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A classification of explosions in dimension one

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Abstract. A discontinuous change in the size of an attractor is the most easily observed type of global bifurcation. More generally, an *explosion* is a discontinuous change in the set of recurrent points. An explosion often results from heteroclinic and homoclinic tangency bifurcations. We prove that, for one-dimensional maps, explosions are generically the result of either tangency or saddle-node bifurcations. Furthermore, we give necessary and sufficient conditions for generic tangency bifurcations to lead to explosions.

1. *Introduction*

For continuously varying one-parameter families of iterated maps in $\mathbb{R}^n$, discontinuous changes in the size of an attractor are the most easily observed type of global bifurcations, including changes in the basin boundary (metamorphosis). These changes can occur as the result of a change in stability of the recurrent set. A more general situation occurs when discontinuous changes in attractors occur as the result of a discontinuous change in the size of the recurrent set itself. Such a global bifurcation is called an *explosion*. For the last several years, we have been studying explosions and their properties, including a classification of explosions at heteroclinic tangencies for planar diffeomorphisms [**1**], and more recently a numerical study of the statistical properties of a certain kind of explosion that occurs in dimension three and higher [**2**]. The first major result in this paper is a full classification of which types of tangencies give rise to explosions for interval maps, a result which we think is important and new.

Our research, as well as that of many others [**14, 24**], has been guided by a 1976 conjecture of Newhouse and Palis [**19**]. (See also the restatements in [**10, 21**].) For over thirty years, this conjecture has managed to elude proof. The conjecture says that a first bifurcation of a Morse–Smale system is generically either the result of a non-hyperbolic periodic orbit, or the result of a tangency between stable and unstable manifolds of fixed or periodic orbits. We make the following reformulation of this conjecture for global bifurcations: for generic planar diffeomorphisms, all explosions occur through the following two local bifurcations: saddle-node bifurcations and tangency bifurcations. The second major result of this paper is that the the analogous conjecture is true for generic smooth interval maps. This is in contrast with a recent result of Horita *et al* [**14**], showing that in a probabilistic sense, the Newhouse–Palis conjecture is not true for circle maps. In a broader sense, we are hopeful that the insight gained from the one-dimensional case will give rise to insights leading to a better understanding of explosions in two dimensions.

It is clear that an isolated saddle-node bifurcation for either a fixed point or periodic orbit gives rise to a local explosion, since new periodic points appear. However, in many cases in both one and higher dimensions, a saddle-node bifurcation also gives rise to a global bifurcation. The set of recurrent points changes discontinuously as the parameter is varied at points not contained in the saddle-node periodic orbit. For example, the period three window in the chaotic attractor for the logistic map corresponds to an explosion: for parameter values less than a bifurcation value, there is a global attractor which comprises the full recurrent set. It consists of an interval. After the bifurcation parameter, the global attractor consists only of a period three orbit, and the full recurrent set is a nowhere dense Cantor set within the attractor interval. An explosion occurs at the bifurcation parameter, and it is due to a saddle-node bifurcation at a point inside the global attractor. Saddle-node bifurcations on invariant circles are another well-studied example of this phenomenon.

In all dimensions, explosions due to a homoclinic or a heteroclinic bifurcation essentially occur due to the creation of a homoclinic tangle when stable and unstable manifolds intersect transversally. In one and two dimensions, the transition between no intersection and transverse intersections involve tangencies between the stable and unstable manifolds of fixed or periodic points (cf. [**10**]). For example, the Hénon map and Ikeda map contain well-studied examples of explosions which are a result of homoclinic bifurcations. In higher dimensions, such bifurcations can occur without tangencies [**13**]. Such a bifurcation leads to unstable dimension variability [**2**].

In our previous work for planar maps, we gave a precise classification for which types of tangencies for heteroclinic cycles will result in explosions. We called this class of cycles *crossing cycles*, because the different stable and unstable manifolds involved in the cycle lie *across* the tangency from each other. We show here that the same results hold for interval maps. Our main results are as follows.

THEOREM 1. (Explosions at tangencies) *For generic one-parameter families of smooth maps of the interval with homoclinic or heteroclinic tangencies (hypotheses H1–H6), explosions occur if and only if there is an isolated crossing orbit.*

THEOREM 2. (General explosion classification) *Explosions within generic one-parameter families of smooth maps of the interval (hypotheses H1–H3) are the result of either a tangency between stable and unstable manifolds of fixed or periodic points or a saddle-node bifurcation of a fixed or periodic point.*

The paper proceeds as follows: in §2, we give basic definitions of explosions and homoclinic tangencies. We are considering the particular recurrence class of chain recurrent points, defined in this section. A motivation for the choice of setting is given in the section as well. In §3, we prove the explosions at tangencies theorem. In §4, we prove the General explosion classification theorem. Our results rely on a very sophisticated and well-developed theory for the dynamics of interval maps. We briefly state the necessary background and results in the course of the proof.

2. *Basic definitions*

We now give some formal definitions of concepts described in the introduction. Let $f : (I \times J) \subset (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ be a smooth one-parameter family of maps. We exchangeably write two types of notation: $f(x, \lambda) = f_\lambda(x)$. For the definition of an explosion it is more natural to use the concept of chain recurrence rather than recurrence. The relationship between chain recurrence and other types of recurrence is discussed in Remark 1 and in [**10**].

Definition 1. For an iterated function g, there is an *ϵ-chain* from x to y when there is a finite sequence $(z_0, z_1, \ldots, z_N)$ such that $z_0 = x$, $z_N = y$, and $d(g(z_{n-1}), z_n) < \epsilon$ for all n.

If there is an ϵ-chain from x to itself for every $\epsilon > 0$ (where $N > 0$), then x is said to be *chain recurrent* [**11, 12**]. The *chain-recurrent set* is the set of all chain-recurrent points. For a one-parameter family f_λ, we say (x, λ) is chain recurrent if x is chain recurrent for f_λ.

If for every $\epsilon > 0$ there is an ϵ-chain from x to y and an ϵ-chain from y to x, then x and y are said to be in the same *chain component* of the chain-recurrent set.

The chain-recurrent set and the chain components are invariant under forward iteration. We now define an explosion bifurcation in the chain-recurrent set. Such a definition can be formulated for the non-wandering set as well.

