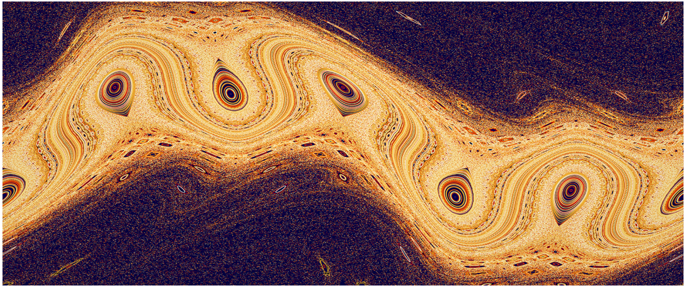
Introduction to Nonlinear Dynamics and Chaos



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Outline

**Picture on title page by George Miloshevich, Physics graduate student at UT. The image won third place in UT's annual "Visualizing Science" competition, 2016.*

Here are some concepts I'd like to introduce today:

- 1 - Nonlinearity and deterministic chaos
- 2 - Fixed points and stable/unstable equilibrium
- 3 - Sensitivity to initial conditions
- 4 - The utility of computers in understanding mathematics

To motivate our study of the logistic map, let’s observe a real world example of a dynamical system transitioning from orderly, predictable behavior to chaotic behavior.

Fluid Transition to Turbulence

The following video shows the behavior of fluid flow in a pipe as it transitions from an orderly, predictable flow (called *laminar* flow) to a chaotic, seemingly random *turbulent* flow.

▸ Fluid transition to turbulence

Simple Dynamical Systems

Exponential Population Growth

Imagine a bug species with the property that the population n_{t+1} in year $t + 1$ is uniquely determined by the population n_t in the preceding year t.

$$n_{t+1} = f(n_t)$$

Let's call this model a growth equation.

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Exponential Population Growth

A simple population model is the *exponential model*.

Let r be some fixed number, and let

$$n_{t+1} = r\,n_t.$$

Problem 1: *Let the population in year zero equal ten, so that* $n_0 = 10$*, and let* $r = 2.0$*. What will the population in year five be?*

$$n_5 = ?$$

Problem 2: *Let the population in year zero equal forty-eight, so that* $n_0 = 48$*, and let* $r = 0.5$*. What will the population in year four be?*

$$n_4 = ?$$

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Exponential Population Model

Answer to P1: 320
Answer to P2: 3

$$n_{t+1} = r\, n_t$$

Problem 3: *Let n_0 be the initial population of the bug species. Can you find the pattern and say what the population is at $t = 100$?**

$$n_1 = r\, n_0 \quad (1)$$
$$n_2 = \,? \quad (2)$$
$$\dots \quad (3)$$
$$n_{100} = \,? \quad (4)$$

(*if you have previously seen mathematical induction, you can prove this)

Exponential Population Model

Answer to P3:

$$n_{100} = r\, n_t \quad (5)$$
$$= r^{100}\, n_0 \quad (6)$$

In general,

$$n_{t+1} = r^{t+1} n_0$$

Problem 4: *Find conditions on r such that, when $t \to \infty$, (i) the population grows and takes over the world, (ii) becomes extinct, (iii) remains constant.*

Logistic Map

- The exponential population model with $r > 1$ can be realistic for the initial growth of many populations, but of course no real population can grow forever. Eventually the growth must slow (due to, e.g., overcrowding or shortage of food).
- A simple modification is to add an extra term to the model. Fix $N > 0$, and consider

$$n_{t+1} = r\, n_t(1 - n_t/N)$$

We call this the *logistic map*.

Logistic Map

$$n_{t+1} = r\,n_t(1 - n_t/N)$$

Features of the logistic map:

- Mimics dynamics of exponential model when

$$n_t/N \ll 1 \iff n_t \ll N \iff n_t/N \approx 0$$

- Ensures population never grows larger than N. If $n_t = N$, then

$$n_{t+1} = r\,n_t\,\overbrace{(1 - n_t/N)}^{=0} = 0$$

- It is *nonlinear*:

$$n_{t+1} = r\,n_t - (r/N)\,\underbrace{n_t^2}$$

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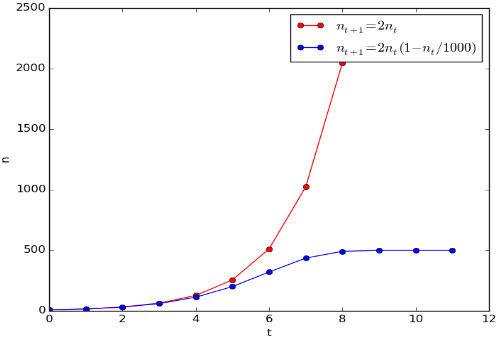
The Logistic Map Example

