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Review for Final Exam

Chapter 1. Functions, Graphs, and Limits

- A *function* is a rule that assigns to *each* objects in a set A *exactly one* object in a set B. The set A is called the *domain* of the function. The set of assigned abjects in B is called the *range*.
- **(Domain Convention)** If a formula (or several formulas) is used to define a function f, then we assume that the domain of f to be the set of all numbers for which $f(x)$ is defined.
- A *demand function* $p = D(x)$ is a function that relates the unit price p for a particular commodity to the number of units x demanded by consumers at that price. The *total revenue* is

  $$\begin{aligned} R(x) &= (\text{number of items sold})(\text{price per item}) \\ &= xp = xD(x) \end{aligned}$$

  If $C(x)$ is the *total cost* of producing the x units, then the *profit* derived from their sale is

  $$P(x) = R(x) - C(x) = xD(x) - C(x).$$

- Given functions $f(u)$ and $g(x)$, the *composition* $f(g(x))$ is the function of x formed by substituting $u = g(x)$ for u in the formula for $f(u)$.
- The *graph* of a function f consists of all points (x, y) where x is in the domain of f and $y = f(x)$; that is, all points of the form $(x, f(x))$.
- The points (if any) where a graph crosses the x axis are called x *intercepts*, and a y *intercept* is a point where the graph crosses the y axis.
- A *power function* is a function of the form $f(x) = x^n$.
- A *polynomial* is a function of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.
- A quotient $\dfrac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ is called a *rational function*.
- A *linear function* is a function that changes at a constant rate with respect to its independent variable.
- The *slope* of the nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is the ratio

  $$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **(The Slope-Intercept Form)** The equation

  $$y = mx + b$$

  is the equation of the line whose slope is m and whose y intercept is $(0, b)$.
- **(The Point-Slope Form)** The equation

  $$y - y_0 = m(x - x_0)$$

  is the equation of the line that passes through the point (x_0, y_0) and that has slope equal to m.

- Let m_1 and m_2 be the slopes of the nonvertical lines L_1 and L_2. Then
  - L_1 and L_2 are *parallel* if and only if $m_1 = m_2$.
  - L_1 and L_2 are *perpendicular* if and only if $m_2 = \dfrac{-1}{m_1}$.
- The quantity Q is said to be:
  - *directly proportional* to x if $Q = kx$ for some constant k
  - *inversely proportional* to x if $Q = k/x$ for some constant k
  - *jointly proportional* to x and y if $Q = kxy$ for some constant k
- *Demand function* $D(x)$: the unit price at which all x units are demanded (sold) in the marketplace. *Supply function* $S(x)$: the unit price at which producers are willing to supply x units to the marketplace.
- **(The Law of supply and demand)** In a competitive market environment, supply tends to equal demand. When this occurs, the market is said to be in *equilibrium*. The corresponding unit price is called the *equilibrium price*. When the market is not in equilibrium, it has a *shortage* when demand exceeds supply and a *surplus* when supply exceeds demand.
- The point at which the total revenue curve and the total cost curve cross is called the *break-even point*.
- If $p(x)$ and $q(x)$ are polynomials, then

  $$\lim_{x \to c} p(x) = p(c)$$

  and

  $$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \text{ if } q(c) \neq 0$$

- **(Reciprocal Power Rules)** If A and k are constants with $k > 0$ and x^k is defined for all x, then

  $$\lim_{x \to +\infty} \frac{A}{x^k} = 0 \text{ and } \lim_{x \to -\infty} \frac{A}{x^k} = 0$$

- **Procedure for evaluating a limit at infinity of** $f(x) = \dfrac{p(x)}{q(x)}$
  1. Divide each terms in $f(x)$ by the highest power x^k in the denominator.
  2. Compute the limit using algebraic properties of limits and the reciprocal power rules.
- **(Existence of a Limit)** The two-sided limit $\lim_{x \to c} f(x)$ exists if and only if the two one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist and are equal, and then

  $$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

- A function f is *continuous* at c if all three of these conditions are satisfied:

  1. $f(c)$ is defined
  2. $\lim_{x \to c} f(x)$ exists
  3. $\lim_{x \to c} f(x) = f(c)$

  If $f(x)$ is not continuous at c, it is said to have a *discontinuity* there.