Definition 2. (Chain explosions) A *chain-explosion point* (x, λ_0) is a point such that x is chain recurrent for f_{λ_0}, but there is a neighborhood N of x such that on one side of λ_0 (i.e. either for all $\lambda < \lambda_0$ or for all $\lambda > \lambda_0$) no point in N is chain recurrent for f_λ. (All explosion points in this paper are chain explosions, so we sometimes drop the qualifier chain.)

Remark 1. (Bifurcations and explosions) Explosions have been used in the past to study global bifurcations for planar diffeomorphisms [**1, 20–23**]. They are closely related to the loss of structural stability, since interesting dynamics only occur where there is recurrence. If f_{λ_0} is a structurally stable function in a one-parameter family on a compact set, then for any x, (x, λ_0) is not an explosion point. For smooth maps on compact domains, the chain-recurrent set is upper semicontinuous, whereas the closure of the set of hyperbolic periodic points is lower semicontinuous. If the sets are equal, then they change continuously [**10**].

In studying global bifurcations, the setting of explosions has an advantage over considering structurally stable systems in that we are not restricted to considering first bifurcations. Explosions can occur for example at the limit point of a sequence of bifurcations. However, there are bifurcations which are not explosions. For example, a point of period doubling is not an explosion point, although it is a bifurcation. In this paper, we do not discuss local bifurcations.

Our work considers chain explosions, whereas others have considered explosions in the non-wandering set. These concepts are quite closely related: the set of periodic points is contained in the recurrent set, which is contained in the non-wandering set. The non-wandering set is contained in the chain-recurrent set. There are examples showing that all the containments are strict. However, for a generic smooth diffeomorphism on a compact manifold, the chain-recurrent set and the non-wandering set are both equal to the closure of the set of hyperbolic periodic orbits (cf. [**10**, §10.2]).

Our choice of chain explosions as opposed to non-wandering explosions is motivated by the following useful recharacterization of the property of upper semicontinuity: if (x, λ) is a chain-explosion point, then x is chain recurrent for parameter λ. This is not true for the non-wandering set or for the closure of the hyperbolic periodic points. Using this property, in our technical arguments we are able to separate out the dynamics and the parameter change, allowing us to confine many of our arguments to studying the chain-recurrent set at the parameter for which the chain explosion occurs. Our investigation is practically equivalent to studying maps (without parameters) for which the chain-recurrent set is strictly bigger than the recurrent set, provided that the map is sufficiently nice that it could be seen in generic one-parameter families.

Remark 2. In order to show that (y, λ) is *not* a chain-explosion point, it is sufficient to show that y is in the closure of the hyperbolic periodic orbits for f_λ.

Remark 3. In the above definition, at f_{λ_0}, x is not necessarily an isolated point of the chain-recurrent set. For example, at a saddle-node bifurcation on an invariant circle, the chain-recurrent set consists of two fixed points prior to bifurcation and the whole circle at and in many cases after bifurcation.

The chain-recurrent set is not invariant under backwards iteration of a non-invertible map. Thus explosion points are not preserved under iteration, forward or backward. The following remark states what is guaranteed by the fact that chain recurrence is preserved under forward iteration.

Remark 4. Let (x, λ_0) be a chain-explosion point for f. Specifically, there exists $\delta > 0$ such that there is no chain-recurrent point in $B_\delta(x)$ for all $\lambda < \lambda_0$, but x is chain recurrent at λ_0. Then $f(x)$ is chain recurrent at λ_0, but $f(x)$ may also be chain recurrent for $\lambda < \lambda_0$. In contrast, if x_{-1} is a preimage of x, then there is a $\delta_{-1} > 0$ such that no point in $B_{\delta_{-1}}(x_{-1})$ is chain recurrent for all $\lambda < \lambda_0$. Note that x_{-1} may not be chain recurrent at λ_0.

We now give definitions of homoclinic and heteroclinic points. Note that for a diffeomorphism, homoclinic and heteroclinic orbits require the existence of saddle points with stable and unstable manifolds of dimension at least one. Thus they can only occur in dimension two or greater. However, for non-invertible maps, it is possible to have a fixed or periodic point with a one-dimensional unstable manifold and a zero-dimensional stable manifold. Marotto terms such points snap-back repellers [**17**]. It is not possible to reverse these stable and unstable manifold dimensions; the existence of a homoclinic orbit to an attracting fixed point requires a multivalued map [**25**]. The following definition of homoclinic points for interval maps is depicted in Figure 1.

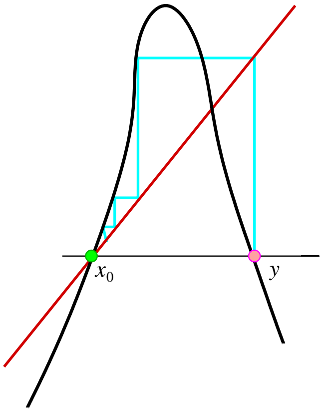

FIGURE 1. A one-dimensional map with a repelling fixed point x_0 where $f'(x_0) > 0$. The point y is a homoclinic point, since $f^k(y) = x_0$ (for $k = 1$), and there exists a sequence of successive preimages of y converging to x_0.

Definition 3. (Homoclinic points) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with a repelling fixed point x_0. Let y be a point in the unstable manifold of x_0 and an integer $K > 0$ such that $f^K(y) = x_0$. Then y is a homoclinic point to x_0. If x_0 is periodic with least period m, then the same definition applies by replacing f with f^m.

Remark 5. If p is a hyperbolic fixed point, and the forward limit set of a point z in the unstable manifold of p includes p, then there are two possibilities: (1) a finite iterate of z is equal to p, (2) the limit set of z contains points other than p. In this second case, z is not referred to as a homoclinic point, since its limit set is larger than just p. This case is considered in later sections of this paper.

For diffeomorphisms, all orbits through homoclinic points are homoclinic orbits. For one-dimensional maps, there may be many non-homoclinic orbits through a homoclinic point.

Definition 4. (Homoclinic orbits) Let f, x_0, and y be as in the above definition of a homoclinic point. An orbit $(z_{-k})_{k=0}^{\infty}$ is a homoclinic orbit through y if the following conditions are satisfied: $z_0 = x_0$, $z_{-K} = y$ for some K, and for all $k \in N$, $f(z_{-k}) = z_{-k+1}$, and $\lim_{k\to\infty} z_{-k} = x_0$.

Since the stable manifold of a homoclinic point is zero-dimensional, a homoclinic tangency is a tangency of the graph of the map at a homoclinic point. Homoclinic tangencies are depicted in Figures 2 and 3.

