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Period-doubling cascades galore

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Abstract. The appearance of numerous period-doubling cascades is among the most prominent features of *parametrized maps*, that is, smooth one-parameter families of maps $F : \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$, where $\mathfrak{M}$ is a smooth locally compact manifold without boundary, typically $\mathbb{R}^N$. Each cascade has infinitely many period-doubling bifurcations, and it is typical to observe that, whenever there are any cascades, there are infinitely many cascades. We develop a general theory of cascades for generic F.

1. *Introduction*

A major goal in dynamical systems is to better explain what is seen. There are many reports of numerically observed (period-doubling) cascades in the contexts of maps, ordinary differential equations and partial differential equations, and even in physical experiments (cf. [**24**]). In each of these contexts, whenever one cascade is seen, an infinite number are observed. There is also considerable numerical evidence that cascades occur as a dynamical system becomes more chaotic. The concept of cascades is usually associated with the orderly picture of the attractor structure of a quadratic map depicted in Figure 1. The structure of quadratic maps is well understood, and it is not hard in this context to show rigorously that cascades exist. However, most dynamical systems have bifurcation structures that are a good deal more complicated than that of a quadratic map (cf. the double-well Duffing equation in Figure 2), implying that the simple explanation for cascades of the quadratic map does not generalize. In this paper, we develop a series of general criteria for the existence of cascades in the context of parametrized maps. We explain why cascades occur with infinite multiplicity. Our results give a rigorous explanation for the link between cascades and chaos. The method of approach lays a general framework for understanding cascades, even for observable systems for which the underlying model equation is unknown. Our approach is very different from the celebrated approach to cascades using scaling and renormalization theory, and this allows us to come to new conclusions connecting cascades and chaos for maps and differential equations.

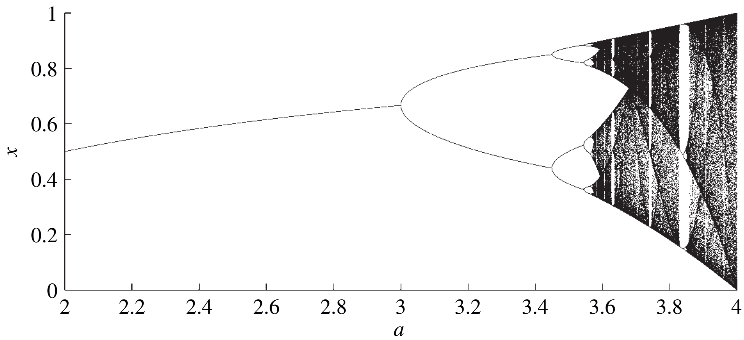

FIGURE 1. The attracting set for the logistic map: $F(a, x) = ax(1 - x)$. That is, for each fixed parameter value, the attracting set in [0, 1] is shown. There are infinitely many cascades of attractors. This is the bifurcation diagram most frequently displayed to illustrate the phenomenon of period-doubling cascades. However, cascades occur for much more complex dynamical systems that are completely unrelated to quadratic maps.

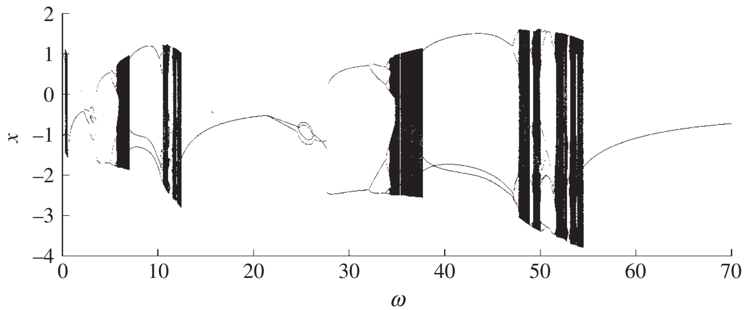

FIGURE 2. The attracting set for the double-well Duffing equation: $u''(t) + 0.3u'(t) - u(t) + (u(t))^3 + 0.01 = \omega \sin t$. This equation is periodically forced with period 2π. Therefore the time-2π map is a diffeomorphism on $\mathbb{R}^2$ parametrized by ω. Depicted here is the attracting set of F, projected to the $(\omega, u(t))$-plane. The constant 0.01 has been added to destroy symmetry in order to avoid non-generic symmetry-breaking bifurcations.

We give a more detailed comparison to previous theoretical work on cascades at the end of this introduction. We establish the connection between cascades and chaos as a concrete mathematical object. Namely, we establish a connecting arc of periodic orbits between each cascade and a point in the chaotic region. In a second paper, we will give a series of examples for which these methods rigorously show the existence of cascades. These examples include large-scale perturbations of families of polynomials, large coupled quadratic systems, high-dimensional maps with horseshoes, as well as flows with well-defined Poincaré sections exhibiting chaos such as the double-well Duffing pictured in Figure 2.

We now give a heuristic description of our results. We call a period-k orbit of a one-parameter family $F : \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$ a *flip* orbit if its Jacobian matrix $D_x F^k(\lambda, x)$ has an odd number of eigenvalues less than -1, and -1 is not an eigenvalue. Otherwise the orbit is *non-flip*. We denote the space of non-flip orbits of F in $\mathbb{R} \times \mathfrak{M}$ under the Hausdorff metric by $\mathrm{PO}_{\text{non-flip}}(F)$. We restrict our attention to a specific residual set (which will be precisely defined) of the C^∞ maps $F : \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$. We call maps in this residual set *generic*. All our results are for generic maps.

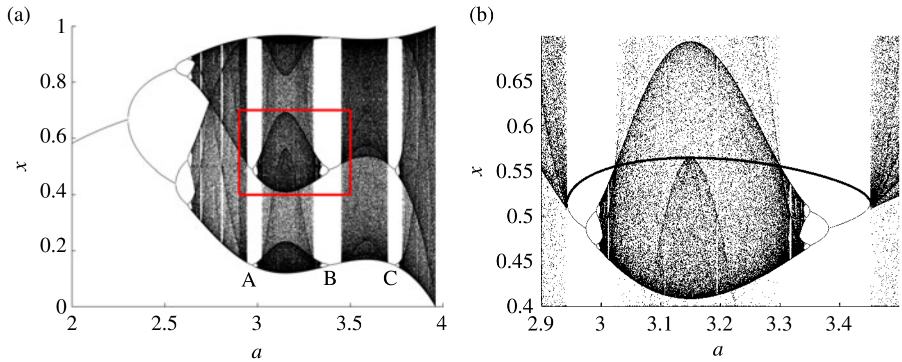

FIGURE 3. Bounded cascades come in pairs. The numerically computed attracting set for a modified logistic map $F(a, x) = h(a)x(1 - x)$, where $h(a) = a(1.18 + 0.17\cos(2.4a))$. Here h is not monotonic. As the parameter a increases, $h(a)$ increases, then decreases, and again increases. Hence $h(a)$ passes three times over the region where the logistic map has a period-three cascade. Panel (a) displays three 'windows' (the parameter ranges labeled A, B, and C) of period-doubling sequences starting from period-three orbits. When parameter a yields a one-piece chaotic attractor, the upper edge of the chaotic set is the image of $x = 1/2$, corresponding to $h(a)/4$, the maximum value of $F(a, \cdot)$. Panel (b) shows a blowup of the box in (a). We have added a curve of unstable non-flip period-three points revealing that the two cascades are in the same component. Hence we see three period-three cascades, revealing a phenomenon that often happens in much more complicated systems. Namely, that bounded cascades can be created or destroyed in *pairs that are in the same component*.