Let $n_0 = 4$, $r = 2$, and $N = 1000$. Compare the dynamics of the exponential model and the logistic model :

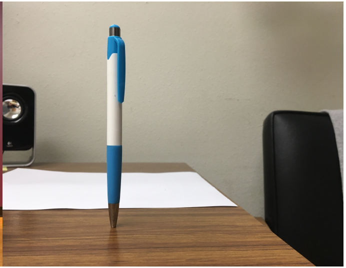


Equilibrium

In the previous figure, the population of the logistic model seemed to level out, and become constant after a few years.

We say the population has approached an *equilibrium*.

An interesting question we might ask is: when is a population equilibrium *stable*? When is it *unstable*?



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Relative Population

To study the stability of the population models, we focus on the *relative population*:

$$x = n/N.$$

Relative population intuitively measures the *percentage* of population that is alive:

$$\frac{\text{population that is alive}}{\text{maximum possible population}}$$

Let's divide both sides of the original equation by N.

Original equation for **total population**:

$$n_{t+1} = r\,n_t\,(1 - n_t/N)$$

New, rescaled equation for **relative population**:

$$x_{t+1} = f(x_t) = r\,x_t(1 - x_t)$$

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Relative Population

$$x_{t+1} = f(x_t) = r\,x_t(1 - x_t)$$

Recall: in the logistic model, the total population could *NOT grow larger than* N:

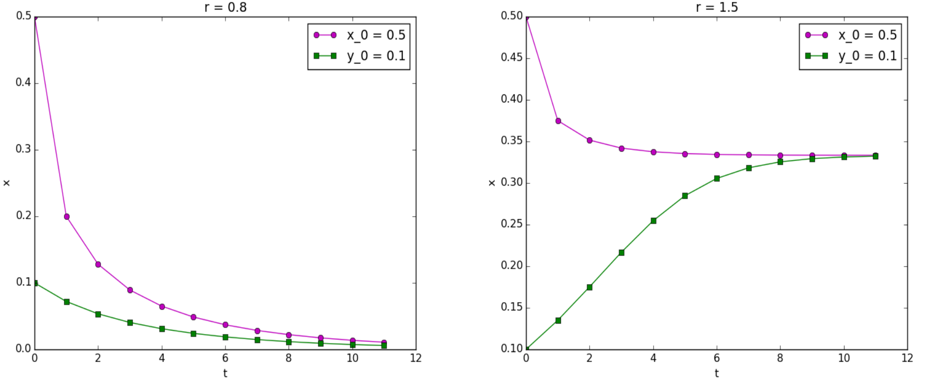
$$0 \leq n_t \leq N$$

Since $x_t = n_t/N$, this implies:

$$0 \leq x_t \leq 1.$$

Relative Population Examples

In the first example, $r = 0.8$, and for both initial conditions the population dies out. In the second example, $r = 1.5$, and an equilibrium of ≈ 0.33 is reached.



Logistic Map

To study the behavior of the logistic map, we will think now of the continuous version:

$$f(x) = r\,x(1-x).$$

In order to ensure $0 \leq x \leq 1$, we need to *consider only*

$$0 \leq r \leq 4$$

(if $r > 4$, then we can easily get $x > 1$).

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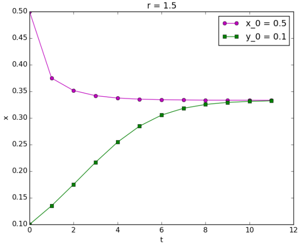
Fixed Points

For the previous example of $r = 1.5$,



the population approaches an equilibrium of $x \approx 0.33$.

Fixed Points

If a population is in equilibrium, then

- it does not change from year to year
- $x_{t+1} = x_t = x_{t-1} = x_{t-2} \ldots$
- $x_{t+1} = f(x_t)$.

We define any point x^* such that

$$x^* = f(x^*)$$

to be a *fixed point*.

Once a population hits a fixed point, it stays there *for all time*.

