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Solution of Quiz 5

$\boxed{1}$ Let f be a function defined by $f(x) = x^5 - 5x^4 - 3x + 2$.

(a) Find intervals on which the graph of f is concave up and concave down.

Solution. The first derivative of f is

$$f'(x) = 5x^4 - 20x^3 - 3$$

and the second derivative of f is

$$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3).$$

The second derivative $f''(x)$ is continuous for all x and $f''(x) = 0$ for $x = 0$ and $x = 3$. These numbers divide the x axis into three intervals; namely $x < 0$, $0 < x < 3$, and $x > 3$. Evaluating $f''(x)$ at test numbers in each of these intervals (say, $x = -1$, $x = 1$, and $x = 4$, respectively), we find

$$f''(-1) = 20(-1)^2(-1-3) < 0, \quad f''(1) = 20(1)^2(1-3) < 0, \quad f''(4) = 20(4)^2(4-3) > 0.$$

Thus, the graph of $f(x)$ is concave down for $x < 0$ and for $0 < x < 3$ and concave up for $x > 3$.

(b) Find x-coordinates of all inflection points of f.

Solution. Since the concavity does not change at $x = 0$, $(0, f(0))$ is not an inflection point. Since the concavity changes from downward to upward at $x = 3$, the graph of $f(x)$ has an inflection point at $x = 3$.

$\boxed{2}$ Find all vertical and horizontal asymptotes of the graph of the function $\dfrac{x^2+2x-3}{x^2-1}$.

Solution. The denominator of the given function is 0 when $x = 1$ and $x = -1$. Since

$$\lim_{x\to -1-} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to -1-} \frac{(x+3)(x-1)}{(x+1)(x-1)} = +\infty$$

$x = -1$ is a vertical asymptote of the given graph. Since

$$\lim_{x\to 1-} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1-} \frac{(x+3)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1-} \frac{x+3}{x+1} = \frac{4}{2} = 2$$

and

$$\lim_{x\to 1+} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1+} \frac{(x+3)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1+} \frac{x+3}{x+1} = \frac{4}{2} = 2,$$

$x = 1$ is not a vertical asymptote. Therefore $x = -1$ is the only vertical asymptote of the given function.
Since

$$\lim_{x\to +\infty} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to +\infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2}}{1 - \frac{1}{x^2}} = 1 \quad \text{and} \quad \lim_{x\to -\infty} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to -\infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2}}{1 - \frac{1}{x^2}} = 1$$

$y = 1$ is a horizontal asymptote of the given graph.