Definition 5. (Homoclinic tangencies) Let f, x_0, y, and $(z_{-k})_{k=0}^{\infty}$ be as in the above definition of a homoclinic orbit. The point $w = z_{-L}$ is a homoclinic tangency point if the graph of f is tangent to the horizontal line at w.

3. *Explosions at homoclinic tangencies*

This section classifies explosions occurring via homoclinic tangencies. The results are stated for fixed points, but the same results hold for homoclinic orbits for periodic points

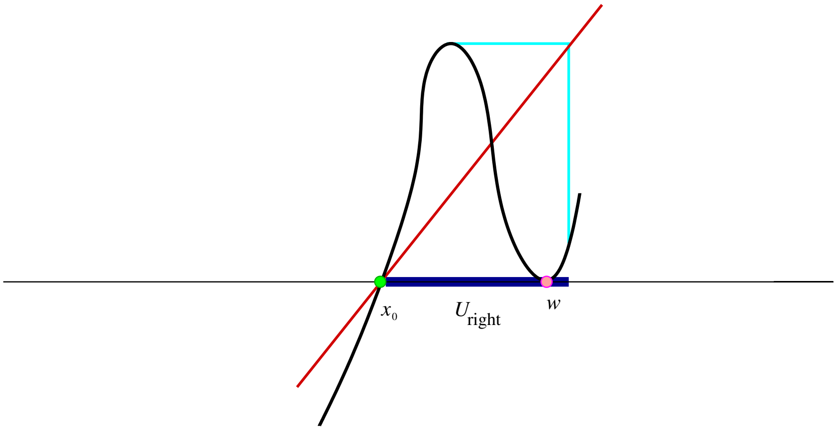

FIGURE 2. A one-dimensional map with a repelling fixed point x_0 with $f'(x_0) > 0$. The point w is a homoclinic tangency point. For the particular tangency depicted here, w is contained in a *non-crossing orbit* (Definition 7), and thus by Theorem 5 w is not an explosion point.

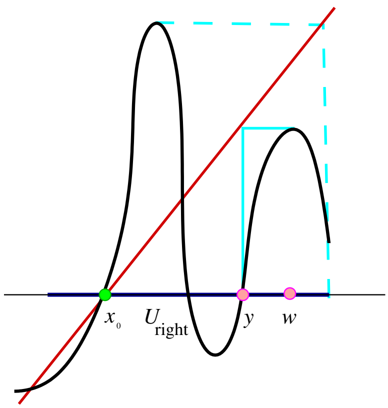

FIGURE 3. In a homoclinic orbit as in Figure 2, the tangency point w does not need to be the immediate preimage of the fixed point. Here, w is the second preimage of x_0.

with least period m if f is replaced by $g = f^m$. Throughout this section, we make the following hypotheses. The first hypothesis is a smoothness assumption. The second and third hypotheses are generic assumptions for one-parameter families. The fourth is a notational convention for the existence of a homoclinic orbit. The fifth hypothesis is generic for one-parameter families containing a homoclinic orbit.

H1 $f : (I \times J) \subset (\mathbb{R} \times \mathbb{R}) \mapsto I$ is a C^1 smooth family of C^2 interval maps. For a fixed parameter λ_0, denote $f_0 := f_{\lambda_0}$.

H2 Assume H1. Assume that there are no intervals on which f_0 is constant. Assume also that critical points of f_0 are non-flat.

H3 Assume H1. Assume that for f_0, there is at most one of the following.

(a) A non-hyperbolic fixed point or periodic orbit. We assume generic behavior as a parameter is varied. That is, any non-hyperbolic periodic point is a codimension one period-doubling or saddle-node bifurcation.

(b) One critical point which comprises a tangency between stable and unstable manifolds of fixed points or periodic orbits. We assume generic behavior as a parameter is varied. That is, with variation of parameter, the critical point moves from above to below (or below to above) the placement of the periodic point.

H4 Assume H1, and that for each λ, x_λ is a repelling fixed point for f_λ. Denote $x_0 := x_{\lambda_0}$. Assume that for f_0, y is homoclinic to x_0.

H5 Assume H1 and H4. At $\lambda = \lambda_0$, the homoclinic orbit containing y contains only one critical point.

The following theorem says that if at a repelling periodic point a map has negative derivative then no homoclinic points are explosion points.

THEOREM 3. (No explosions for orbits with negative derivatives) *Assume that f_λ is a family of maps satisfying H1–H5, and $f'_{\lambda_0}(x_0) < 0$, then (y, λ_0) is not an explosion point.*

Proof. We show that y is contained in the closure of the hyperbolic periodic points of f_{λ_0}.

If y is contained in a homoclinic orbit without a tangency, then the homoclinic orbit is preserved under perturbation, which automatically implies that (y, λ_0) is not an explosion point. Thus we assume that y is contained in a homoclinic orbit containing a homoclinic tangency point w. As before, denote this orbit by $(z_{-k})_{k=0}^{\infty}$, where $z_0 = x_0$, $\lim_{k\to\infty} z_{-k} = x_0$, and K and L are such that $y = z_{-K}$ and $w = z_{-L}$.

Fix a neighborhood U of y. By H5, there exists a sequence of neighborhoods $U_k \ni z_{-k}$, such that: (i) $U_K \subset U$; (ii) for $k \geq L$, $f_0(U_{k+1}) = U_k$ and the map is injective; and (iii) for $0 \leq k < L$, $U_{L-k} = f_0^k(U_L)$. Let $V_1 = f_0^L(U_L \setminus w) = U_0 \setminus x_0$. By H2, V_1 is an interval on one side of x_0. By assumption $f'_0(x_0) < 0$, so if the $\{U_k\}$ are sufficiently small, $V_2 = f_0^{L+1}(U_L \setminus w)$ is an interval on the other side of x_0. That is, $V_1 \cup V_2 \cup \{x_0\}$ is a neighborhood of x_0.

For some sufficiently large J, $U_J \subset U_0$. In addition, $f^J(U_J) = U_0$. Thus for $0 \leq k \leq J$, U_k contains a periodic point. Thus there is a periodic point in U. Since U was arbitrary, there are periodic points p_j with period n_j such that $\lim_{j\to\infty} p_j = y$ and $\lim_{j\to\infty} n_j = \infty$. By H3, for j sufficiently large, the periodic points are hyperbolic. Therefore (y, λ_0) is not an explosion point. □

Now consider the positive derivative case. Let w be a homoclinic tangency point contained in homoclinic orbit $(z_{-k})_{k=0}^{\infty}$. Since the eigenvalue of x_0 is positive, there is an M sufficiently large that $(z_{-k})_{k=M}^{\infty}$ lies entirely on one side of x_0. That is, the homoclinic orbit converges to the fixed point along one branch of the unstable manifold. We denote this by saying that the homoclinic orbit is contained in the local right or left branch of x_0, as formalized in the following definition.