- A polynomial or a rational function is continuous *wherever it is defined.*

- A function $f(x)$ is said to be *continuous on an open interval* $a < x < b$ if it is continuous at each point $x = c$ in that interval. Moreover, f is *continuous on the closed interval* $a \le x \le b$ if it is continuous on the open interval $a < x < b$ and

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b)$$

Sample questions (from Review Problems)
(9th edition) 1, 2, 3a, 3b, 4a, 4c, 6, 9a, 9b, 10a, 10d, 10e, 11, 14, 15, 17, 19, 21, 24, 26, 31, 33, 35, 37, 39, 41, 43, 45, 47, 48
(10th edition) 1, 3, 6a, 6c, 11, 13a, 13c, 14c, 15, 16, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 38, 39, 42, 43, 45, 49, 52, 54, 58

Chapter 2. Differentiation: Basic Concepts

- The *derivative* of the function $f(x)$ with respect to x is the function $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

  The process of computing the derivative is called *differentiation*, and we say that $f(x)$ is *differentiable* at $x = c$ if $f'(c)$ exists.

- The *slope of the tangent line* to the curve $y = f(x)$ at the point $(c, f(c))$ is $f'(c)$.

- The *rate of change* of $f(x)$ with respect to x when $x = c$ is given by $f'(c)$.

- If the function f is differentiable at $x = c$, then

  - f is *increasing* at $x = c$ if $f'(c) > 0$,
  - f is *decreasing* at $x = c$ if $f'(c) < 0$.

- The derivative $f'(x)$ of $y = f(x)$ is sometimes written as $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$.

  In this notation, $f'(c)$ is written as $\left.\dfrac{dy}{dx}\right|_{x=c}$ or $\left.\dfrac{df}{dx}\right|_{x=c}$.

- **(The Constant Rule)** $\frac{d}{dx}[c] = 0$
- **(The Power Rule)** $\frac{d}{dx}[x^n] = nx^{n-1}$
- **(The Constant Multiple Rule)** $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
- **(The Sum Rule)** $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
- The *relative rate of change* of a quantity $Q(x)$ with respect to x is $\frac{Q'(x)}{Q(x)}$.

  The corresponding *percentage rate of change* of $Q(x)$ with respect to x is $\frac{100Q'(x)}{Q(x)}$.
- Motion of an object along a line is called *rectilinear motion*.
  - If the *position* at time t of an object moving along a straight line is given by $s(t)$, then the object has *velocity* $v(t) = s'(t) = \frac{dx}{dt}$ and *acceleration* $a(t) = v'(t) = \frac{dv}{dt}$.
  - The object is *advancing* when $v(t) > 0$, *retreating* when $v(t) < 0$, and *stationary* when $v(t) = 0$.
  - It is *accelerating* when $a(t) > 0$ and *decelerating* when $a(t) < 0$.
- **(The Product Rule)** $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$
- **(The Quotient Rule)** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g^2(x)}$
- The *nth derivative* of a function $f(x)$ is obtained from $f(x)$ by differentiating successively n times, and is denoted by $\frac{d^n y}{dx^n}$ or $f^{(n)}(x)$.
- **(The Chain Rule)** $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or, equivalently, $\frac{dy}{dx} = f'(g(x))g'(x)$.
- **(The General Power Rule)** $\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1}\frac{d}{dx}[h(x)]$
- **(Implicit Differentiation)** Suppose an equation defines y implicitly as a differentiable function of x. To find the derivative of y,
  1. Differentiate both sides of the equation with respect to x. Remember that y is really a *function of x* and use the chain rule when differentiating terms containing y.
  2. Solve the differentiated equation algebraically for $\frac{dy}{dx}$.

Sample questions (from Review Problems)
(9th edition) 3, 5, 7, 9, 11, 13, 15, 17, 18b, 18c, 19a, 19c, 20a, 21a, 22a, 22c, 22e, 23c, 23d, 24b, 24c, 27, 28
(10th edition) 3, 5, 7, 9, 11, 13, 15, 17, 18b, 19a, 20a, 22a, 25a, 26a, 26c, 27c, 28b, 29b, 30b, 31a, 35, 36

Chapter 3. Additional applications of the derivative

- If $f'(x) > 0$ on an interval I, $f(x)$ is *increasing* on I.
  If $f'(x) < 0$ on an interval I, $f(x)$ is *decreasing* on I.
- A number c *in the domain of* $f(x)$ is called a *critical number* if either $f'(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a *critical point* for $f(x)$.
- (*The First Derivative Test for Relative Extrema*) Let c be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is
  - A *relative maximum* if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
  - A *relative minimum* if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c.
  - *Not a relative extremum* if $f'(x)$ has the same sign on both sides of c.
- f is *concave upward* on an interval I if f' is increasing on I, i.e., $f''(x) > 0$.
  f is *concave downward* on an interval I if f' is decreasing on I, i.e., $f''(x) < 0$.
- An *inflection point* is a point $(c, f(c))$ on the graph of f where the concavity changes.
- (*The Second Derivative Test*) Suppose $f''(x)$ exists on an open interval containing $x = c$ and that $f'(c) = 0$.
  - If $f''(c) > 0$, then f has a relative minimum at $x = c$.
  - If $f''(c) < 0$, then f has a relative maximum at $x = c$.

