Math 108, Solution of Midterm Exam 3

$\boxed{1}$ Find an equation of the tangent line to the curve $x^3 + y^3 = 2xy$ at the point $(1, 1)$.

Solution. Differentiating both sides of the given equation with respect to x, we get

$$\begin{aligned}
\frac{d}{dx}\left[x^3 + y^3\right] &= \frac{d}{dx}\left[2xy\right] \\
3x^2 + 3y^2\frac{dy}{dx} &= 2y\frac{d}{dx}(x) + 2x\frac{d}{dx}(y) \\
3x^2 + 3y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\
(3y^2 - 2x)\frac{dy}{dx} &= 2y - 3x^2 \\
\frac{dy}{dx} &= \frac{2y - 3x^2}{3y^2 - 2x}.
\end{aligned}$$

So, the slope of the tanget line is

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = -1$$

and hence the equation of the tangent line is

$$\begin{aligned}
y - 1 &= -1(x - 1) \\
y &= -x + 2
\end{aligned}$$

$\boxed{2}$ Let f be a function defined by $f(x) = x^5 - 5x^4 + 3$.

(a) Find intervals on which f is increasing and decreasing.

Solution. The derivative of f is

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[x^5 - 5x^4 + 3\right] \\
&= 5x^4 - 20x^3 \\
&= 5x^3(x - 4)
\end{aligned}$$

which is continuous everywhere, with $f'(x) = 0$ where $x = 0$ and $x = 4$. The numbers 0 and 4 divide the x axis into three open intervals; namely, $x < 0$, $0 < x < 4$, and $x > 4$. Choose a test number from each of these intervals; say $c = -1$ from $x < 0$, $c = 1$ from $0 < x < 4$, and $c = 5$ from $x > 4$. Then, evaluate $f'(c)$ for each test number:

$$\begin{aligned}
f'(-1) &= 5(-1)^3(-1 - 4) > 0, \\
f'(1) &= 5(1)^3(1 - 4) < 0, \\
f'(5) &= 5(5)^3(5 - 4) > 0.
\end{aligned}$$

So, $f(x)$ is increasing on $x < 0$ and $x > 4$ and decreasing on $0 < x < 4$.

(b) Find x-coordinates of all critical points of f and classify each of them as a relative maximum, a relative minimum, or neither.

Solution. Since $f'(x)$ exists for all x, the only critical numbers are where $f'(x) = 0$, i.e., $x = 0$ and $x = 4$. Since $f'(x) > 0$ to the left of $x = 0$ and $f'(x) < 0$ to the right of $x = 0$, the critical point where $x = 0$ is a relative maximum. Since $f'(x) < 0$ to the left of $x = 4$ and $f'(x) > 0$ to the right of $x = 4$, the critical point where $x = 4$ is a relative minimum.

(c) Find intervals on which the graph of f is concave up and concave down.

Solution. Since the first derivative of f is

$$f'(x) = 5x^4 - 20x^3,$$

the second derivative of f is

$$\begin{aligned} f''(x) &= 20x^3 - 60x^2 \\ &= 20x^2(x-3) \end{aligned}$$

The second derivative $f''(x)$ is continuous for all x and $f''(x) = 0$ for $x = 0$ and $x = 3$. These numbers divide the x axis into three intervals; namely $x < 0$, $0 < x < 3$, and $x > 3$. Evaluating $f''(x)$ at test numbers in each of these intervals (say, $x = -1$, $x = 1$, and $x = 4$, respectively), we find

$$\begin{aligned} f''(-1) &= 20(-1)^2(-1-3) < 0, \\ f''(1) &= 20(1)^2(1-3) < 0, \\ f''(4) &= 20(4)^2(4-3) > 0. \end{aligned}$$

Thus, the graph of $f(x)$ is concave down for $x < 0$ and $0 < x < 3$ and concave up for $x > 3$.

(d) Find x-coordinates of all inflection points of f.

Solution. Since the concavity does not change at $x = 0$, f(x) does not an inflection point at $x = 0$. Since the concavity changes from downward to upward at $x = 3$, the graph of $f(x)$ has an inflection point at $x = 3$.

$\boxed{\mathbf{3}}$ Let f be a function defined everywhere except $x = 1$ and whose derivative is given by

$$f'(x) = \frac{x^2 - 7x + 10}{x - 1}.$$

(a) Find all critical numbers of f and classify each of them as a relative maximum, a relative minimum, or neither.

Solution. Since
$$f'(x) = \frac{x^2 - 7x + 10}{x - 1} = \frac{(x-2)(x-5)}{x-1}$$
is continuous everywhere except $x = 1$ which is not in the domain of $f(x)$ and $f'(x) = 0$ when $x = 2$ or $x = 5$, the critical numbers are $x = 2$ and $x = 5$. The critical numbers 2 and 5 together with $x = 1$ divide the x axis into four open intervals; namely, $x < 1$, $1 < x < 2$, $2 < x < 5$, and $x > 5$. Evaluating $f'(x)$ at test numbers in each of these intervals (say, $x = 0$, $x = \frac{3}{2}$, $x = 3$, and $x = 6$, respectively), we find
$$\begin{aligned} f'(0) &= \frac{(0-2)(0-5)}{0-1} < 0, \\ f'\left(\frac{3}{2}\right) &= \frac{(\frac{3}{2}-2)(\frac{3}{2}-5)}{\frac{3}{2}-1} > 0, \\ f'(3) &= \frac{(3-2)(3-5)}{3-1} < 0, \\ f'(6) &= \frac{(6-2)(6-5)}{6-1} > 0. \end{aligned}$$
Since $f'(x) > 0$ to the left of $x = 2$ and $f'(x) < 0$ to the right of $x = 2$, the critical point where $x = 2$ is a relative maximum. Since $f'(x) < 0$ to the left of $x = 5$ and $f'(x) > 0$ to the right of $x = 5$, the critical point where $x = 5$ is a relative minimum.