Definition 6. (Two branches of the unstable manifold) Let x_0 be a repelling fixed point for $f \in C^2$, with $f'(x_0) > 0$. The local left and right branches of the unstable manifold of x_0 are disjoint. Define U_{left} and U_{right} to be the respective unions of images of local left and right manifold branches.

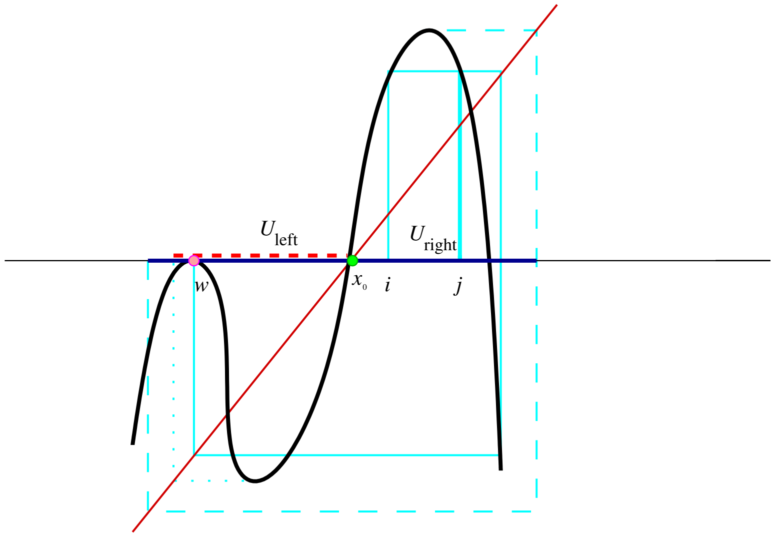

FIGURE 4. The tangency point w is contained in $U_{\text{left}} \cap U_{\text{right}}$, and is thus not an explosion point. However, the preimages i and j of w are explosion points.

Remark 6. The union of U_{left} and U_{right} is the entire unstable manifold of x_0. By the intermediate value theorem U_{left} and U_{right} are intervals. If $(U_{\text{left}} \cap U_{\text{right}}) \setminus x_0$ is not empty, then the intersection must contain either U_{left} or U_{right}. For example, U_{right} may contain points both to the left and to the right of U_{left}. See Figures 4 and 5.

From the proof of Theorem 3, it is clear that to study chain explosions in homoclinic orbits, it is sufficient to consider homoclinic orbits containing tangency points. We formalize the notation in the following hypothesis:

H6 For a family satisfying H1 and H4, at f_0, point w is a homoclinic tangency point to x_0 contained in (at least one) homoclinic orbit $(z_{-k})_{k=0}^{\infty}$. Let L be such that $w = z_{-L}$.

LEMMA 1. (No explosions when manifold branches intersect) *Assume H1–H6. Assume that $f_0'(x_0) > 0$ and that $w \in U_{\text{right}}$. If for any neighborhood $N \ni w$, the sets $f_0^L(N)$ and the local right branch of x_0 contain points in common, then (w, λ_0) is not an explosion point.*

Proof. The details of this proof are similar to the negative derivative case. Preimages of any small neighborhood N of w are contracting and in the local righthand manifold branch. Thus the Lth image of N includes a shrunk preimage of N. Therefore N contains a periodic point, which by H3 and H5 is hyperbolic. Since N is arbitrary, (w, λ_0) is not an explosion point. □

Remark 7. The theorem above is also true when U_{right} is replaced by U_{left}.

THEOREM 4. *Assume H1–H6, and that w is contained in $U_{\text{right}} \cap U_{\text{left}}$. Then (w, λ_0) is not an explosion point.*

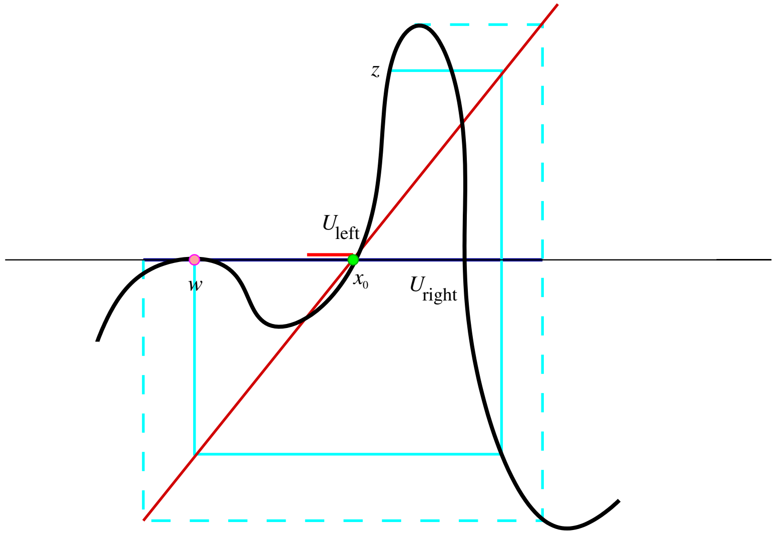

FIGURE 5. In this figure, w is a homoclinic tangency point contained in a crossing orbit. Thus w lies in U_{right}, but not U_{left}. By Theorem 6, w is an explosion point. As depicted here, the preimage z of w is also an explosion point.

Proof. Take a small neighborhood N of w. $f_0^L(N)$ is either a local left or a local right branch of x_0. Since w is contained in both U_{left} and U_{right}, there is a shrunk preimage of N contained in $f_0^L(N)$. Therefore by Lemma 1, (w, λ_0) is not an explosion point. □

As mentioned in the introduction, in our previous work we gave a useful geometric method of approaching chain explosions in homoclinic orbits in two and three dimensions, termed crossing cycles. In two dimensions, we showed that a crossing cycle is necessary and sufficient for a chain explosion to occur [**1**]. The analogous statements are true in one dimension, as in the theorem below. Crossing and non-crossing orbits are shown in Figures 5 and 2 respectively.