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Fixed Points of the Logistic Map

For the logistic map, a fixed point x^* must satisfy

$$x^* = f(x^*) \iff x^* = r\,x^*(1 - x^*).$$

Problem 5: *Find all of the fixed points* x^* *of the logistic map.*

Fixed Points of the Logistic Map

Answer to P5: the two fixed points are

$$x^* = 0 \qquad \text{and} \qquad x^* = \frac{r-1}{r}.$$

Note that when $r < 1$, this means $x^* < 0$, which is nonsensical (population cannot be negative). So, for $r < 1$, there is only one fixed point.

Stability of Fixed Points

By definition, if hit a fixed point, we will remain there *for all time*:

$$x^* = f(x^*).$$

What if, however, we find ourselves *very close to* a fixed point?

$$x_t = x^* + \epsilon_t$$

What happens at $x_{t+1} = f(x^* + \epsilon_t)$? If a fixed point is **stable**, then we will quickly return to the fixed point

$$x_{t+1} \to x^* \iff \epsilon_{t+1} \to 0.$$

If the fixed point is **unstable**, then x_{t+1} will move away from the fixed point x^*

$$x_{t+1} \not\to x^* \iff \epsilon_{t+1} \not\to 0.$$

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Stability of Fixed Points

$$\epsilon_t = x_t - x^*$$

We can predict exactly when this will happen*:

$$x_{t+1} = f(x_t) = f(x^* + \epsilon_t) \quad (7)$$
$$\approx f(x^*) + \lambda\,\epsilon_t = x^* + \lambda\,\epsilon_t \quad (8)$$

Now subtract x^* from both sides:

$$x_{t+1} - x^* \approx \lambda\,\epsilon_t$$

$$\implies \epsilon_{t+1} \approx \lambda\,\epsilon_t$$

(*warning: calculus required to fully follow the argument)

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Stability of Fixed Points

$$\epsilon_{t+1} = \lambda \, \epsilon_t$$

This looks like the exponential population model from earlier!

From **Problem 4**, we know that if $|\lambda| < 1$, then $\epsilon_t \to 0$ and the fixed point is stable.

If $|\lambda| > 1$, however, then $\epsilon_t \to \infty$ and our fixed point is unstable.

Stability of Fixed Points

For the logistic map,

$$\lambda = r\left(1 - 2x^*\right).$$

Recall the two fixed points we found were $x^* = 0$ and $x^* = (r-1)/r$.

Problem 6: *When $x^* = 0$, determine the values of r that give us $|\lambda| < 1$ and $|\lambda| > 1$.*

Problem 7: *When $x^* = (r-1)/r$, determine the values of r that give us $|\lambda| < 1$ and $|\lambda| > 1$.*

Remember: we only consider $0 \leq r \leq 4$.

Stability of Fixed Points

Answer to P6: When $r < 1$, the fixed point $x^* = 0$ is stable. When $r > 1$, the fixed point is unstable.

Answer to P7: When $1 < r < 3$, the fixed point $x^* = (r-1)/r$ is stable. When $r > 3$ it is unstable.

Fixed Point Diagram

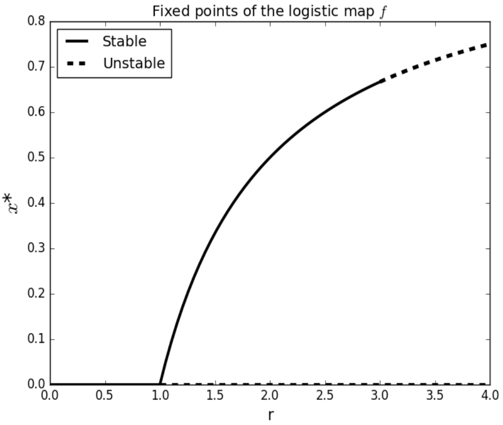
There is a convenient way to visually summarize this analysis. Recall that

$$x^* = 0 \qquad \text{and} \qquad x^* = \frac{r-1}{r}.$$

Problem 8: *Make a plot of x^* versus r. When x^* is stable, use a solid line. When x^* is unstable, use a dashed line.*

Fixed Diagram

Answer to P8:



What Happens when $r > 3$?

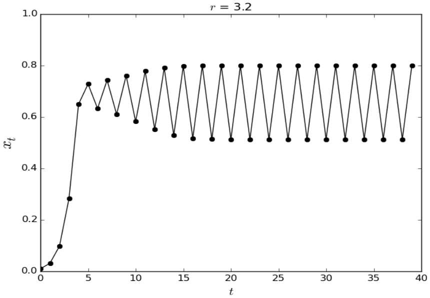
The previous diagram tells the whole story of fixed points of the logistic map. It does not tell, however, what happens when $r > 3$. We know then that the fixed point is *unstable*, but how can we expect the logistic map to behave in this case?