Components of $\mathrm{PO}_{\text{non-flip}}(F)$ *are one-manifolds for generic* F. For a specific residual set of F, which we call *generic* F, we show in Theorem 1 (M_1) that all connected components of $\mathrm{PO}_{\text{non-flip}}(F)$ are one-manifolds; that is, they are homeomorphic either to circles or to open intervals. Furthermore, the ratio of the periods of two orbits in a component of $\mathrm{PO}_{\text{non-flip}}(F)$ is always a power of 2. This result may seem counterintuitive to those familiar with cases where periodic points are dense in a space. But our result is about orbits, not points. This result is one of the keys to understanding cascades. From now on, we use the term *component* to denote a connected component of $\mathrm{PO}_{\text{non-flip}}(F)$. When a component is homeomorphic to an open interval, we call it an *open arc*.

Definition of cascade. Cascades were first reported by Myrberg [**21**], and later by May [**18**]. We define a *cascade* as the following type of subarc of an open arc A: let k denote the smallest period of the orbits in A. A cascade is a half-open subarc that contains orbits with all of the periods $k, 2k, 4k, 8k, \ldots$ such that it contains precisely one orbit of period k. We refer to such a cascade as a *period-k cascade*. A cascade is homeomorphic to $[0, 1)$ where 0 maps to the single period-k orbit.

We call an open arc A *bounded* if there is a compact subset of $\mathbb{R} \times \mathfrak{M}$ that contains all the orbits of A. Otherwise A is *unbounded*. If a cascade is contained in an unbounded open arc, we call it *unbounded*; otherwise we call it *bounded*. We show that if a component is a bounded open arc, then it always contains two disjoint cascades, and we refer to these as *paired cascades*. See Figure 3.

The orbit index. An essential tool for studying cascades is the *orbit index*, a topological index of a periodic orbit taking on a value in $\{0, -1, +1\}$. A periodic orbit in $\mathrm{PO}_{\text{non-flip}}(F)$, has an orbit index of either -1 or $+1$. In Theorem 1 (M_2), we state and prove that each component has a preferred orientation that can be determined at each hyperbolic orbit by computing its orbit index.

Section 4 contains Theorem 2, establishing existence of cascades in a bounded parameter region based on a small amount of information about the types of periodic orbits on the boundary. The result is established by using the preferred orientation via the orbit index to show that an open arc in a bounded parameter region is bounded and therefore contains a cascade.

The literature on general existence of cascades is scant. Once a cascade is known to exist, it can be understood using scaling and renormalization theory found in the work from the early 1980s of Feigenbaum [**8**] and others [**1–7, 10–12, 16**]. Recent results include [**14, 15**]. The goal of these methods is to show that, when a period-one cascade exists for a parametrized map that is nearly quadratic, the procession of period-doubling values has a regular scaling behavior. In contrast, our goal is to establish criteria for the existence of infinitely many cascades. Our methods are topological and say nothing about scaling. For a much more detailed comparison between our results and those given by scaling theory, see [**24**, §6].

In the theory of one-dimensional quadratic maps $F(\lambda, x)$, the existence of cascades became a folk theorem based on the property of *monotonicity*: namely, that, as the parameter increases, new orbits can appear but no orbits are destroyed. This monotonicity was originally proved by Douady and Hubbard in the complex analytic setting. See [**19**] for a proof. When there is monotonicity, the existence of cascades is quite straightforward. Monotonicity of the quadratic map implies that the only possible periodic-orbit bifurcations are those saddle-node and period-doubling bifurcations in which periodic attractors are created. Periodic attractors do not persist as attractors as the parameter λ increases because there are no periodic attractors for $\lambda \geq 2$; in order for a periodic attractor to become unstable as λ increases, it must undergo a period-doubling bifurcation, creating a new periodic attractor with double the period. This new periodic attractor must cease to exist before $\lambda = 2$, so the new higher period attractor undergoes a period doubling as well, etc. The attractors must undergo infinitely many period-doubling bifurcations as λ increases. Hence a cascade exists.

General one-dimensional systems need not be monotonic, and higher-dimensional systems tend never to be monotonic as shown in [**13**]. In these cases there are many more possible bifurcations, which both create and destroy orbits. See Figure 3. However, our results demonstrate that the existence of cascades is in no way dependent on either dimension one, nor on monotonicity, nor on having attractors.

Yorke and Alligood [**25**] discuss in detail a case where the cascade of period doublings involves attractors, and so they restrict attention to the case of systems where trajectories are at most one-dimensionally unstable. We make no such restriction. In higher dimensions, there is no reason for attractors to be present in cascades, and attractorless cascades do exist.

The paper proceeds as follows. In §2, we classify the set of generic bifurcations of orbits. In §3, we develop the orbit index, in order to investigate the 'index orientation' on each component. Section 4 contains our main result on cascades in Theorem 2.

2. *Components in the space of orbits*

This section introduces formal versions of the main concepts described in the introduction, including the space of orbits under the Hausdorff metric, the set of flip orbits, the set of non-flip orbits, and cascades viewed as subsets of this space. Throughout this paper we will assume the following context. To avoid technicalities, we follow a C^∞ approach similar to Milnor's treatment of Sard's theorem [**20**], proving our results for C^∞ rather than for C^r for a specific r.

HYPOTHESIS 1. (The setting) *Let $F : \mathbb{R} \times \mathfrak{M} \to \mathfrak{M}$ be C^∞-smooth. We refer to it as a parametrized map on $\mathfrak{M}$.*

Definition 1. (Orbits, flip orbits, and non-flip orbits) Write $[x]$ for the orbit of the periodic point x. In this paper, *orbit* always means *periodic orbit*. By *period* of an orbit or point, we mean its least period. If x is a periodic point for $F(\lambda, \cdot)$, then we sometimes say $\sigma = (\lambda, x)$ is a periodic point and write $[\sigma]$ or $(\lambda, [x])$ for its orbit. Let $\sigma = (\lambda, x)$ be a periodic point of period p of a smooth map $G = F(\lambda, \cdot)$. We refer to the *eigenvalues of σ or $[\sigma]$* as shorthand for the eigenvalues of Jacobian matrix $DG^p(x)$. Of course, all the points of an orbit have the same eigenvalues. We say that $[\sigma]$ is *hyperbolic* if none of its eigenvalues have absolute value 1. We say it is a *flip orbit* if the number of its eigenvalues (adding multiplicities) less than -1 is odd, and -1 is not an eigenvalue. We call all other orbits *non-flip orbits*. Define

$$\mathrm{PO}(F) = \{[\sigma] : [\sigma] \text{ is an orbit for } F\}$$

and

$$\mathrm{PO}_{\text{non-flip}}(F) = \{[\sigma] \in \mathrm{PO}(F) : [\sigma] \text{ is a non-flip orbit for } F\}.$$

The distance between two orbits in the space $\mathrm{PO}(F)$ or $\mathrm{PO}_{\text{non-flip}}(F)$ is defined using the Hausdorff metric. We say that two orbits are close in $\mathfrak{M}$ if every point of each orbit is close to some point of the other orbit. The periods of the two orbits need not be the same. This is made precise in the following definition.

