Lecturer: Sangwook Kim
Office : Science & Tech I, 226D
math.gmu.edu/~skim22

Solution of Quiz 4

$\boxed{\mathbf{1}}$ Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = x^2 + y$.

Solution. Differentiating both sides of the given equation with respect to x, we get

$$\begin{aligned} \frac{d}{dx}\left[x^3 + y^3\right] &= \frac{d}{dx}\left[x^2 + y\right] \\ 3x^2 + 3y^2\frac{dy}{dx} &= 2x + 1\frac{dy}{dx} \\ (3y^2 - 1)\frac{dy}{dx} &= 2x - 3x^2 \\ \frac{dy}{dx} &= \frac{2x - 3x^2}{3y^2 - 1}. \end{aligned}$$

$\boxed{\mathbf{2}}$ Let f be a function defined by $f(x) = x^4 - 4x^3 + 2$.

(a) Find intervals on which f is increasing and decreasing.

Solution. The derivative of f is

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

which is continuous everywhere, with $f'(x) = 0$ where $x = 0$ and $x = 3$. The numbers 0 and 3 divide the x axis into three open intervals; namely, $x < 0$, $0 < x < 3$, and $x > 3$. Choose a test number from each of these intervals; say $c = -1$ from $x < 0$, $c = 2$ from $0 < x < 3$, and $c = 4$ from $x > 3$. Then, evaluate $f'(c)$ for each test number:

$$f'(-1) = 4(-1)^2(-1 - 3) < 0, \quad f'(1) = 4(1)^2(1 - 3) < 0, \quad f'(4) = 4(4)^2(4 - 3) > 0.$$

So, $f(x)$ is increasing on $x > 3$ and decreasing on $x < 0$ and $0 < x < 3$.

(b) Determine the critical numbers of f and classify each critical points as a relative maximum, a relative minimum, or neither.

Solution. Since $f'(x) = 4x^2(x - 3)$ exists for all x, the only critical numbers are where $f'(x) = 0$, that is, $x = 0$ and $x = 3$. Corresponding critical points are $(0, 2)$ and $(3, -25)$. Since $f'(x)$ has the same sign (minus) on both sides of $x = 0$, the critical point $(0, 2)$ is not a relative extremum. Since $f'(x) < 0$ to the left of $x = 3$ and $f'(x) > 0$ to the right of $x = 3$, the critical point $(3, -25)$ is a relative minimum.