It's best to consider specific examples *computationally* (i.e. using a computer).

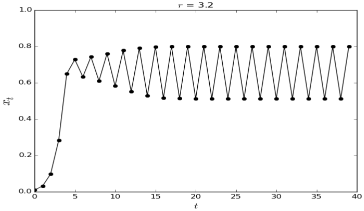
Logistic Map Example–Period Doubling

Consider $r = 3.2$ and $x_0 = 0.01$. Notice that

- $r > 3$
- initial point $x_0 = 0.01$ is near fixed point $x^* = 0$
- x_t quickly moves away from $x^* = 0$
- after some initial behavior (called *transients*), the behavior of x_t is periodic with period 2.



Period Doubling



- For $r < 3$, the system evolved towards a single fixed point x^*
- For $r = 3.2$, the system eventually oscillates between two points x_a and x_b, where

$$f(x_a) = x_b \quad \text{and} \quad f(x_b) = x_a$$

Period Doubling

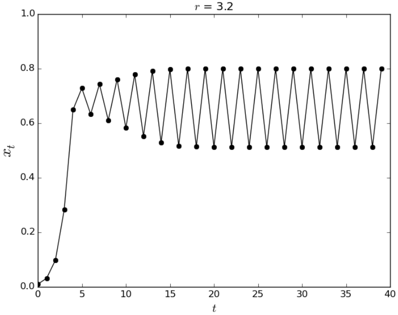
- x_a and x_b are not fixed points of f
- Instead they are fixed points of the *double map*
- Let $g(x) = f(f(x))$:
- **Claim**: x_a and x_b are fixed points of the double map
- **Proof**:

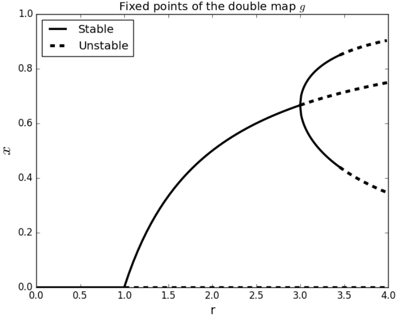
$$g(x_a) = f(f(x_a)) = f(x_b) = x_a$$

- Similar proof holds for x_b

Bifurcation Diagram

Recall the fixed point diagram from earlier. We can make the same diagram for the double map $g(x) = f(f(x))$:





- Notice the two separate fixed points at $r = 3.2$
- They correspond to the two cycle of the logistic map f

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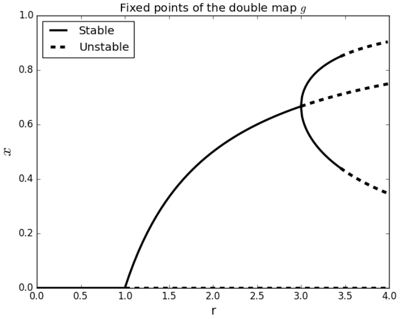
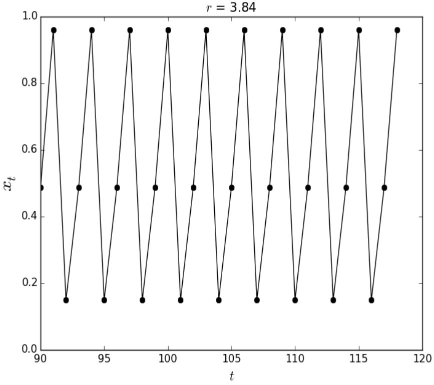
Bifurcation Diagram

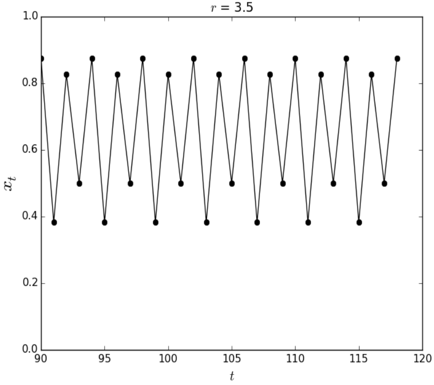


- For the double map, as one fixed point becomes unstable, two more stable ones appear
- We say that the map *bifurcates*
- If we look more closely, we can find *a lot* more bifurcations