Definition 2. (Hausdorff metric on sets) For a compact set S of $\mathfrak{M}$ and $\epsilon > 0$, let $B(\epsilon, S)$ be the closed ϵ neighborhood of S. Let S_1 and S_2 be compact subsets of $\mathfrak{M}$. (We are only interested in the case where these sets are orbits.) Assume ϵ is chosen as small as possible such that $S_1 \subset B(\epsilon, S_2)$ and $S_2 \subset B(\epsilon, S_1)$. Then the *Hausdorff distance* $\mathrm{dist}(S_1, S_2)$ between S_1 and S_2 is defined to be ϵ.

Let $\sigma_j = (\lambda_j, x_j)$ for $j = 1, 2$ be orbits. We define the distance between $[\sigma_1]$ and $[\sigma_2]$ to be

$$\mathrm{dist}([\sigma_1], [\sigma_2]) = \mathrm{dist}([x_1], [x_2]) + |\lambda_1 - \lambda_2|.$$

For example, if $(\lambda, [x(\lambda)])$ is a family of period-$2p$ orbits that bifurcate from the period-p orbit $(\lambda_*, [x_*])$ due to a period-doubling bifurcation, then

$$\mathrm{dist}((\lambda, [x(\lambda)]), (\lambda_*, [x_*])) \to 0 \quad \text{as } \lambda \to \lambda_*.$$

Remark 1. Generally one expects periodic points to be dense in a compact chaotic set. Using the Hausdorff metric changes the geometry. Every $\mathbb{R} \times \mathfrak{M}$ neighborhood of a saddle fixed point (λ, x) with a transverse homoclinic intersection has infinitely many periodic points y_m (of arbitrarily large period). However, since x is a hyperbolic saddle point, some points in the orbit of each y_m are far from x. Thus the orbits $[y_m]$ do not converge to $[x]$ in the Hausdorff metric.

Definition 3. (Cascade of period m) The term *component* means a connected component of $\mathrm{PO}_{\text{non-flip}}(F)$ in the Hausdorff metric. An *arc* is a set that is homeomorphic to an interval. We call it an *open arc* if the interval is open or *half-open* if that describes the interval.

A *(period-doubling) cascade of period m* is a half-open arc C in $\mathrm{PO}_{\text{non-flip}}(F)$ with the following properties. Let $h : [0, 1) \to C$ be a homeomorphism.

(i) The set of periods of orbits in C is $m, 2m, 4m, 8m, \dots$.

(ii) The number m is the minimum period of orbits in the component that contains C.

(iii) C has no proper connected subset with properties (i) and (ii). Note that $h(0)$ will be the only orbit of period m in C and it will be a period-doubling bifurcation orbit.

If a component contains a cascade, we refer to the component as a *cascade component*.

Remark 2. The cascades we discuss in this paper each lie in a compact subset of $\mathbb{R} \times \mathfrak{M}$ (though the cascade's component may be unbounded).

For generic F, we will show in Proposition 3 that such cascades have the following additional property.

(iv) If $\{p_k\}_1^\infty$ is the sequence of periods of the orbits, ordered so that for each k the $k + 1$ orbit lies 'between' (using the ordering induced from $[0, 1)$) the k orbit and the $k + 2$ orbit, then no period will occur more than a finite number of times. That implies

$$\lim_{k\to\infty} p_k = \infty.$$

Examples of arcs and a cascade. There is a simple example of a subset of the orbits in $\mathrm{PO}_{\text{non-flip}}(F)$ that is homeomorphic to an interval. Let $F = \lambda - x^2$. The smallest λ for which there is an orbit is $-1/4$, and that is a saddle-node fixed point $Q = (-1/4, -1/2)$. There is a unique periodic attractor for each $\lambda \in J = (-1/4, \lambda_{\text{Feig}})$ where λ_{Feig} is the end of the first cascade. We often refer to λ_{Feig} as the 'Feigenbaum limit parameter'. The attractors for $\lambda \in J$ constitute a component C of the attractors in $\mathrm{PO}_{\text{non-flip}}(F)$. For each $\lambda \in J$ there is a unique attracting orbit x in C, and trivially for each orbit in C there is a unique λ in J. In fact the map on C defined by

$$(\lambda, [x]) \mapsto \lambda$$

is a homeomorphism. Therefore C is an open arc since it is homeomorphic to the interval J. We shall see below that C is not maximal.

The cascade. There is a subarc $C_1 \subset C$ that is a cascade. For $\lambda = 3/4$ the orbit in C is the period-doubling fixed point $Q_1 = (3/4, 1/2)$. For $\lambda > 3/4$, the orbits in C have period greater than 1. Let $C_1 = \{$the orbits in C for which $\lambda \geq 3/4\}$. Then the cascade C_1 is the smallest subarc of C for which all periods 2^k occur.

The component containing C. The arc C is not a maximal arc. There is another arc, an arc of unstable (derivative $>+1$) fixed points $(\lambda, y(\lambda))$ defined for $\lambda \in (-1/4, +\infty)$, where $y(\lambda) = (-1 + \sqrt{1 + 4\lambda})/2$. The two arcs terminate at the saddle-node fixed point Q. Taking the union of the two arcs plus Q yields the component containing C. It is maximal because on one extreme the period goes to ∞ and on the other extreme, $\lambda \to \infty$. Each point of the arc is a different orbit. Since the set of λ values in the maximal arc is unbounded, the cascade is an unbounded cascade.

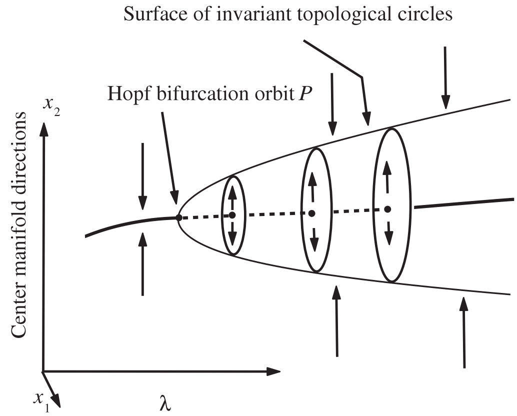

FIGURE 4. The Hopf bifurcation: all the local invariant sets of a generic Hopf bifurcation lie on a three-dimensional center manifold. There is a two-dimensional surface consisting of invariant topological circles as well as the arc of periodic points from which it bifurcates. As indicated by vertical arrows, in the case shown here the invariant surface is attracting (when the dynamics are restricted to the center manifold). There also exist generic Hopf bifurcations for which the arrows are all reversed. In this depiction, the surface appears as λ increases, but it could also occur as λ decreases.

In contrast to this straightforward example for quadratic maps, in which there is exactly one saddle-node bifurcation, and exactly one period-doubling bifurcation for each period 2^k, a cascade is generally quite complicated. For example, a bounded component that is a open arc contains two disjoint cascades (one on each end). See Figure 3. Quadratic maps have a *monotonicity* property described in the introduction [**19**]; if an orbit exists at λ_*, then it exists for all $\lambda > \lambda_*$. This leads to cascades that consist only of forward-directed period doublings of attractors. That is, the period always doubles as the parameter increases. In contrast, a cascade for a general map generally does not have such regular behavior, nor does it consist only of attractors. In what follows we will be interested in arcs of orbits with no regard to whether the orbits are attractors.

We now explain what is meant by a generic orbit bifurcation and show that there is a residual set of F for which every orbit is either hyperbolic or is a generic bifurcation orbit (Proposition 1 below).