Definition 7. (Crossing orbits) Assume H1–H6. We call the homoclinic orbit (z_{-k}) a *crossing orbit* if for sufficiently large k, $z_{-k} \in U_{\text{left}} \setminus U_{\text{right}}$ (respectively $U_{\text{right}} \setminus U_{\text{left}}$), and for any interval M which is a local right (respectively left) branch of the unstable manifold to x_0, there is a sufficiently small neighborhood N of the tangency point z_{-L} such that $f_0^L(N)$ is contained in M. A homoclinic orbit that is not crossing is called a *non-crossing orbit*.

Remark 8. Note that there is a geometric interpretation for a crossing orbit: if a crossing orbit has a tangency point at $w = z_{-1}$, the definition is equivalent to saying that $z_{-k} \in U_{\text{left}} \setminus U_{\text{right}}$ (respectively $U_{\text{right}} \setminus U_{\text{left}}$), and f_0 is locally above (respectively below) the horizontal line at z_{-1}. If $w = z_{-L}$ for $L > 1$, then for $g = f_0^{L-1}$ there is a tangency point at z_{-1}, and one can draw the same geometric conclusion for g at z_{-1}.

THEOREM 5. (Explosions imply crossing orbits) *If H1–H6 hold and (w, λ_0) is an explosion point, then $(z_{-k})_{k=0}^{\infty}$ is a crossing orbit.*

Proof. If at $\lambda = \lambda_0$ there is a non-crossing homoclinic orbit containing w, then under f_0^L the image of a small neighborhood of w contains a shrunk preimage of itself. The result now follows from Lemma 1. □

The results stated so far give necessary conditions for an explosion point. We now give a converse to these.

The following theorem gives sufficient conditions for an explosion.

THEOREM 6. (Crossing bifurcations and explosions) *Assume H1–H6. If every homoclinic orbit containing w is a crossing orbit, then (w, λ_0) is an explosion point.*

Proof. Assume H1–H6 and that every orbit containing the homoclinic tangency point $w = z_{-L}$ is a crossing orbit. The point w is a homoclinic point for f_0, so w is chain recurrent for $\lambda = \lambda_0$. Any sufficiently small neighborhood N of w is such that $f_0^L(N)$ is contained in the local left (respectively right) branch of x_0. If $w \in U_{\text{left}} \cap U_{\text{right}}$, then there exist a set of preimages q_{-k} for $k > L$ of w converging to x_0 from the left (respectively right) such that the union of (q_{-k}) for $k > L$ and (z_{-k}) for $k \leq L$ forms a non-crossing orbit, contradicting our hypothesis. Therefore either $w \in U_{\text{right}} \setminus U_{\text{left}}$ or $w \in U_{\text{left}} \setminus U_{\text{right}}$. For specificity, let $w \in U_{\text{right}} \setminus U_{\text{left}}$. Since U_{right} is an interval, any neighborhood N of w contains points in U_{right} other than w. Choose N sufficiently small such that under f_0^L, the image of N is contained only in the local left branch of x_0. Note that U_{right} is forward invariant. Thus there are points in $f_0^L(N)$ contained in $U_{\text{left}} \cap U_{\text{right}} \setminus \{x_0\}$.

By Remark 6, since $U_{\text{left}} \cap U_{\text{right}}$ contains points other than x_0, either $U_{\text{left}} \subset U_{\text{right}}$ or $U_{\text{right}} \subset U_{\text{left}}$. Since w is contained in U_{right} but not in U_{left}, $U_{\text{left}} \subset U_{\text{right}}$, and the inclusion is strict. Thus the right endpoint of U_{left} is x_0. Furthermore, no points in the interior of U_{left} map to x_0 under f_0, since that would imply a second simultaneous homoclinic tangency, violating H3b.

Now consider the left endpoint x_L of U_{left}. Since U_{left} is forward invariant, $f_0(x_L) \in \overline{U_{\text{left}}}$. This leaves two possibilities.

(i) x_L maps into the interior of U_{left}. If this is the case, there is a neighborhood of x_L which also maps into U_{left}.

(ii) x_L is a fixed point. In this second case, since there is already a homoclinic tangency at w, H3 implies that x_L is a hyperbolic fixed point. We assume x_L is a repelling fixed point (to get a contradiction). Specifically, we will show in the next paragraph that in this case, there is a critical point in U_{left} mapping onto x_L. That means there is a heteroclinic tangency between x_0 and x_L. This is a contradiction, since w was assumed (H3) to be the only tangency point between stable and unstable manifolds. Therefore, we can conclude that if x_L is a fixed point, then it is an attracting fixed point.

Here is the proof that if x_L a repelling fixed point then there is a critical point in U_{left} mapping to x_L: if a point r in the interior of U_{left} maps onto x_L, then r is a critical point. Specifically, since U_{left} is forward invariant, a neighborhood of r maps to the local right branch of x_L. If no such r exists, then any interval of the form $(x_0 - a, x_0)$ contains a sequence of points c_k such that $\lim_{k\to\infty} f_0^k(c_k) = x_L$. The limit approaches x_L from the right since $x_L$is the left endpoint of a forward-invariant set. Since x_L is assumed to be a repelling fixed point, we know that there is an interval $(x_L, x_L + b)$

which is monotone increasing under f_0. This implies that for k sufficiently large, there is a minimum $j_k \geq 1$ such that $f_0^{j_k+1}(c_k) > f_0^{j_k}(c_k)$, and $f_0^{j_k}(c_k) \leq f_0^k(c_k)$. The latter condition implies that $\lim_{k\to\infty} f_0^{j_k}(c_k) = x_L$. Note that since j_k is the minimum such value, $f_0^{j_k-1}(c_k) > f_0^{j_k}(c_k)$. Therefore $f_0^{j_k-1}(c_k)$ is not in $(x_L, x_L + b)$. There is a convergent subsequence of $f_0^{j_k-1}(c_k)$. Call the limit point q. Then $f_0(q) = x_L$, but $q \neq x_L$. Thus there is a critical point in the interior of U_{left} mapping to x_L.