More Logistic Map Examples

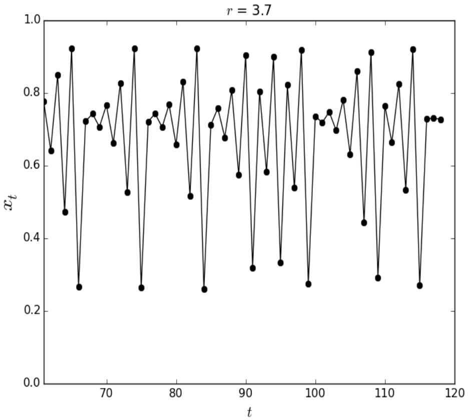
For $r = 3.84$ and $r = 3.5$ we get periodic behavior again, with period 3 and 4, respectively.





More Logistic Map Examples

For $r = 3.7$, however, we *do not* get periodic behavior. Instead, we get *chaos*:



The map *never* repeats itself in a periodic manner.

Bifurcations

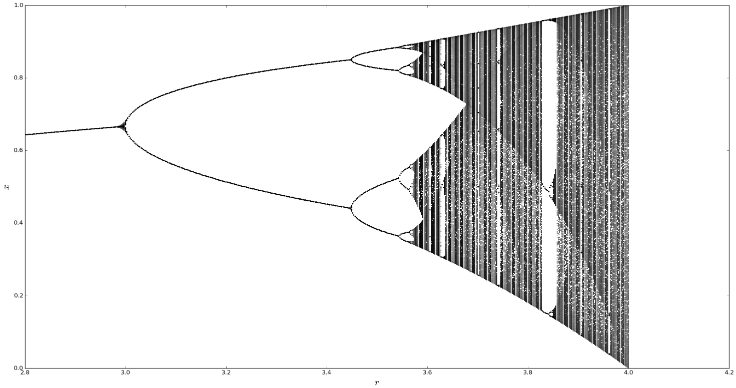
- It turns out *chaotic* behavior is more common than *periodic*, orderly behavior
- With a great bit of help from a computer, we can visualize this nicely . . .

Bifurcation Diagram

- We plot x_t versus the parameter r
- For each r, we start the map at $x_0 = 0.1$
- We let the map run for $t = 200$ time steps
- Then, we start plotting x_t versus r for 1400 more time steps
- (note: there will be **a lot** of points plotted for each r)

Bifurcation Diagram

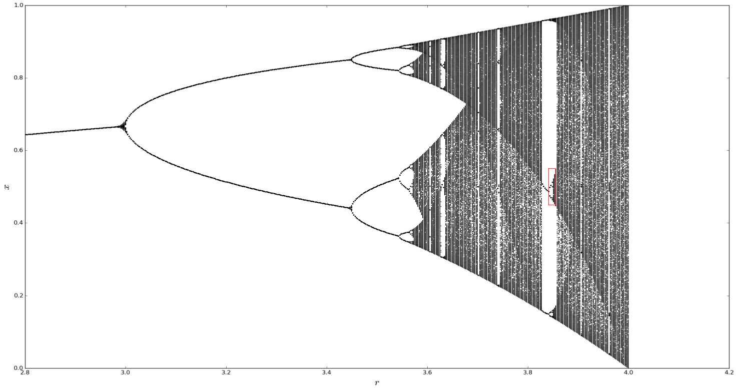
We get a remarkable picture!



What happens if we zoom in one particular section?

Bifurcation Diagram

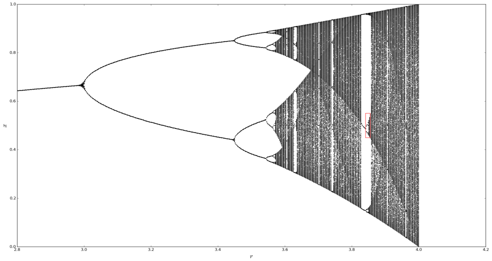
We get a remarkable picture!

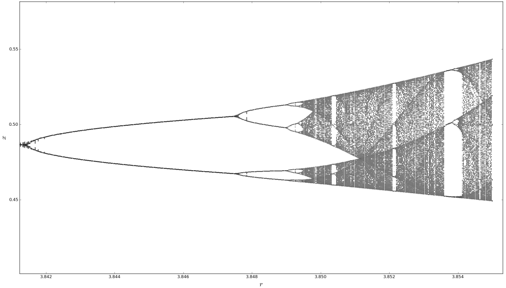


What happens if we zoom in one particular section?