2.1. *Generic orbit bifurcations.* We list the three kinds of generic orbit bifurcations, largely following the treatment in Robinson [**23**]. Cases (i) and (ii) are depicted in Figure 5. Figures 4 and 6–8 depict case (iii).

Definition 4. (Generic orbit bifurcations) Let F satisfy Hypothesis 1. We say that a *bifurcation orbit P of F is generic* if it is one of the following three types (as described in detail by Robinson [**23**]):

(i) a generic saddle-node bifurcation (having eigenvalue $+1$);
(ii) a generic period-doubling bifurcation (having eigenvalue -1);
(iii) a generic Hopf bifurcation with complex conjugate eigenvalues that are not roots of unity.

We say that *F is generic* if each non-hyperbolic orbit is one of the above three types.

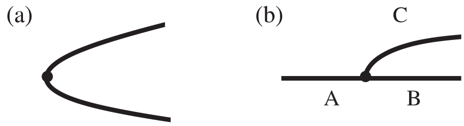

FIGURE 5. A depiction of a sufficiently small neighborhood of (a) the saddle-node bifurcation and (b) the period-doubling bifurcation. The horizontal axis is the parameter, but it can be either increasing or decreasing in this figure. The vertical axis is the space $\mathrm{PO}(F)$, so each orbit is depicted as a single point. Near the bifurcation point in (a) all orbits are either flip orbits or all orbits are non-flip orbits. Near the bifurcation point in (b), exactly one of segments A and B consists of flip orbits, and the other consists of non-flip orbits. The period-doubled segment C always consists of non-flip orbits. Hence exactly two of the segments of a period-doubling bifurcation are contained in $\mathrm{PO}_{\text{non-flip}}(F)$.

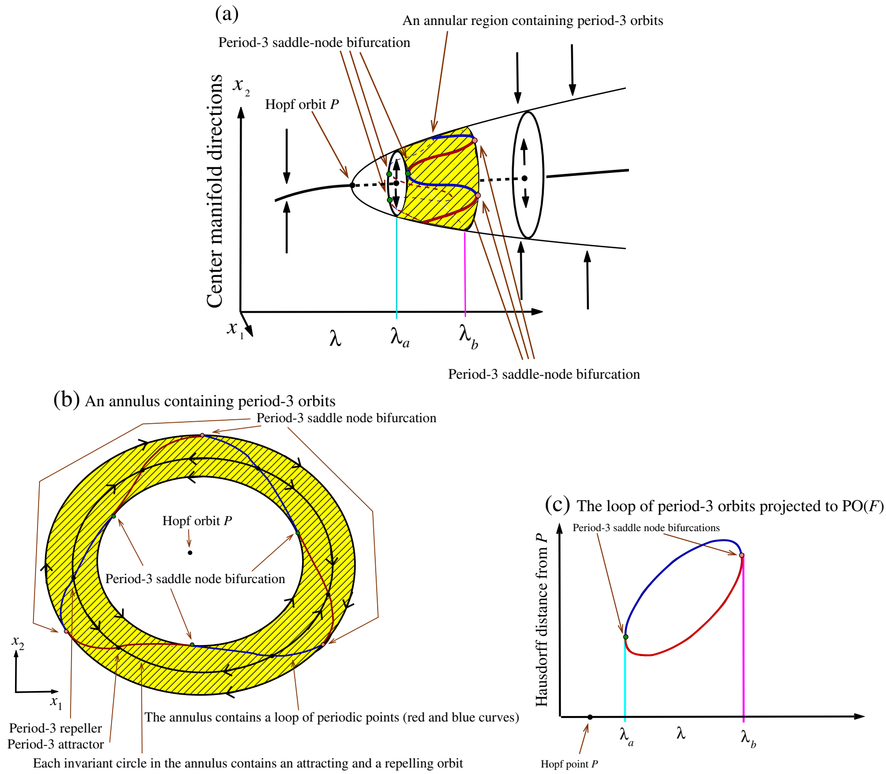

FIGURE 6. Orbits near a Hopf bifurcation. (a) Within the two-dimensional surface of invariant circles near a generic Hopf bifurcation, the topological invariant circles containing orbits of a fixed period form annular regions. (b) Each annular region projects to an annulus when projected to the plane of spatial directions of the center manifold. The annulus consists of invariant topological circles, and each of those circles has an attracting period-k orbit (attracting in the circle but not in the annulus) and a repelling period-k orbit—except for the inner and outer boundary circles of the annulus. The boundary circles contain bifurcating period-k saddle-node orbits. (c) The loop of periodic points is a k-fold cover of the corresponding loop of period-k orbits in $\mathrm{PO}(F)$. The component of the period-k orbits is a topological circle (that is, a loop) in $\mathrm{PO}(F)$.

Note that the standard Hopf bifurcation theorem permits complex conjugate eigenvalues to be higher-order roots of unity. However, we have chosen a more stringent generic bifurcation condition, since a parametrized map with a bifurcation through complex

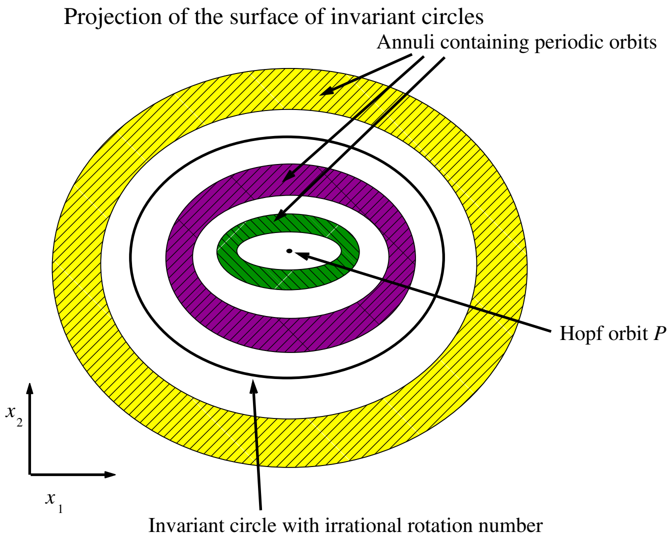

FIGURE 7. Orbits near a Hopf bifurcation lying on the two-dimensional surface in the three-dimensional center manifold: here we project the parabolic region shown in Figure 6 onto a plane. Generically, near a Hopf bifurcation orbit, there are infinitely many annular regions of orbits, each of a fixed period, separated by invariant circles with irrational rotation number.

conjugate pairs that are roots of unity can be rotated by an arbitrarily small perturbation to a family with a bifurcation through one complex conjugate pair that are not roots of unity. Neither 1 nor -1 can be perturbed away in this manner, since these eigenvalues are real, so do not occur in conjugate pairs. Since the roots of unity are countable, the families with Hopf bifurcations for which the eigenvalues are not complex roots of unity are generic. Hence we exclude bifurcations such as period tripling or multiples other than two.

Further, a generic Hopf bifurcation orbit has a neighborhood in $\mathrm{PO}(F)$ in which the only bifurcation orbits are saddle-node bifurcation orbits. See Figures 4–8.

HYPOTHESIS 2. (Generic bifurcations) *Assume Hypothesis 1. Assume that each orbit of F is either hyperbolic or is a generic bifurcation orbit.*

PROPOSITION 1. (Generic F constitute a residual set) *There is a residual set $S \subset C^\infty$ of parametrized maps F satisfying Hypothesis 1 for which all bifurcation orbits are generic.*

This residual set S is C^1 dense in the uniform C^1 topology; that is, for each $F \in C^\infty$, there is a sequence $(F_i) \subset S$ such that $\|F - F_i\|_{C^1} \to 0$ as $i \to \infty$.