(iii) There appears to be a third possibility for x_L: $f_0(x_0) = x_0$. In the setting of this theorem, this third possibility cannot occur by the following argument: as in (ii), if a point r in the interior of U_{left} is such that $f_0(r) = x_L$, then r is a critical point. Since x_L is a homoclinic point, this is ruled out since it would be a simultaneous homoclinic tangency. Assume there is no such r. Since x_L is an endpoint of U_{left}, there is a sequence of points c_k in the interior of U_{left} such that $f_0(c_k)$ converges to x_L. A convergent subsequence of c_k exists, and has a limit point r such that $f_0(r) = x_L$. Unlike in (ii), x_L is not a fixed point, so no fancy argument is needed to show that $r \neq x_L$ (and $r \neq x_0$). Therefore r is in the interior of U_{left}, which contradicts our assumption. This implies that (iii) cannot happen.

So far we have shown that either (i) there is an infimum value of x_L in U_{left}, and x_L maps into the interior of U_L or (ii) x_L is an attracting fixed point. In either case, for λ close to λ_0, there is a continuation of the left endpoint of U_{left}, which we denote x_L^λ. In case (*i*), there is a continuation of the local infimum value x_I^λ. In case (ii), there is a continuation of the fixed point which we denote also by x_I^λ (since the two cases do not appear at the same time). The interior of U_{left}^λ for each λ is contained in the interval $[x_I^\lambda, x_0^\lambda]$. This is clear in case (i). In case (ii), x_L is attracting, and at λ_0 there are no critical points mapping to x_L from inside of U_{left}. Thus small perturbations of the map have the same property. Therefore for small $\epsilon > 0$, no ϵ-chains leave the interval $[x_I^\lambda, x_0^\lambda]$ to the left. In fact, since x_0 is repelling with no tangency points in U_{left}, there exists a $\delta_0 > 0$ such that for any sufficiently close λ, for all $\delta < \delta_0$, there is an $\epsilon > 0$ such that no ϵ-chains leave $[x_I^\lambda, x_0^\lambda - \delta]$.

We have assumed that w is a generic and unique homoclinic tangency point for f_0. Therefore for λ near λ_0 prior to tangency, $f_0^L(N)$ is contained in the interior of the local left branch and not in the local right branch of x_0. Thus for some $\delta < \delta_0$, $f_\lambda^L(N)$ is contained in $[x_I^\lambda, x_0^\lambda - \delta]$. This implies that there is an $\epsilon > 0$ such that no ϵ-chains carry points from N to itself. Thus (w, λ_0) is an explosion point. □

We are interested not only in explosion points which are themselves tangency points, but also in explosion points far from tangencies which are caused by tangencies. Theorem 3 contained such a result, but the other results were specifically about the tangency points. It is now straightforward to combine the previous results with Remark 4 to get results for general homoclinic points. Since the chain-recurrent set is invariant under forward iteration, the image of a non-crossing tangency point is a non-explosion point. However, there may be explosion points with iterates that are non-explosion points. For example, Figure 4 shows points of explosion which are not tangency points but are preimages of tangency points.

THEOREM 7. (Explosions when manifold branches do not intersect) *If H1–H6 hold, and there exists a* $k_* > 0$ *such that* (z_{-k_*}, λ_0) *is an explosion point, then the orbit* (z_{-k}) *is a crossing orbit.*

Proof. If the orbit is a non-crossing orbit, then there are hyperbolic periodic orbits limiting on every point in the orbit. □

THEOREM 8. *If H1–H6 hold, and* (w, λ_0) *is an explosion point, then any pre-image* z_{-k} *in the homoclinic orbit of* w *is such that* (z_{-k}, λ_0) *is an explosion point.*

Proof. If (w, λ_0) is an explosion point, then by Lemma 4, z_{-k} is not chain recurrent for $\lambda < \lambda_0$. At λ_0, z_{-k} is contained in a homoclinic orbit, and thus chain recurrent. □

Remark 9. This theorem only mentions preimages of w which are contained in a homoclinic orbit at λ_0. There may be other preimages of w which are not contained in any homoclinic orbit, and are thus not chain recurrent at λ_0.

Remark 10. Assume H1–H6, and that at λ_0 the tangency point w is contained exclusively in one manifold branch of x_0. Then at $\lambda = \lambda_0$, either tangency point $w \in U_{\text{right}}$, and U_{left} contains no tangencies to fixed points or periodic orbits, or tangency point $w \in U_{\text{left}}$, and U_{right} contains no tangencies to fixed points or periodic orbits. This follows from the fact that w is only contained in one of U_{left} and U_{right}, so by H3 there is no tangency in the other manifold branch.

THEOREM 9. *Assume H1–H6. Assume that for* $\lambda = \lambda_0$, $(z_{-k})_{k=0}^{\infty}$ *is a crossing orbit, but* w *is also contained in some other homoclinic orbit* $(\alpha_{-k})_{k=0}^{\infty}$ *which is a non-crossing orbit. Thus* (w, λ_0) *is not an explosion point. Then the following statements hold:*

(i) *for all* $m > 0$, $(f^m(w), \lambda_0)$ *is not an explosion point;*

(ii) *if there exists* $J > L$ *such that* z_{-J} *is contained neither in* $(\alpha_{-k})_{k=0}^{\infty}$ *nor in any other non-crossing orbit, then for all* $k > J$, (z_{-k}, λ_0) *is an explosion point.*

Proof. The first statement holds since at λ_0, any image of w is contained in $(\alpha_{-k})_{k=0}^{\infty}$, which is a non-crossing orbit. Thus by Theorem 7, this image is not an explosion point.

Let $J > L$ be as in Part (ii) of the theorem. For specificity, assume that $z_{-J} \in U_{\text{right}}$. As in the proof of Theorem 6, since z_{-J} is not contained in any non-crossing orbit, it is not contained in U_{left}. A neighborhood of z_{-J} maps into U_{left} under f_0^J. If the tangency point w is in $U_{\text{right}} \setminus U_{\text{left}}$, then the rest of this proof is exactly the same as the proof of Theorem 6, and we conclude that (z_{-J}, λ_0) is an explosion point.

If $w \in U_{\text{left}} \cap U_{\text{right}}$, then the penultimate forward orbit of w: (x_{-k}), $1 \le k \le L$ is contained in $U_{\text{left}} \cap U_{\text{right}} \setminus \{x_0\}$. Therefore, the options for x_L include the two options listed in the proof of Theorem 6, in addition to the third option that x_L is included in the forward orbit of w. In this case, for λ prior to tangency, the left endpoint of U_{left} moves such that it maps into the interior of U_{left}. Let x_I^{λ} denote the minimum preimage of the continuation of the critical point of w in this third case.