  However, *if* $f''(c) = 0$ *or if* $f''(c)$ *does not exist, the test is inconclusive* and f may have a relative maximum, a relative minimum, or no relative extremum at all at $x = c$.
- The vertical line $x = c$ is a *vertical asymptote* of the graph of $f(x)$ if either

$$\lim_{x\to c^-} f(x) = +\infty \quad (\text{or } -\infty) \quad \text{or} \quad \lim_{x\to c^+} f(x) = +\infty \quad (\text{or } -\infty)$$

- The horizontal line $y = b$ is a *horizontal asymptote* of the graph of $f(x)$ if

$$\lim_{x\to -\infty} f(x) = b \quad \text{or} \quad \lim_{x\to +\infty} f(x) = b$$

- *General Procedure for Sketching the Graph*
  1. Find the domain of $f(x)$.
  2. Find and plot all intercepts.
  3. Determine all vertical and horizontal asymptotes and draw them.
  4. Find $f'(x)$ and determine the critical numbers and intervals of increase and decrease.
  5. Determine all relative extrema and plot them.
  6. Find $f''(x)$ and determine intervals of concavity and points of inflection. Plot inflection points.
  7. Complete the sketch by joining the plotted points.

- (*The Extreme Value Property*) A function $f(x)$ that is continuous on the closed interval $a \leq x \leq b$ attains its absolute extrema on the interval either at an *endpoint* of the interval (a or b) or at a *critical number* c such that $a < c < b$.

- (*The Second Derivative Test for Absolute Extrema*) Suppose that $f(x)$ is continuous on I where $x = c$ is the only critical number and that $f'(c) = 0$. Then
  - if $f''(c) > 0$, the absolute minimum of $f(x)$ on I is $f(c)$.
  - if $f''(c) < 0$, the absolute maximum of $f(x)$ on I is $f(c)$.

Sample questions (from Review Problems)
(9th edition) 1, 5, 7, 9, 13, 15, 25, 27, 28, 29
(10th edition) 1, 5, 7, 9, 13, 15, 25, 27, 28, 29

Chapter 4. Exponential and logarithmic functions

- *Exponential function*: $f(x) = b^x$ for $b > 0$ and $b \neq 1$
- *Properties of exponential functions*
  - The *equality rule*: $b^x = b^y$ if and only if $x = y$
  - The *product rule*: $b^x b^y = b^{x+y}$
  - The *quotient rule*: $\dfrac{b^x}{b^y} = b^{x-y}$
  - The *power rule*: $(b^x)^y = b^{xy}$
  - The *multiplication rule*: $(ab)^x = a^x b^x$
  - The *division rule*: $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- *Logarithmic functions*: $y = \log_b x$ if and only if $b^y = x$
- *Properties of logarithms*
  - The *equality rule*: $\log_b u = \log_b v$ if and only if $u = v$
  - The *product rule*: $\log_b uv = \log_b u + \log_b v$
  - The *quotient rule*: $\log_b \dfrac{u}{v} = \log_b u - \log_b v$
  - The *power rule*: $\log_b u^r = r \log_b u$
  - *Special values*: $log_b 1 = 0$ and $\log_b b = 1$
- The *natural exponential base* e is the number defined by the limit

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n$$

  and its value is $e = 2.71828\ldots$

- *The natural logarithm*: $y = \ln x$ if and only if $e^y = x$
- *Properties of natural logarithms*
  - The *equality rule*: $\ln u = \ln v$ if and only if $u = v$
  - The *product rule*: $\ln(uv) = \ln u + \ln v$
  - The *quotient rule*: $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$
  - The *power rule*: $\ln u^r = r \ln u$
  - *Special values*: $ln 1 = 0$ and $\ln e = 1$
- *Inverse relationship*: $e^{\ln x} = x$ for $x > 0$ and $\ln e^x = x$ for all x
- Derivatives of exponential functions: $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}[e^{u(x)}] = e^{u(x)}u'(x)$
- Derivatives of exponential functions: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}[\ln(u(x)] = \frac{u'(x)}{u(x)}$
- (*Logarithmic Differentiation*) Differentiating a function that involves products, quotients, or powers can often be simplified by first *taking the logarithm of the function.*

Sample questions (from Review Problems)
(9th edition) 5, 7, 9, 11, 13, 15, 17, 23, 29, 31, 35, 37, 39, 41, 43
(10th edition) 5, 7, 9, 11, 13, 15, 17, 23, 29, 31, 35, 37, 39, 41, 43