The proof of this proposition uses standard transversality arguments. See Palis and Takens [**22**]. Minor changes to their methods give the irrational rotation number for Hopf bifurcations.

2.2. *Components are one-manifolds for generic F.*

Definition 5. (Index orientation) Assume that a periodic orbit y of period p of a smooth map G is hyperbolic. Define the *unstable dimension* $\dim_u(y)$ to be the number of real eigenvalues (with multiplicity) having absolute value >1.

Let F satisfy Hypothesis 2, and assume that Q is a component that is a one-manifold. (We show below that this is true for generic F.) Then we know that there is a homeomorphism $h : X \to Q$ where X is either the interval $(-1, +1)$ or a circle, which we will write as $[-1, 1]/\{-1, 1\}$. For each $s \in X$, let $h_\lambda(s)$ denote the projection of $h(s)$ to the corresponding parameter value. Thus h_λ is a map from X to $\mathbb{R}$, and X and λ both inherit an orientation from the real numbers. Therefore we can describe h as increasing or decreasing at s whenever h_λ is increasing or decreasing at s.

We say that the homeomorphism h is an *index orientation* for Q if, whenever $h(s)$ is a hyperbolic orbit, $h_\lambda(s)$ is locally strictly increasing when $\dim_u(h(s))$ is odd and is locally strictly decreasing when $\dim_u(h(s))$ is even.

We now state the main result of this section.

THEOREM 1. (Components are oriented one-manifolds for generic F) *Consider all F as in Hypothesis 1. There is a residual set of such F for which each component (of* $\mathrm{PO}_{\text{non-flip}}(F)$*):*

(M_1) *is a one-manifold, i.e. is either a simple closed curve or is homeomorphic to an open interval; and*

(M_2) *has an index orientation.*

The proof of (M_1) consists of two parts. First, we show that in a neighborhood of any hyperbolic orbit, its component is an arc. Secondly we show the same for each non-hyperbolic orbit. Hence each point in a component has a neighborhood in the component that is an arc, so the component is a one-manifold.

The neighborhood of a hyperbolic orbit. If $\sigma_0 = (\lambda_0, x_0)$ is a period-p hyperbolic point, the implicit function theorem implies that it has a smooth unique *continuation curve* of period-p points $\sigma(\lambda) = (\lambda, x(\lambda))$, defined for λ in some neighborhood of λ_0 with $\sigma(\lambda_0) = \sigma_0$. Furthermore $[\sigma_0]$ has a neighborhood in $\mathrm{PO}(F)$ in which there are no orbits other than those of the continuation. Hence each hyperbolic orbit has a neighborhood in $\mathrm{PO}(F)$ that is an arc.

The neighborhoods of a non-hyperbolic orbit. This part of the proof relies on the fact that F is generic as defined in Definition 4.

The proof of part (M_1) will be complete with Proposition 2, which shows that each non-flip generic bifurcation orbit y has a neighborhood that—when intersected with the component it is in—is an arc that passes through y.

The proof of part (M_2) of Theorem 1 is complete when we finish proving Proposition 4 in §3. There we show that for each homeomorphism h from X to a component, either h or h^* is an index orientation where $h^*(s) = h(-s)$. Recall that $-s \in X$ whenever $s \in X$.

PROPOSITION 2. (The neighborhood of a generic bifurcation orbit) *Assume that F satisfies Hypothesis 2. Assume $P = (\lambda_0, [x_0])$ is a generic bifurcation orbit in* $\mathrm{PO}_{\text{non-flip}}(F)$ *and let C be its component. Then P has a neighborhood in C that is an open arc in which it is the only bifurcation orbit.*

Proof. We consider each of the three generic orbit bifurcations individually.

Case (i). See Figure 5(a). Locally a generic saddle-node orbit P is in an open arc of otherwise hyperbolic orbits. If P is non-flip, then so are the nearby orbits. No other orbits are nearby, so the proposition's assertion is true for these orbits.

Case (ii). See Figure 5(b). Locally a generic period-doubling orbit P of period p consists of an arc of period-p orbits passing through P, plus an arc of period $2p$ that terminates at P. The latter is always non-flip near P. The period-p branch has an eigenvalue that passes through -1 at P. Hence (locally) on one side of P the arc consists of non-flip orbits and flip orbits on the other side. Hence there are two arcs in $\mathrm{PO}_{\text{non-flip}}(F)$ that terminate at P. Hence a neighborhood of P in $\mathrm{PO}_{\text{non-flip}}(F)$ is an arc.

Case (iii). The generic Hopf bifurcation is much more complicated. It is the only generic local bifurcation for which it is possible to have orbits of unbounded periods limiting to P in the Hausdorff metric.

The center manifold theorem guarantees that, for any r in a sufficiently small neighborhood of P, there is a three-dimensional C^r center manifold for P in $\mathbb{R} \times \mathfrak{M}$. The Hopf bifurcation theorem guarantees that, within this center manifold, there is an invariant topological paraboloid that is either attracting, as depicted in Figure 4, or repelling. Within this paraboloid, for each λ value there is an associated rotation number ω_λ of the invariant circle at parameter value λ, as shown in Figures 6 and 7.

By the genericity assumption on Hopf bifurcation points, the limit ω_{λ_0} as $\lambda \to \lambda_0$ of the rotation numbers ω_λ is irrational. If ω_λ is constant, then there are no local orbits, so P would not be a bifurcation orbit. If ω_λ is non-constant, then it varies through an interval that contains both irrational and rational values. Let Y be the local set of orbits in $\mathrm{PO}(F)$ other than $\{(\lambda, [x_\lambda])\}$, the continuation of P. All points in Y are on the invariant paraboloid. Therefore $(\lambda, [y]) \in Y$ is only possible for λ values such that ω_λ is rational. Therefore no point in Y is contained in the same component of $\mathrm{PO}(F)$ as P.

Specifically, take any point in Y. It is not in the same component as $\{(\lambda, [x_\lambda])\}$ since the paraboloid can be separated into two components at every parameter value for which the rotation is irrational. Therefore the curve $\{(\lambda, [x_\lambda])\}$ is isolated in its component of $\mathrm{PO}(F)$. Furthermore, since a Hopf bifurcation changes the number of eigenvalues outside the unit circle by two, the curve $\{\lambda, [x_\lambda]\}$ is either entirely flip orbits or is entirely contained in $\mathrm{PO}_{\text{non-flip}}(F)$. □

Remark 3. Under Hypothesis 2, the period of the orbits in a component C is locally constant near hyperbolic orbits and near saddle-node and Hopf bifurcations. The period can only change at period-doubling bifurcations, in which case it changes by a factor of 2. Hence an arc C in $\mathrm{PO}_{\text{non-flip}}(F)$ is a cascade only if the sequence of periods $\{p_k\}$ of the non-hyperbolic orbits in C (such that orbit $k+1$ is between orbit k and orbit $k+2$) limits to infinity. This occurs if the sequence is infinite and no period occurs more than a finite number of times.