Whether or not x_L is in in the forward orbit of w, for all λ sufficiently near λ_0 prior to tangency, there exists a $\delta_\lambda > 0$ such that for all $\delta < \delta_\lambda$, there is an $\epsilon > 0$ such that no ϵ-chain leaves $[x_I^{\lambda}, x_0^{\lambda} - \delta]$. The rest of the proof that z_{-J} is an explosion point is the same as in Theorem 6. For all $k > J$, z_{-k} has all the same key properties as z_{-J}, implying that (z_{-k}, λ_0) is an explosion point. □

This completes the classification of when homoclinic points are explosion points.

Unlike the planar case, in one dimension the heteroclinic case reduces to the homoclinic case. That is, if there is a transverse heteroclinic cycle including an orbit from $\{p\}$ to $\{q\}$, which are hyperbolic periodic orbits, then both periodic orbits must be repellers. Further, the unstable manifold of $\{p\}$ contains the unstable manifold of $\{q\}$. Since we assume only one tangency, a heteroclinic tangency point is also a homoclinic tangency point.

4. *General explosion classification*

There are many previous results on the structure of ω-limit sets and chain-recurrent sets for interval maps. Block and Coppel [**5**] showed that the chain-recurrent set for maps of the interval can be classified as the set of points $\{x \mid x \in Q(x)\}$, where $Q(x)$ is the intersection of all asymptotically stable sets containing the limit set $\omega(x)$. They showed that $Q(x)$ is either an asymptotically stable periodic orbit, a set of asymptotically stable iterated intervals, or a special type of set known as a solenoidal set. However, this classification is not as useful as it would appear, since $Q(x)$ is not the set of points in the chain component containing x. Block [**4**] also proved that an interval map has a homoclinic point if and only if it has a periodic point with period not a power of two. Block and Hart [**6**] improved on this result to show the existence of a homoclinic point to a given power of two implies a cascade of homoclinic bifurcations. Further, if a family of maps changes from zero to positive entropy, then there is a cascade of homoclinic bifurcations. Of most relevance to the topic of this paper are the works of Sarkovskii, Mañé [**16**], Blokh [**7, 9**], and Blokh, Bruckner, Humke, and Smítal [**8**], where the detailed structure of all possible ω-limit sets is studied. The ω-limit set can be a Cantor set known as a basic set (Definition 9). Another interesting case occurs when the ω-limit set of a point is a limit of a period-doubling cascade, known as a solenoid (Definition 10). See [**3, 15**] for a detailed characterization of solenoids. Blokh [**9**] showed that for C^2-smooth maps, ω-limit sets are either periodic orbits, periodic transitive intervals, subsets of basic sets, or solenoids. We use this result to systematically show that for all possible explosions, there are saddle-node or tangency bifurcations.

The following definition describes points at which jumps in an ϵ-chain are required. We call these points barricades, as they serve to obstruct orbits. For example, at the bifurcation parameter, a saddle-node point blocks the points on one side from reaching the other side.

Definition 8. (Barricade) Assume H1 and H2. Let z be any point. Let $S_\epsilon = \omega(B_\epsilon(z))$. A point $y \in \lim_{\epsilon\to 0} S_\epsilon$ is called a *barricade* for z if it is blocking the orbit of z. That is, let $Z_\epsilon = \omega(B_\epsilon(y))$. Then $\lim_{\epsilon\to 0} Z_\epsilon$ contains points not contained in $\lim_{\epsilon\to 0} S_\epsilon$.

We consider a point such that the ω-limit set is a fixed point or periodic orbit.

THEOREM 10. *Assume H1–H3. Assume that for f_0, z is a point such that $\omega(z)$ is equal to a fixed point or periodic orbit $\{p\}$, and $\{p\}$ is a barricade. Then $\{p\}$ is either non-hyperbolic or is the image of a critical point.*

Proof. Assume that p is a fixed point, since otherwise we can let $g = f_0^n$. Since p is a barricade, $p \in S_\epsilon$, and $\lim_{\epsilon\to 0} Z_\epsilon \neq \lim_{\epsilon\to 0} S_\epsilon$ (using the notation from Definition 8). This implies that p cannot be an attracting fixed point, since for an attracting fixed point $\lim_{\epsilon\to 0} Z_\epsilon = p$. Thus if p is hyperbolic, it must be repelling, meaning that z is a preimage of p. Define K by $f^K(z) = p$. Since p is a barricade, for small ϵ, $f^K(B_\epsilon(z))$ is an interval on one side of p, implying that p is in the orbit of a critical point. □

The previous theorem only indicates that a critical point exists. It still remains to be shown that the critical point is actually a homoclinic or heteroclinic point. We use the fact that all ω-limit sets for interval maps have been classified. The following theorem is useful in the proof of several subsequent theorems.

THEOREM 11. *Assume H1–H3, and that M is an invariant interval under f_0, such that for all $\epsilon > 0$, there is an ϵ-chain from a point $x \in M$ to a point $y \notin M$. Then there is an ϵ-chain from an endpoint e of M to y, where e is fixed or period two. Furthermore, if x is not an endpoint of M, then either e is non-hyperbolic; or e is repelling, and there is an orbit of a critical point in M mapping onto e.*

Proof. Since $f(M) = M$, the only way for an ϵ-chain to exit M is through an ϵ-jump across one of the endpoints. Call the endpoint e. Thus there is a chain from e to y. Assume e does not map to itself or to the other endpoint of M. Then $f(e)$ is contained in the interior of M. But this means that no small ϵ-jump at e exits M, which is a contradiction. Thus e can be chosen to be either a fixed point or a period two point.

The orbit of e cannot be attracting, since then there would not be ϵ-chains from e to any other point. If the orbit of e is hyperbolic, then it is repelling, and there is a chain from x to e. If the orbit of x includes e, and x is not an endpoint of M, then the orbit of x contains a critical point, since M is invariant. If the orbit of x does not include e, then the limit set of x contains more than just a repelling periodic orbit. □

The following result shows what happens if the backward limit set of a point contains a periodic orbit.

