Math 108, Solution of Midterm Exam 1

$\boxed{1}$ Specify the domain of each of the following functions.

(a) $f(x) = \dfrac{x^2 - 4x + 3}{x^2 + x - 2}$

Solution. Since division by any nonzero number is possible, the domain of f is the set of all numbers satisfying $x^2 + x - 2 \neq 0$. Since $x^2 + x - 2 = (x-1)(x+2)$, the domain of f is the set of all real numbers x except $x = 1$ and $x = -2$.

(b) $g(x) = x^2 - 1 + \sqrt{4 - x^2}$

Solution. Since negative numbers do not have real square roots, $g(x)$ can be defined only when $4 - x^2 \geq 0$, i.e., $x^2 - 4 \leq 0$. Since $x^2 - 4 = (x-2)(x+2)$, the domain of g is the set of all real numbers x such that $-2 \leq x \leq 2$.

$\boxed{2}$ Suppose the total cost in dollars of manufacturing q units of a certain commodity is

$$C(q) = q^2 + 10q + 40.$$

Compute the cost of manufacturing the 10th unit.

Solution. The cost of manufacturing the 10th unit is the difference between the cost of manufacturing 10 units and the cost of manufacturing 9 units. That is,

$$\begin{aligned}
\text{Cost of 10th unit} &= C(10) - C(9) \\
&= \Big[(10)^2 + 10(10) + 40\Big] - \Big[(9)^2 + 10(9) + 40\Big] \\
&= 240 - 211 \\
&= \$29
\end{aligned}$$

$\boxed{3}$ Find all values of x such that $f(g(x)) = g(f(x))$, where $f(x) = x^2 - 2$ and $g(x) = 1 - x$.

Solution. Replace x by $g(x) = 1 - x$ in the formula for $f(x)$ to get

$$f(g(x)) = (1-x)^2 - 2 = 1 - 2x + x^2 - 2 = x^2 - 2x - 1.$$

Similarly, we replace x by $f(x) = x^2 + 2$ in the formula for $g(x)$ to get

$$g(f(x)) = 1 - (x^2 - 2) = 3 - x^2.$$

By solving the equation $f(g(x)) = g(f(x))$, we get

$$\begin{aligned} x^2 - 2x - 1 &= 3 - x^2 \\ 2x^2 - 2x - 4 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \end{aligned}$$

Thus $f(g(x)) = g(f(x))$ only when $x = -1$ or $x = 2$.

$\boxed{4}$ Find all x and y intercepts of the function

$$f(x) = \begin{cases} 2x + 6 & \text{if } x \le 1 \\ -x^2 + 4x + 5 & \text{if } x > 1 \end{cases}$$

Solution. Since $f(0) = 6$, the y intercept is $(0, 6)$.

To find the x intercepts, solve the equation $f(x) = 0$. Solving $2x + 6 = 0$, we get $x = -3$. Since $x = -3 \le 1$, $(-3, 0)$ is an x intercept. Next, we solve $-x^2 + 4x + 5 = 0$. Dividing by -1 and factoring, we find that

$$\begin{aligned} x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \end{aligned}$$

Since $x = 5 > 1$, $(5, 0)$ is another x intercept. But $(-1, 0)$ is not an x intercept because $x = -1 \not> 1$. Thus $f(x)$ has two x-intercepts $(-3, 0)$ and $(5, 0)$.

$\boxed{5}$ Find all points of intersection of the graphs of $f(x) = \dfrac{x+6}{x-2}$ and $g(x) = x + 1$.

Solution. You must solve the equation $f(x) = g(x)$. Since

$$\begin{aligned} \frac{x+6}{x-2} &= x + 1 \\ x + 6 &= x^2 - x - 2 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \end{aligned}$$

the solutions are $x = -2$ and $x = 4$. Computing the corresponding y coordinates from the equation $y = x + 1$, the points of intersections are $(-2, -1)$ and $(4, 5)$.

$\boxed{6}$ Find the slope-intercept form of the equation for each of the following lines:

(a) Line L_1 that passes through $(-1, 2)$ and $(2, -1)$

Solution. The slope is

$$m_1 = \frac{(-1) - 2}{2 - (-1)} = \frac{-3}{3} = -1.$$

Then use the point-slope formula with $(-1, 2)$ to get

$$y - 2 = -1[x - (-1)].$$

Thus the slope-intercept form of the equation of the given line is

$$y = -x + 1.$$

(b) Line L_2 that passes through $(1, 5)$ and is perpendicular to the line $L_3 : 2x + 4y = 3$.

Solution. By rewriting the equation $2x + 4y = 3$ in the slope-intercept form $y = -\frac{1}{2}x + \frac{3}{4}$, we see that the line L_3 has slope $m_3 = -\frac{1}{2}$. Thus the line L_2 has slope $m_2 = -\frac{1}{m_3} = 2$. Since the line L_2 contains $(1, 5)$, we have

$$\begin{aligned} y - 5 &= 2(x - 1) \\ y &= 2x + 3 \end{aligned}$$

$\boxed{7}$ Market research indicates that manufacturers will supply x units of a particular commodity to the marketplace when the price is $p = S(x)$ dollars per unit and that the same number of units will be demanded (bought) by consumers when the price is $p = D(x)$ dollars per unit, where the supply and demand functions are given by

$$S(x) = x^2 + 24 \qquad D(x) = 204 - 8x$$

At what level of production x is market equilibrium achieved?