2.3. *Bounded arcs and cascades.* For a generic map F, let A be a component that is an open arc. Then there is a homeomorphism $h : (-1, 1) \to A$. Let m denote the minimum

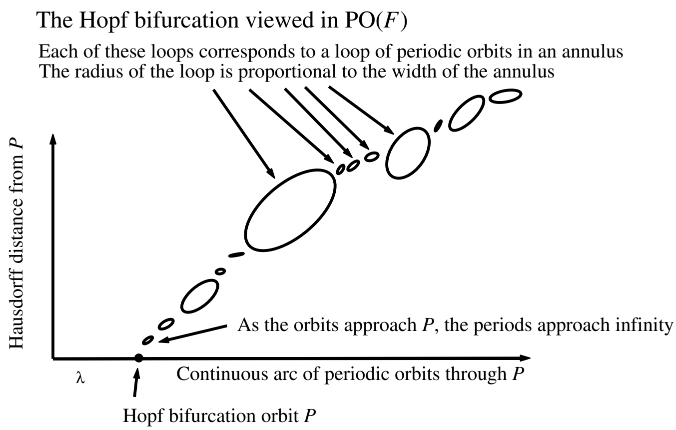

FIGURE 8. Orbits near a Hopf bifurcation in PO(F): a neighborhood in PO(F) of a generic Hopf bifurcation orbit P consists of (i) an arc of orbits (the horizontal axis) with the same period as P, and (ii) a collection of components, each of which is a loop of orbits, i.e., a simple closed curve. All the orbits in each of these loops have the same period. As the loops approach P in PO(F), the periods go to infinity.

period of the orbits in A. Without loss of generality, we can assume that $h(0)$ is a hyperbolic orbit with period m. Write $A^- = h((-1, 0))$ and $A^+ = h((0, +1))$. We say that a set of orbits in $\mathrm{PO}_{\text{non-flip}}(F)$ is *bounded* if the union of its orbits lies in a compact subset of $\mathbb{R} \times \mathfrak{M}$. We say that an open arc A has a *bounded end* if either A^- or A^+ is bounded. While we are splitting A at a rather particular orbit, the property of boundedness of an end is independent of where the arc A is split. Note that, by our assumption, the point $h(0)$ is not in a cascade, since a cascade contains only one orbit of smallest period, meaning that it is not hyperbolic. We refer to A^- and A^+ as the *ends of the component*.

PROPOSITION 3. (Bounded cascades) *Assume Hypothesis 2. If a component A is an open arc, and one of its ends is bounded, then that end contains a cascade. If the entire component A is bounded, then A contains two cascades, and these are disjoint.*

Write $\mathrm{Per}(u)$ for the period of an orbit $u \in \mathrm{PO}(F)$. We adopt the above notation for m, h, A, A^- and A^+. For brevity, we give the proof for the case where A is bounded, since the case of one end being bounded uses the exact same method of proof.

Proof. Write $J = (-1, 1)$. Let $(t_j)_1^\infty \subset J$ be a sequence that converges to either $+1$ or -1. Since A is a component, it is closed in $\mathrm{PO}_{\text{non-flip}}(F)$, so the sequence has no limit points. Let m_j denote $\mathrm{Per}(h(t_j))$ for each j. If a subsequence of (m_j) were bounded, $h(t_j)$ would have a limit point, which it does not, so $\lim_{j\to\infty} m_j = \infty$. Let $m = \min_j\{m_j\}$. Let $S = t \in J : \mathrm{Per}(h(t)) = m$. Note that S is compact. Let $t_{\sup} = \sup S$ and $t_{\inf} = \inf S$. Note that $-1 < t_{\inf} < 0 < t_{\sup} < 1$. Write $J_1 := (-1, t_{\inf}]$ and $J_2 := [t_{\sup}, +1)$. We claim that $A_1 = h(J_1)$ and $A_2 = h(J_2)$ are cascades. Note that they are disjoint and each is homeomorphic to a half-open interval under h. Notice that, if the $\mathrm{Per}(h(t))$ changes discontinuously at t, then $h(t)$ is a period-doubling orbit and the change is precisely by a factor of 2 (cf. Proposition 2). Hence since the periods are unbounded in A_1 and A_2, the periods of orbits in each must be $\{2^k m : k = 0, 1, 2, \ldots\}$, as required by item (i) in

Definition 3. Since m is the smallest period in A, item (ii) is also satisfied. Item (iii) is satisfied since, by our choice of $t_{\inf}$ and $t_{\sup}$, A_1 and A_2 both have only one orbit with period m. The two cascades are disjoint because J_1 and J_2 are disjoint.

We note that $\mathrm{Per}(h(t)) \to \infty$ as $|t| \to 1$, as mentioned in item (iv). □

The previous result gives a criterion for cascades in terms of finding components that are bounded arcs, but it remains unclear how to demonstrate that components are bounded arcs. The key to solving that problem lies in the next section, which provides a natural orientation for each component using the index orientation.

Note that one interpretation of the fact that A has no limit points is the following property.

Isolation of generic bifurcation orbits of period $\leq p$. Assume Hypothesis 2. For each period p, F has at most a finite number of non-hyperbolic orbits of period p in each bounded region of $\mathbb{R} \times \mathfrak{M}$. To prove this, note that, if there were an infinite number of bifurcation orbits of period $\leq p$ in a bounded set, then there would be an accumulation point of these bifurcation orbits in $\mathrm{PO}_{\text{non-flip}}(F)$, which would have to be non-hyperbolic. However, this cannot occur, since generic orbit bifurcations are isolated from bifurcation orbits of bounded period.

3. *An orientation for components*

An open arc component of $\mathrm{PO}_{\text{non-flip}}(F)$ has two orientations. This section establishes that one of these two orientations is consistent with a specific topological invariant called the orbit index. We establish the behavior of this index near each generic bifurcation. From this, we are able to conclude that cascades occur, detailed in the main theorem in §4.

The following concept of the *orbit index* was developed in [**17**], where it is defined for all isolated orbits (for flows).

Definition 6. (Orbit index) Assume that an orbit y of period p of a smooth map G is hyperbolic. Based on the eigenvalues of y, we define

$$\sigma^+ = \sigma^+(y) = \text{the number of real eigenvalues (with multiplicity) in } (1, \infty),$$
$$\sigma^- = \sigma^-(y) = \text{the number of real eigenvalues (with multiplicity) in } (-\infty, -1).$$

The *fixed point index* of y is defined as $\mathrm{ind}(y) = (-1)^{\sigma^+}$. From the definition of fixed point index, it follows that

$$\begin{aligned} \mathrm{ind}(x, G^{pm}) &= (-1)^{\sigma^+} && \text{for } m \text{ odd}, \\ &= (-1)^{\sigma^+ + \sigma^-} && \text{for } m \text{ even}. \end{aligned}$$

Since σ^+ and σ^- are the same for each point of an orbit, we can define the *orbit index* of a hyperbolic orbit

$$\phi([x]) = \begin{cases} (-1)^{\sigma^+} & \text{if } \sigma^- \text{ is even}, \\ 0 & \text{if } \sigma^- \text{ is odd}. \end{cases} \tag{1}$$

Hence if $[x]$ is a non-flip hyperbolic orbit,

$$\phi([x]) = \mathrm{ind}([x]).$$

Note that a hyperbolic orbit is a flip orbit if and only if its orbit index is zero. Thus for every hyperbolic orbit $[x]$ that is non-flip, i.e., $[x] \in \mathrm{PO}_{\text{non-flip}}(F)$, $\phi([x])$ is ± 1 (never zero).

The following proposition is a stronger version of part (M_2) of Theorem 1 and is used to prove (M_2). It states that each component has an index orientation.

