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Solution of Quiz 5

$\boxed{1}$ The first derivative of a certain function $f(x)$ is given by

$$f'(x) = x^3(2x-5)^2.$$

(a) Find intervals on which f is increasing and decreasing.

Solution. The derivative of f is continuous everywhere with $f'(x) = 0$ where $x = 0$ and $x = \frac{5}{2}$. The numbers 0 and $\frac{5}{2}$ divide the x axis into three open intervals; namely, $x < 0$, $0 < x < \frac{5}{2}$, and $x > \frac{5}{2}$. Choose a test number from each of these intervals; say $c = -1$ from $x < 0$, $c = 1$ from $0 < x < \frac{5}{2}$, and $c = 3$ from $x > \frac{5}{2}$. Then, evaluate $f'(c)$ for each test number:

$$\begin{aligned} f'(-1) &= (-1)^3[2(-1)-5]^2 < 0, \\ f'(1) &= (1)^3[2(1)-5]^2 > 0, \\ f'(3) &= (3)^3[2(3)-5]^2 > 0. \end{aligned}$$

So, $f(x)$ is decreasing on $x < 0$, and increasing on $0 < x < \frac{5}{2}$ and $x > \frac{5}{2}$.

(b) Determine values of x for which relative maxima and minima occur on the graph of $f(x)$.

Solution. Since $f'(x) < 0$ to the left of $x = 0$ and $f'(x) > 0$ to the right of $x = 0$, the critical point where $x = 0$ is a relative minimum. Since $f'(x)$ has the same sign (plus) on both sides of $x = \frac{5}{2}$, the critical point corresponding to $x = \frac{5}{2}$ is not a relative extremum.

(c) Find intervals on which the graph of f is concave up and concave down.

Solution. The second derivative of f is

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[x^3(2x-5)^2\right] \\ &= x^3\frac{d}{dx}(2x-5)^2 + (2x-5)^2\frac{d}{dx}(x^3) \\ &= x^3(2)(2x-5)(2) + (2x-5)^2(3)x^2 \\ &= x^2(2x-5)(4x+6x-15) \\ &= x^2(2x-5)(10x-15) \\ &= 5x^2(2x-5)(2x-3) \end{aligned}$$

The second derivative $f''(x)$ is continuous for all x and $f''(x) = 0$ for $x = 0, x = \frac{3}{2}$ and $x = \frac{5}{2}$. These numbers divide the x axis into four intervals; namely $x < 0$, $0 < x < \frac{3}{2}$, $\frac{3}{2} < x < \frac{5}{2}$, and $x > \frac{5}{2}$. Evaluating $f''(x)$ at test numbers in each of these intervals (say, $x = -1$, $x = 1$, $x = 2$, and $x = 3$, respectively), we find

$$\begin{aligned} f''(-1) &= 5(-1)^2[2(-1)-5][2(-1)-3] > 0, \\ f''(1) &= 5(1)^2[2(1)-5][2(1)-3] > 0, \\ f''(2) &= 5(2)^2[2(2)-5][2(2)-3] < 0, \\ f''(3) &= 5(3)^2[2(3)-5][2(3)-3] > 0. \end{aligned}$$

Thus, the graph of $f(x)$ is concave up for $x < 0$, $0 < x < \frac{3}{2}$, and $x > \frac{5}{2}$, and concave down for $\frac{3}{2} < x < \frac{5}{2}$.

(d) Find x-coordinates of all inflection points of f.

Solution. Since the concavity does not change at $x = 0$, $f(x)$ does not an inflection point at $x = 0$. Since the concavity changes from upward to downward at $x = \frac{3}{2}$, the graph of $f(x)$ has an inflection point at $x = \frac{3}{2}$. Since the concavity changes from downward to upward at $x = \frac{5}{2}$, the graph of $f(x)$ has an inflection point at $x = \frac{5}{2}$.