Solution. Market equilibrium occurs when

$$\begin{aligned} S(x) &= D(x) \\ x^2 + 24 &= 204 - 8x \\ x^2 + 8x - 180 &= 0 \\ (x + 18)(x - 10) &= 0 \end{aligned}$$

Since only positive values of the production level x are meaningful, the market equilibrium is achieved when $x = 10$.

$\boxed{8}$ A manufacturer can sell tables for \$100 apiece. Total cost consists of a fixed overhead of \$3,000 plus production costs of \$40 apiece.

(a) How many tables must the manufacturer sell to break even?

Solution. If x is the number of tables manufactured and sold, the total revenue is given by $R(x) = 100x$ and the total cost by $C(x) = 3,000 + 40x$. To find the break-even point, set $R(x)$ equal to $C(x)$ and solve:

$$100x = 3,000 + 40x$$
$$60x = 3,000$$
$$x = 50$$

So, the manufacturer will have to sell 50 tables to break even.

(b) What is the manufacturer's profit or loss if 30 tables are sold?

Solution. The profit $P(x)$ is revenue minus cost. Thus,

$$P(x) = R(x) - C(x) = 100x - (3,000 + 40x) = 60x - 3,000$$

The profit from the sale of 30 tables is

$$P(30) = 60(30) - 3,000 = 1,800 - 3,000 = -1,200$$

This the manufacturer will lose \$1,200 if 30 tables are sold.

$\boxed{9}$ An open box with a square base is to have a volume of 200 cubic inches. The sides of the box will cost \$2 per square inch, and the base will cost \$3 per square inch. Express the construction cost of the box as a function of the length of its base.

Solution. Let x denote the length of the base, y the height of the box, and C the construction cost of the box. Since the volume of the box is 200 cubic inches, we have

$$\text{Volume of the box} = x^2y = 200$$

and

$$C = \text{Cost of base} + \text{Cost of sides}$$

Since the area of the base is x^2 and the cost per square inch of the base is \$3,

$$\text{Cost of base} = 3 \cdot x^2.$$

Since there are four sides with the area xy each and the cost per square inch of the sides is \$2,

$$\text{Cost of sides} = 4 \cdot 2 \cdot xy.$$

Thus the construction cost is

$$C = 3x^2 + 8xy.$$

Since $y = \dfrac{200}{x^2}$, the construction cost is

$$C(x) = 3x^2 + 8x\left(\frac{200}{x^2}\right) = 3x^2 + \frac{1600}{x}.$$

$\boxed{\textbf{10}}$ Find the following limit or show it does not exist. If the limit is infinite, indicate whether it is $+\infty$ or $-\infty$.

$$\lim_{x\to 2}\frac{3x^2-5x-2}{x^2-x-2}$$

Solution. As $x \to 2$, both the numerator and the denominator approach zero. Since the numerator and the denominator have a common factor $x-2$, we have

$$\begin{aligned}\lim_{x\to 2}\frac{3x^2-5x-2}{x^2-x-2} &= \lim_{x\to 2}\frac{(x-2)(3x+1)}{(x-2)(x+1)}\\ &= \lim_{x\to 2}\frac{3x+1}{x+1}\\ &= \frac{3\cdot 2+1}{2+1} = \frac{7}{3}.\end{aligned}$$

$\boxed{\textbf{11}}$ Find each of the following limits or show it does not exist. If the limit is infinite, indicate whether it is $+\infty$ or $-\infty$.

(a) $\displaystyle\lim_{x\to+\infty}\frac{x^3-x+2}{x^3-x-2}$

Solution. The highest power in the denominator is x^3. Divide the numerator and the demonimator by x^3 and use the reciprocal power rules and algebraic properties of limits to get

$$\begin{aligned}\lim_{x\to+\infty}\frac{x^3-x+2}{x^3-x-2} &= \lim_{x\to+\infty}\frac{1-\dfrac{1}{x^2}+\dfrac{2}{x^3}}{1-\dfrac{1}{x^2}-\dfrac{2}{x^3}}\\ &= \frac{\displaystyle\lim_{x\to+\infty}1-\lim_{x\to+\infty}\frac{1}{x^2}+\lim_{x\to+\infty}\frac{2}{x^3}}{\displaystyle\lim_{x\to+\infty}1-\lim_{x\to+\infty}\frac{1}{x^2}-\lim_{x\to+\infty}\frac{2}{x^3}}\\ &= \frac{1-0+0}{1-0-0}\\ &= 1\end{aligned}$$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^3 - x + 2}{x^2 - x - 2}$

Solution. The highest power in the denominator is x^2. Divide the numerator and the demonimator by x^2 to get

$$\lim_{x\to-\infty} \frac{x^3 - x + 2}{x^2 - x - 2} = \lim_{x\to-\infty} \frac{x - 1/x + 2/x^2}{1 - 1/x - 2/x^2}$$

Since

$$\lim_{x\to-\infty} \left(x - 1/x + 2/x^2\right) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} \left(1 - 1/x - 2/x^2\right) = 1$$

it follows that

$$\lim_{x\to-\infty} \frac{x^3 - x + 2}{x^2 - x - 2} = -\infty$$