PROPOSITION 4. (Each component has an index orientation) *Let F satisfy Hypothesis 2, and let Q be a component. Let $\psi : X \to Q$ be a homeomorphism where X is the circle or interval in Definition 5. Define the homeomorphism $\psi^* : X \to Q$ by $\psi^*(s) = \psi(-s)$ for all $s \in X$. Then either ψ or ψ^* is an index orientation.*

Proof. On each non-flip orbit y (i.e., where $\sigma^-(y)$ is even), $\phi(y) = (-1)^{\sigma^+(y)}$. Hence on hyperbolic non-flip orbits, $\phi(y) = (-1)^{\dim_u(y)}$. Hence on a hyperbolic non-flip orbit y,

$$\dim_u(y) \text{ is odd if and only if } \phi(y) = -1.$$

Hence in the definition of 'index orientation' we will substitute '$\phi(h(s)) = -1$ (or $+1$)' for '$\dim_u(h(s))$ is odd (or even, respectively)'.

Let $\psi, \psi^* : X \to Q$ be as in the statement of the proposition. The component Q consists of pairs of segments of hyperbolic orbits connected at generic bifurcation orbits. Pick $s \in X$ such that $P = \psi(s) = \psi^*(-s)$ is hyperbolic. Since both the direction of the arc and the orbit index are fixed on a hyperbolic segment, either ψ or ψ^* is an index orientation on the segment of hyperbolic orbits containing P. Assume without loss of generality that this occurs for ψ. We show below that, at each type of orbit bifurcation, there is a 'consistent' index orientation, as depicted in Figure 10. That is, two consecutive segments of hyperbolic orbits (separated by a bifurcation orbit) have the same index orientation, ψ or ψ^*. One of the two segments always leads toward the bifurcation orbit and one segment leads away as s increases. Thus continuing by induction through all of its hyperbolic segments, ψ is consistent with ϕ on all hyperbolic segments. We can then conclude that ψ is an index orientation.

We now must show only that at each type of generic orbit bifurcation, there is a consistent index orientation. Here and subsequently, since the indices are fixed on hyperbolic segments, we denote the indices of a segment when we mean the indices of any orbit on that segment.

Part (i): saddle-node bifurcations. Let $y_0 = (\lambda_0, [x_0])$ be a generic period-p saddle-node bifurcation orbit. In some neighborhood of y_0, on one side there will be two segments of hyperbolic period-p orbits y_a and y_b, and on the other side there are no orbits, as depicted in Figure 5(a). The σ^+ values of y_a and y_b will differ by 1, since along the arc an eigenvalue passes through 1, but their σ^- values will be equal. Hence y_a and y_b have opposite fixed point index $\mathrm{ind}(y_a, G^p) = -\mathrm{ind}(y_b, G^p)$, both possibly 0. Also $\mathrm{ind}(y_a, G^{2p}) = -\mathrm{ind}(y_b, G^{2p})$. Hence from equation (1),

$$\phi(y_a) + \phi(y_b) = 0. \tag{2}$$

Thus there is a consistent index orientation at this bifurcation.

Part (ii): period-doubling bifurcations. Let $y_0 = (\lambda_0, [x_0])$ be a generic period-p period-doubling bifurcation orbit. In some neighborhood of y_0, on one side there will be a segment of hyperbolic period-$2p$ orbits. We denote the orbits by y_c, as in Figure 5(b). For λ close to λ_0, $D_x F^{2p}$ is approximately $(D_x F^p)^2$ and so it has no real eigenvalues less than -1, and, in particular, $\phi(y_c) \neq 0$. Hence

$$\phi(y_c) = \text{ind}(y_0, G^{2p}). \tag{3}$$

On the same side there must be a segment of hyperbolic period-p orbits, which we denote y_b. We write y_a for the segment of period-p orbits on the other side from y_c. The invariance of the total fixed point index at a bifurcation yields

$$\begin{aligned} \text{ind}(y_a, G^p) &= \text{ind}(y_b, G^p), \\ \text{ind}(y_a, G^{2p}) &= \text{ind}(y_b, G^{2p}) + 2\,\text{ind}(y_c, G^{2p}). \end{aligned}$$

We substitute ϕ for ind using equation (3), take the average of the left-hand sides of the above equations, and set that equal to the average of the right-hand sides. This yields

$$\phi(y_a) = \phi(y_b) + \phi(y_c). \tag{4}$$

Since each of these has values in $\{-1, 0, +1\}$, and $\phi(y_c)$ is not zero, there are two cases:

$$\phi(y_a) = 0 \quad \text{and} \quad \phi(y_b) = -\phi(y_c)$$

or

$$\phi(y_b) = 0 \quad \text{and} \quad \phi(y_a) = +\phi(y_c).$$

Hence there are two segments on which ϕ is non-zero. If both non-zero segments are on the same side of the bifurcation point, they have opposite orientation. If the segments are on opposite sides of the bifurcation point, they have the same orientation. Thus there is a consistent index orientation at this bifurcation orbit.

Part (iii): Hopf bifurcations. If an arc of orbits has a Hopf bifurcation, σ^+ and σ^- are the same on the two sides of the bifurcation, so ϕ does not change. If a pair of complex values become real as λ is varied, σ^+ or σ^- can change by $+2$ or -2, which has no effect on ϕ and ind. Thus there is a consistent index orientation at this bifurcation.

This completes the proof of the proposition. □

Figure 9 depicts a typical oriented arc in $\text{PO}_{\text{non-flip}}(F)$ as described in the above result. A proof similar to the proof of the above proposition also shows that the orbit index is a bifurcation invariant for generic bifurcations; see [**17**]. Figure 10 depicts all generic bifurcations.

4. *Theorem of cascades from boundaries*

4.1. *Oriented arcs entering or exiting regions.* We now describe the restriction of oriented arcs to a region U with a bounded parameter range.

HYPOTHESIS 3. (Orbits near the boundary) *Let F satisfy Hypothesis 2. Let $\lambda_0 < \lambda_1$, and let $U = [\lambda_0, \lambda_1] \times \mathfrak{M}$ and $\partial U = \{\lambda_0, \lambda_1\} \times \mathfrak{M}$. Assume that all orbits in ∂U are hyperbolic. Assume that all orbits in U are contained in a compact subset of $\mathfrak{M}$.*

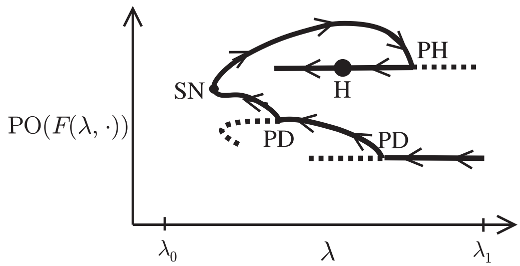

FIGURE 9. Part of oriented component Q with period-doubling (PD), period-halving (PH), saddle-node (SN), and Hopf (H) bifurcations. If the homeomorphism $h : X \to \mathrm{PO}_{\text{non-flip}}(F)$ is an index orientation, then, as $s \in X$ increases, the λ coordinate of $h(s)$ increases or decreases and the arrows on the segments indicate which. Left-pointing arrows correspond to orbit index $\phi = -1$, and right-pointing to $+1$. One of the two adjacent segments always leads toward the intervening bifurcation orbit and one segment leads away as s increases. The dotted lines indicate flip orbits (which are not in Q).