(b) Find x-coordinates of all inflection points of f.

Solution. By the quotient rule,
$$\begin{aligned} f''(x) &= \frac{(x-1)\frac{d}{dx}(x^2 - 7x + 10) - (x^2 - 7x - 10)\frac{d}{dx}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(2x-7) - (x^2 - 7x + 10)(1)}{(x-1)^2} \\ &= \frac{2x^2 - 7x - 2x + 7 - x^2 + 7x - 10}{(x-1)^2} \\ &= \frac{x^2 - 2x - 3}{(x-1)^2} \\ &= \frac{(x+1)(x-3)}{(x-1)^2} \end{aligned}$$
Since $f''(x)$ is not continuous at $x = 1$ and $f''(x) = 0$ when $x = -1$ or $x = 3$, the x-axis is divided into four intervals; namely, $x < -1$, $-1 < x < 1$, $1 < x < 3$, and $x > 3$. Evaluating $f''(x)$ at test

numbers in each of these intervals (say, $x = -2$, $x = 0$, $x = 2$, and $x = 4$, respectively), we find

$$\begin{aligned} f'(-2) &= \frac{(-2+1)(-2-3)}{(-2-1)^2} > 0, \\ f'(0) &= \frac{(0+1)(0-3)}{(0-1)^2} < 0, \\ f'(2) &= \frac{(2+1)(2-3)}{(2-1)^2} < 0, \\ f'(4) &= \frac{(4+1)(4-3)}{(4-1)^2} > 0. \end{aligned}$$

Since the concavity changes from upward to downward at $x = -1$, the graph of $f(x)$ has an inflection point at $x = -1$. Since the concavity changes from downward to upward at $x = 3$, the graph of $f(x)$ has an inflection point at $x = 3$.

$\boxed{4}$ Find all vertical and horizontal asymptotes of the graph of the function $\dfrac{2x^2 + x - 3}{x^2 + x - 2}$.

Solution. Since the denominator of the given function can be factored into $(x-1)(x+2)$, it is 0 when $x = 1$ or $x = -2$. Since

$$\lim_{x \to 1-} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to 1-} \frac{(x-1)(2x+3)}{(x-1)(x+2)} = \lim_{x \to 1-} \frac{2x+3}{x+2} = \frac{5}{3}$$

and

$$\lim_{x \to 1+} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to 1+} \frac{(x-1)(2x+3)}{(x-1)(x+2)} = \lim_{x \to 1+} \frac{2x+3}{x+2} = \frac{5}{3},$$

$x = 1$ is not a vertical asymptote. Since

$$\lim_{x \to -2+} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to -2+} \frac{(x-1)(2x+3)}{(x-1)(x+2)} = \lim_{x \to -2+} \frac{2x+3}{x+2} = -\infty$$

$x = -2$ is a vertical asymptote of the given graph. Therefore $x = -2$ is the only vertical asymptote of the given function.

Since

$$\lim_{x \to +\infty} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to +\infty} \frac{2 + \frac{1}{x} - \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = 2$$

and

$$\lim_{x \to -\infty} \frac{x^2 + 2x - 3}{x^2 + x - 2} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x} - \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = 2$$

$y = 2$ is a horizontal asymptote of the given graph.

$\boxed{\mathbf{5}}$ Find the absolute maximum and absolute minimum (if any) of the function $f(x) = x^3 - 3x^2 - 9x + 10$ on the interval $-2 \leq x \leq 2$.

Solution. From the derivative

$$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$$

we see that the critical numbers are $x = -1$ and $x = 3$. Of these, only $x = -1$ lies in the interval $-2 < x < 2$. Compute $f(x)$ at the critical number $x = -1$ and at endpoints $x = -2$ and $x = 2$.

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) + 10 = 15 \\ f(-2) &= (-2)^3 - 3(-2)^2 - 9(-2) + 10 = 8 \\ f(2) &= (2)^3 - 3(2)^2 - 9(2) + 10 = -12 \end{aligned}$$

Compare these values to conclude that the absolute maximum of $f(x)$ on the interval $-1 \leq x \leq 3$ is $f(-1) = 15$ and the absolute minimum is $f(2) = -12$.

$\boxed{\mathbf{6}}$ Find the absolute maximum and absolute minimum (if any) of the function $f(x) = x^3 - 12x + 20$ on the interval $x \geq 0$.

Solution. Since $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$, $x = 2$ is the only critical number of $f(x)$ on $x \geq 0$. Since $f'(x) < 0$ for $0 < x < 2$ and $f'(x) > 0$ for $x > 2$, the graph of f is decreasing for $0 < x < 2$ and increasing for $x > 2$. It follows that $f(2) = (2)^3 - 12(2) + 20 = 4$ is the absolute minimum of f on the interval $x \geq 0$ and that there is no absolute maximum.