Bifurcation Diagram

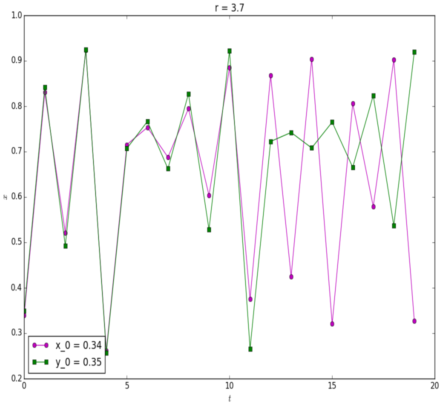
The plot exhibits *self-similarity*.

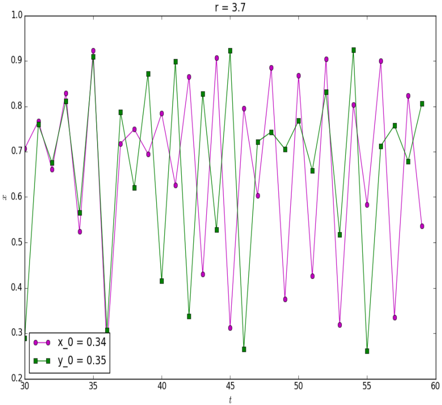




Chaos – Sensitivity to Initial Conditions

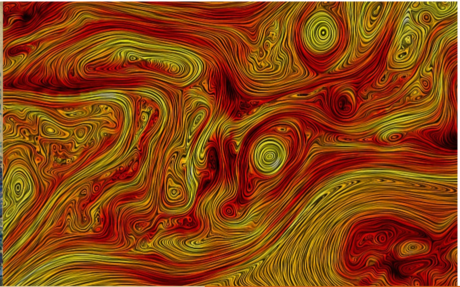
Another defining characteristic of chaotic behavior is *sensitivity to initial conditions*. Consider two different plots of the logistic map, again with $r = 3.7$, with slightly different initial conditions $x_0 = 0.34$ and $y_0 = 0.35$:





More Examples of Chaos

- There are many, many more examples of chaotic dynamical systems
- One example from everyday life is the phenomenon of *fluid turbulence*.
- The behavior of fluids exhibits some of the same features as the logistic map:
  - sensitivity to initial conditions
  - unstable equilibrium
  - seemingly random behavior from deterministic rules
  - beautiful pictures



Fluid Turbulence

Fluids behave according to a complicated set of *partial differential equations* (PDEs) called the *Navier-Stokes equations.*

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\,u) + \nabla p - \left(\frac{1}{Re}\right)\Delta u = 0 \tag{9}$$

$$\nabla \cdot u = 0 \tag{10}$$

- u is a velocity field–it meaures how fast and in what direction a fluid is moving at point in time and space
- p measures pressure at a point in time and space
- **important:** the equation is *nonlinear* (notice the $u\,u$ term)
- Re is a *constant*, and measures how turbulent a fluid is
- Similar to the parameter r in the logistic map, for different values of Re a fluid can behave quite differently

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Transition to Turbulence–Unstable Equilibrium

Recall the video from earlier showing the flow of fluid in a pipe as Re increases. We'll see orderly, predictable flow (called *laminar* flow) transition to chaotic, seemingly random *turbulent* flow.

▸ Fluid transition to turbulence

This is similar to the logistic map behaving predictably for $r < 3$ and chaotically for $r > 3$. Unfortunately, for the more complicated Navier-Stokes system, it is much more difficult to predict in general the value of Re for which this transition occurs.

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Fluid Turbulence

Here is a simulation of fluid convection (called *Rayleigh-Benard convection*)–warm fluid at the bottom rises and mixes with cold fluid at the top and creates a beautiful, turbulent mess.

▸ Rayleigh-Benard Convection

Turbulence–$1 Million Problem

- Despite its ubiquity, turbulence is an unsolved problem in mathematics
- The Clay Institute offers a $1 million dollar prize to anyone who can prove whether the Navier-Stokes equations possess unique, smooth solutions for all time
- ▸ Clay Institute Millennium Problem

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\,u) + \nabla p - \left(\frac{1}{Re}\right)\Delta u = 0 \tag{11}$$

$$\nabla \cdot u = 0 \tag{12}$$

Maybe *you* can solve the problem and claim the prize.

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Thanks for listening!

Questions?