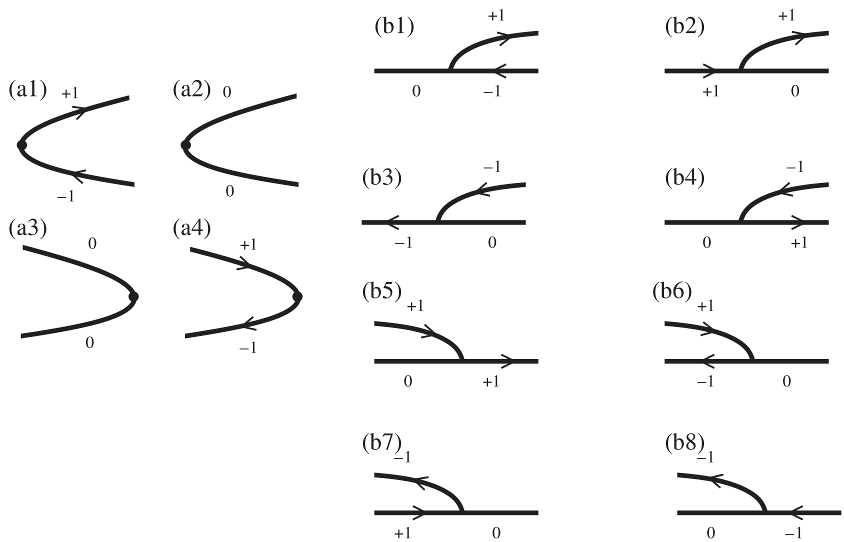

FIGURE 10. Generic bifurcations: this figure depicts all possible generic (a) saddle-node and (b) period-doubling (or period-halving) bifurcations with their orbit indices, the numbers near each segment. Each point denotes an orbit. In this symbolic representation, the horizontal axis is the parameter, and the vertical axis is PO(F). The arrows have the same meaning as in Figure 9 but segments of flip orbits are indicated here by having no arrows (and by having orbit index 0). The one-dimensional quadratic map $\lambda - x^2$ has only bifurcations of types (a1) and (b2).

Definition 7. Assume Hypothesis 3 and its notation. Let $p \in \mathrm{PO}_{\text{non-flip}}(F)$ be a hyperbolic orbit in ∂U. If p is oriented 'into' the region U by index orientation, then it is called an *entry orbit* of U. That is, p is an entry orbit if either $\lambda = \lambda_0$ and $\phi(p) = +1$, or $\lambda = \lambda_1$ and $\phi(p) = -1$. Otherwise, it is called an *exit orbit* of U.

A cascade is said to be *essentially in U* if all but a finite number of its bifurcation orbits are in U.

THEOREM 2. (Cascades from boundaries) *Assume Hypothesis 3. Let IN be the set of entry orbits. Let OUT be the set of exit orbits. Assume that IN contains K elements, and OUT* *contains J elements. We allow one but not both of the sets to have an infinite number of elements.*

(C_K) *If $K < J$, then all but possibly K orbits in OUT are contained in distinct components, each of which contains a cascade that is essentially in U.*
Likewise, if $J < K$, then all but possibly J orbits in IN are contained in distinct components, each of which contains a cascade that is essentially in U.

(C_0) *If $J = 0$ or $K = 0$, then the non-flip orbits of ∂U are in one-to-one correspondence with the components that intersect the boundary. Each of these components has one cascade that is essentially in U.*

Proof. Assume Hypothesis 3. For simplicity, we specify that $J < K$. The proof of the other case is similar.

Let q be an orbit of *IN*. Then q is an entry orbit and it lies in a component Q, which by Theorem 1 is a one-manifold. Let $\sigma : X \to Q$ be the index orientation for Q. Starting at q and following the component in the forward direction (using the index orientation), $\sigma(s)$ initially enters the interior of U. If it leaves U, it does so through an exit orbit. Let exit(q) denote the first such exit orbit encountered. Distinct entry orbits q that exit yield distinct exit(q) of which there are at most J. Hence all but J entry orbits are in components that do not leave U for increasing s. Such a component must be an open arc and the component must have a bounded end. By Proposition 3, it contains a cascade, a cascade that is essentially in U. Hence (C_K) is proved.

If $J = 0$, then each entry orbit is in a component that crosses the boundary only once. Furthermore any component that crosses the boundary must do so at an entry orbit. Each such component has an end in U and so has a component that is essentially in U, proving (C_0). □

4.2. *Related results.* Our abstract results in the previous section build on the work in [**25**] and Franks [**9**]. We now compare our results to these two previous results.

The papers [**25, 26**] proved the existence of cascades of attracting periodic points for area-contracting maps and for elliptic periodic points for area-preserving maps, in a particular case of Theorem 2, part (C_0), without assuming genericity. Whereas our current result applies to parametrized maps with a large number of unstable dimensions, their result only considered maps with at most one unstable dimension. The existence of attractors relies on having at most one unstable dimension, because this implies that there are no Hopf bifurcations. For parametrized maps with more than one unstable dimension, cascades do not in general contain attractors. Both results involve snakes in the generic case, followed by smooth convergence arguments to show the general case. These convergence arguments no longer apply when there is more than one unstable dimension.

The *Morse index* is the number of unstable eigenvalues. If the Morse index is even, the orbit index is either 0 or $+1$. If the Morse index is odd, the orbit index is either 0 or -1.

Franks [**9**] proves there are cascades under the following conditions. Let d be an odd integer. Assume that, for every non-negative integer k, every orbit of period $2^k d$ has a Morse index with the same 'parity' (all are odd or all even) at $F(\lambda_0, \cdot)$, and the opposite parity at $F(\lambda_1, \cdot)$. In our notation, this corresponds approximately to saying that, on the boundary $F(\lambda_0, \cdot) \cup F(\lambda_1, \cdot)$, all orbits are entry orbits (or alternatively all are exit orbits);

we ignore flip orbits. Our theorem relaxes this condition; we only assume that the numbers of entry and exit orbits differ.

Franks' proof uses the Lefschetz trace formula, which allows the smoothness of F to be relaxed; F is assumed to be a continuous parametrized map, where $F(\lambda_0, \cdot)$ and $F(\lambda_1, \cdot)$ are smooth maps. However, this lack of smoothness has implications. The theorem does not give information about the bifurcations, only assuring that the component of $\mathrm{PO}(F)$ containing the original hyperbolic orbit of period $2^r d$ (d odd) contains flip orbits of period $2^k d$ for all $k \in N$ on the boundary $F(\lambda_0, \cdot) \cup F(\lambda_1, \cdot)$. The theorem says nothing about the way these orbits bifurcate. A cascade is usually viewed as a sequence of events with some separation, but in the context of Franks, a portion of the cascade can occur at a single parameter value. For example, in non-generic maps with dimension larger than one, two eigenvalues can simultaneously bifurcate through -1. Thus a fixed point can bifurcate to a period-four orbit, missing a bifurcation through period two. A more extreme case of this phenomenon is shown in the following example using a slight adaptation of Franks' result. The example is a one-dimensional but non-smooth map, in which an entire generalized cascade occurs at one parameter value.

Example 1. (The parametrized tent map) Consider the parametrized map consisting of tent maps of slopes of absolute value λ, with λ increasing to the standard value of 2. Orbits of all the periods 2^k (for $k \geq 0$) appear at exactly $\lambda = 1$, or, more precisely, each exists for all $\lambda > 1$. In fact, for every $k > 2$, the period-k cascades appear in this manner. All of these are cascades in the sense of Franks' theorem.

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