim_{x\to c} g(x) \neq 0$$

$$\lim_{x\to c}[f(x)]^p = [\lim_{x\to c} f(x)]^p \quad \text{if } [\lim_{x\to c} f(x)]^p \text{ exists}$$

Computation of Limits

Limits of two linear functions

For any constant k,

$$\lim_{x \to c} k = k \text{ and } \lim_{x \to c} x = c$$

Example

Find $\lim_{x \to 2} (x^2 - 4x + 7)$.

Computation of Limits

Limits of Polynomials and Rational functions

If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c)$$

and

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

Example

Find $\lim_{x \to 1} \frac{x+3}{2x+1}$.

Computation of Limits

Example

Find $\lim_{x \to 2} \dfrac{2x + 3}{x - 2}$.

Example

Find $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$.

Limits involving Infinity

Limits at infinity

If the values of $f(x)$ approach the number L as x increases without bound,

$$\lim_{x \to +\infty} f(x) = L$$

Similarly, we write

$$\lim_{x \to -\infty} f(x) = M$$

when the functional values $f(x)$ approach the number M as x decreases without bound.

Limits involving Infinity

Reciprocal Power Rules

If A and k are constants with $k > 0$ and x^k is defined for all x, then

$$\lim_{x \to +\infty} \frac{A}{x^k} = 0 \text{ and } \lim_{x \to -\infty} \frac{A}{x^k} = 0$$

Procedure for evaluating a limit at infinity of $f(x) = \dfrac{p(x)}{q(x)}$

Step. 1 Divide each terms in $f(x)$ by the highest power x^k in the denominator.

Step. 2 Compute the limit using algebraic properties of limits and the reciprocal power rules.

Limits involving Infinity

Example

Find $\lim_{x \to +\infty} \dfrac{1 - 2x^3}{2x^3 - 5x + 4}$.

Example

Find $\lim_{x \to -\infty} \dfrac{x^2 + 2x - 3}{1 - 3x - x^3}$.

Limits involving Infinity

Infinite Limits

If $f(x)$ increases without bound as $x \to c$, we write

$$\lim_{x \to c} f(x) = +\infty.$$

Also, if $f(x)$ decreases without bound as $x \to c$, then

$$\lim_{x \to c} f(x) = -\infty.$$

Example

Find $\displaystyle\lim_{x \to +\infty} \frac{1 - 3x^3}{2x + 1}$.