THEOREM 12. *Assume H1–H3. Let $\{p\}$ be a fixed point or periodic orbit for f_0 which is hyperbolic. Let z be a point such that for all $\epsilon > 0$ there is an ϵ-chain from $\{p\}$ to z, but z is not in the unstable manifold of $\{p\}$. Then there is a homoclinic tangency to a periodic orbit.*

Proof. Assume without loss of generality that p is a fixed point (since otherwise, we can use the same proof for an iterate of f_0). Since p is hyperbolic, it is repelling. The unstable manifold of p is an invariant interval, denoted $U(p)$. Since $z \notin U(p)$, there is a barricade point for p. By Theorem 11, there must be a barricade point which is a periodic endpoint of $U(p)$, with a critical point in $U(p)$ mapping to the endpoint. That is, there is a homoclinic tangency point for the periodic endpoint. □

Combining Theorems 10 and 12, we conclude that if an explosion occurs at a point (z, λ_0) such that $\omega(z)$ is equal to a periodic orbit, then there is either a saddle-node bifurcation point or a tangency between stable and unstable manifolds. We now consider more general ω-limit sets. First consider the case where the ω-limit set of a point is an interval.

THEOREM 13. *Assume H1–H3, for f_0, z is such that $\omega(z)$ is an interval M, and that there exists $x \in M$ such that e is a barricade for z. Then e is an endpoint of the interval, e is fixed or period two, and if e is hyperbolic, then there is a homoclinic tangency to e.*

Proof. Since $\omega(z) = M$, f_0 is transitive on M. Therefore the unstable manifold of any repelling periodic orbit in M contains all of M. A barricade must not be in the interior of $\omega(z)$. Thus e is an endpoint of M. The result now follows from Theorem 11. □

We now consider the case of an ω-limit set that is contained in an invariant interval but is nowhere dense.

Definition 9. (Basic set [**7**]) Assume H1. Let $M = \bigcup_{k=1}^{n} I_k$ be an n-periodic cycle of intervals for a function f_0. Define $B(M, f_0) = \{x \in M \mid$ for every relative neighborhood U of x in M, $\overline{\mathrm{orb}(U)} = M\}$. If $B(M, f_0)$ is infinite, then it is called a basic set.

THEOREM 14. *Assume H1, H2, and H3a. Assume that* (z, λ_0) *is an explosion point, and* $\omega(z)$ *is nowhere dense and is contained in a basic set* $B(M, f_0)$. *Then* $\omega(z)$ *is a periodic orbit.*

Proof. Assume that (z, λ_0) is an explosion point, and that $\omega(z) \subset B(M, f_0)$. Assume that z is contained in an interval complementary to $B(M, f_0)$.

Blokh [**7**] considers the class of interval maps such that if I is a wandering interval, then $\omega(I)$ is a periodic orbit. This class of maps includes smooth interval maps [**7**] with non-flat critical points [**18**]. Under the assumption that $\omega(z)$ is a nowhere dense set contained in a basic set $B(M, f_0)$, [**7**, Property C] shows that if a point z is not contained in $B(M, f_0)$ but $\omega(z)$ is, then $\omega(z)$ is a periodic orbit for f_0.

Assume $\omega(z)$ is non-periodic, meaning z is contained in the basic set. By [**9**], $B(M, f_0)$ is contained in the closure of the periodic orbits for f_0. Using H3a, there is a sequence of hyperbolic periodic points converging to z. Thus (z, λ_0) is not an explosion point. □

By the above theorem combined with Theorems 10 and 12, if an explosion point has an ω-limit set which is a basic set, then there is either a saddle-node point or a tangency. The last possibility for an ω-limit set is a solenoid, as in the following definition.

Definition 10. (Solenoid) Assume H1. Let $M_j = \bigcup_{k=1}^{n_j} I_k^j$ be a nested sequence of cycles of intervals for a function f_0 with least period n_j, $\lim_{j\to\infty} n_j = \infty$. Thus $f_0^{n_j}(I_1^j) = I_1^j$, n_j is increasing, and for each j, $I_1^{j+1} \subset I_1^j$. If the set $S = \bigcap_{j=1}^{\infty} M_j$ is nowhere dense, then S is called a solenoid or Feigenbaum-like set.

Jiménez López has shown that solenoids are the boundary of chaos and order [**15**]. Blokh [**9**] demonstrated that solenoids and basic sets are disjoint. We prove the following result.

THEOREM 15. *Assume H1, H2, and H3a. Assume that* (z, λ_0) *is an explosion point, and* $\omega(z)$ *is a solenoid* S. *Then there is an infinite sequence of periodic orbits which are barricades with associated tangencies.*

Proof. Since solenoids and basic sets are closed, invariant, and disjoint, there is a neighborhood of solenoid S containing no basic sets. By definition, there is an infinite nested sequence of invariant cycles of intervals M_j containing S. As before, for interval maps such that a non-wandering interval I has $\omega(I)$ periodic, Blokh [**7**] proved that the periodic orbits are dense in a neighborhood of S, meaning that z is not contained in S. Thus for all j sufficiently large, for all $\epsilon > 0$ there is an ϵ-chain from points in M_j to z, but z is not contained in M_j. By Theorem 11, there exists e_j, an endpoint of an interval the cycle of M_j which is periodic. By hypothesis H3a, for sufficiently large j, e_j is hyperbolic. Since for all $\epsilon > 0$ there is an ϵ-chain from e_j to z, e_j is a repeller for large j. There is an orbit of a critical point in M_j mapping onto e_j. Furthermore, z is not contained in the unstable manifold of e_j, since there is a nested sequence of invariant M_j not containing z. Theorem 12 implies that the critical point to e_j is a point of homoclinic tangency. □

Remark 11. If f_0 has a finite number of homoclinic and heteroclinic tangencies, as assumed in H3b, then the above theorem shows that there are no forward chains from solenoid S to a point outside of S.

Remark 12. We have shown that there is either a tangency or a non-hyperbolic critical point contained in the same chain component as z. Under a generic hypothesis (H3a), a non-hyperbolic periodic orbit is either codimension-one saddle-node or period-doubling bifurcation. In fact, such an orbit is not a period-doubling point, since the periodic orbit at a period-doubling bifurcation point is attracting.

We now combine the results of this section to give a proof of the General explosion classification theorem.

Proof of the General explosion classification theorem. Assume that (z, λ_0) is an explosion point. The only possibilities for $\omega(z)$ are a periodic orbit, a cycle of intervals, a nowhere-dense basic set, and a solenoid. Above, we have shown that in any of these cases, there is a periodic barricade point for z which is either non-hyperbolic, or there is a homoclinic or heteroclinic tangency. In fact, the case of a solenoid is ruled out by H3b. By Remark 12, a non-hyperbolic periodic orbit must necessarily be a saddle-node point. This completes the proof the theorem. □

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