Math 108, Solution of Midterm Exam 2

1 List all values of x for which the following function $f(x)$ is not continuous. Explain the reason.

$$f(x) = \begin{cases} \dfrac{x^2+2x-3}{x^2-4x+3} & \text{if } x < 1 \\ 2x-4 & \text{if } 1 \le x < 3 \\ x^2-2x-3 & \text{if } x \ge 3 \end{cases}$$

Solution. Since $\dfrac{x^2+2x-3}{x^2-4x+3}$ is a rational function defined everywhere except at $x=1$ and $x=3$, and $2x-4$ and x^2-2x-3 are polynomials defined everywhere, $f(x)$ is continuous everywhere possibly except $x=1$ and at $x=3$. Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2+2x-3}{x^2-4x+3} = \lim_{x\to 1^-} \frac{(x-1)(x+3)}{(x-1)(x-3)} = \lim_{x\to 1^-} \frac{x+3}{x-3} = \frac{1+3}{1-3} = \frac{4}{-2} = -2$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-4) = 2(1)-4 = -2,$$

$\lim_{x\to 1} f(x) = -2$. Since $f(1) = 2(1)-4 = -2$, $f(x)$ is continuous at $x=1$.

For $x=3$,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (2x-4) = 2(3)-4 = 2$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^2-2x-3) = (3)^2-2(3)-3 = 0.$$

Thus $\lim_{x\to 3} f(x)$ does not exist, and so the function $f(x)$ is not continuous at $x=3$.

Hence, $f(x)$ is not continuous only at $x=3$.

2 The position at time t of a ball moving along a line is given by $s(t) = 4t^2 - 12t + 8$.

(a) Find the velocity of the ball.

Solution. The velocity is

$$\begin{aligned} v(t) = \frac{ds}{dt} &= \frac{d}{dt}\left[4t^2 - 12t + 8\right] \\ &= 4\frac{d}{dt}\left[t^2\right] - 12\frac{d}{dt}[t] + \frac{d}{dt}[8] \\ &= 4 \cdot 2t - 12 \\ &= 8t - 12 \end{aligned}$$

(b) Find the total distance traveled by the ball between $t = 0$ and $t = 4$.

Solution. Since $v(t) = 8t - 12$, the ball is either advancing or retreating, as described in the following table.

Interval	Sign of $v(t)$	From	To
$0 < x < \frac{3}{2}$	$-$	$s(0) = 8$	$s\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 8 = -1$
$\frac{3}{2} < x < 4$	$+$	$s\left(\frac{3}{2}\right) = -1$	$s(4) = 4(4)^2 - 12(4) + 8 = 24$

Thus, the ball moves backward from $s(0) = 8$ to $s\left(\frac{3}{2}\right) = -1$, then moves forward to $s(4) = 24$. Therefore, the total distance traveled by the ball is

$$D = |-1 - 8| + |24 - (-1)| = 34.$$

3 Find the relative rate of change of the function $f(t) = (t^3 - t + 2)(t^2 - 3t + 1)$ with respect to t when $t = 1$.

Solution. By the product rule, the derivative of the given function with respect to t is

$$\begin{aligned} f'(t) &= (t^3 - t + 2)\frac{d}{dt}\left[t^2 - 3t + 1\right] + (t^2 - 3t + 1)\frac{d}{dt}\left[t^3 - t + 2\right] \\ &= (t^3 - t + 2)(2t - 3) + (t^2 - 3t + 1)(3t^2 - 1) \end{aligned}$$

Thus the relative rate of change of $f(t)$ when $t = 1$ is

$$\begin{aligned} \frac{f'(1)}{f(1)} &= \frac{[(1)^3 - 1 + 2][2(1) - 3] + [(1)^2 - 3(1) + 1][3(1)^2 - 1]}{[(1)^3 - 1 + 2][(1)^2 - 3(1) + 1]} \\ &= \frac{[2][-1] + [-1][2]}{[2][-1]} \\ &= 2 \end{aligned}$$

4 Find all points on the graph of $\dfrac{x^2 - x + 1}{x^2 + 1}$ where the tangent line is horizontal.

Solution. By the quotient rule, the derivative of $f(x)$ is

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{x^2-x+1}{x^2+1}\right] \\ &= \frac{(x^2+1)\frac{d}{dx}\left[x^2-x+1\right]-(x^2-x+1)\frac{d}{dx}\left[x^2+1\right]}{(x^2+1)^2} \\ &= \frac{(x^2+1)(2x-1)-(x^2-x+1)(2x)}{(x^2+1)^2} \\ &= \frac{2x^3-x^2+2x-1-2x^3+2x^2-2x}{(x^2+1)^2} \\ &= \frac{x^2-1}{(x^2+1)^2}.\end{aligned}$$

Since the tangent line is horizontal exactly when its slope is 0,

$$\begin{aligned}f'(x) = \frac{x^2-1}{(x^2+1)^2} &= 0 \\ \frac{(x-1)(x+1)}{(x^2+1)^2} &= 0\end{aligned}$$

implies that the tangent line is horizontal at $(-1, f(-1)) = \left(-1, \frac{3}{2}\right)$ and $(1, f(1)) = \left(1, \frac{1}{2}\right)$.

5 Find the second derivative of the function $f(x) = 4\sqrt{x} + \dfrac{x^3-2}{x}$.

Solution. The first derivative is

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[4x^{1/2}+x^2-2x^{-1}\right] \\ &= 4\frac{d}{dx}\left[x^{1/2}\right]+\frac{d}{dx}\left[x^2\right]-2\frac{d}{dx}\left[x^{-1}\right] \\ &= 4\cdot\frac{1}{2}x^{-1/2}+2x^1-2(-1)x^{-2} \\ &= 2x^{-1/2}+2x++2x^{-2}\end{aligned}$$

and the second derivative is

$$\begin{aligned}f''(x) &= \frac{d}{dx}\left[2x^{-1/2}+2x+2x^{-2}\right] \\ &= 2\frac{d}{dx}\left[x^{-1/2}\right]+2\frac{d}{dx}\left[x\right]+2\frac{d}{dx}\left[x^{-2}\right] \\ &= 2\left(-\frac{1}{2}\right)x^{-3/2}+2(1)+2(-2)x^{-3} \\ &= -\frac{1}{x\sqrt{x}}+2-\frac{4}{x^3}\end{aligned}$$

6 Find the equation of the tangent line to the curve $y = (x^2+x+1)^4$ at the point where $x = -1$.

Solution. When $x = -1$, the corresponding y coordinate on the given curve is

$$[(-1)^2 + (-1) + 1]^4 = 1^4 = 1,$$

so the point of tangency is $(-1, 1)$.

Apply the general power rule to get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[(x^2+x+1)^4\right] \\ &= 4(x^2+x+1)^3 \frac{d}{dx}\left(x^2+x+1\right) \\ &= 4(x^2+x+1)^3(2x+1). \end{aligned}$$

Thus the slope of the tangent line to the given curve at the point $(-1, 1)$ is given by

$$\left.\frac{dy}{dx}\right|_{x=-1} = 4[(-1)^2 + (-1) + 1]^3[2(-1) + 1] = 4(1)^3(-1) = -4$$

Therefore the equation of the tangent line is

$$\begin{aligned} y - 1 &= -4(x - (-1)) \\ y &= -4x - 3 \end{aligned}$$