# Math 413 – Exam 2 – Wednesday, November 29

Name: SOLUTION

Directions: Show calculation steps if you want partial credit. There are six problems, two worth 10 points and four each worth 20 points. Use this paper for your answers and work.

1. (10 points) In creating our model for an elastic bar, the operator $A$ became the derivative $\frac{d}{dx}$. Explain what $A^T$ is and why that is reasonable.

$A^T = -\frac{d}{dx}$ This follows from our "dot product" and integration by parts:

If $v, u$ are given, $\int_a^b u' v \, dx = \int_0^1 (Au) \, v \, dx$

$= uv\Big|_a^b - \int_a^b u v' \, dx = \underset{\substack{\uparrow \\ \text{if B.C.} \\ \text{are good}}}{0} + \int_0^1 (A^T v) \, u \, dx$

2. (10 points) In 2-D, the operator $A$ became the gradient. Explain what $A^T$ is and why that is reasonable. For the domain, you may use a rectangle in $x - y$.

Here $u$ is scalar, $\vec{w}$ is $\underline{\text{vector}}$

$Au = \nabla u$
$A^T = -\underline{\text{div}}$

$$\iint \vec{w} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} dx\,dy = \iint \left( w_1 \frac{\partial u}{\partial x} + w_2 \frac{\partial u}{\partial y} \right) dx\,dy$$

after integration by parts in $x$ or $y$ (with boundary terms $= 0$)

we get $\iint \left( -\frac{\partial w_1}{\partial x} u - \frac{\partial w_2}{\partial y} u \right) dx\,dy$

$= \iint \left( -\text{div}\, \vec{w} \right) u \, dx\,dy$

3. (20 points) Consider an elastic bar with $f(x) = \frac{3}{2} - x$ and $c(x) = 1 - (x/2)$. Find the displacement using the usual equations and boundary conditions $u(0) = 0$, $w(1) = 0$.

usual eq'ns $-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f$

so $-\frac{dw}{dx} = f \Rightarrow w(x) = \int_x^1 \left(\frac{3}{2} - t\right) dt = \frac{3}{2}(1-x) - \frac{1}{2}(1-x^2)$

$= (1-x)\left[\frac{3}{2} - \frac{1}{2}(1+x)\right] = (1-x)\left(1-\frac{1}{2}x\right)$

So $c(x)\frac{du}{dx} = w(x) = (1-x)\left(1-\frac{1}{2}x\right)$ implies $\frac{du}{dx} = 1-x$

Then $u = x - x^2/2$ using $u(0) = 0$.

4. (20 points) A beam with fixed ends has boundary conditions $u(0) = 0$, $u'(0) = 0$, $u(1) = 0$, $u'(1) = 0$. Use $c = 1$ for the bending stiffness. With applied force $f = 1$, solve for the displacement $u(x)$.

$$u'''' = f = 1 \Rightarrow u(x) = A + Bx + Cx^2 + Dx^3 + \frac{x^4}{4!}$$

From $u(0) = u'(0) = 0$ we get $A = 0$, $B = 0$

So $u(x) = Cx^2 + Dx^3 + x^4/4!$

$$\left.\begin{aligned} u(1) &= C + D + 1/4! = 0 \\ u'(1) &= 2C + 3D + 1/3! = 0 \end{aligned}\right\} \text{2 eq's for } C, D$$

use elimination

$$C + D = -1/24 \rightarrow 3C + 3D = -3/24 \Rightarrow C = -3/24 + 1/6 = \frac{1}{24}$$

$$2C + 3D = -1/6 \qquad D = -2/24 = -\frac{1}{12}$$

solution $u(x) = \frac{1}{24}x^2 - \frac{1}{12}x^3 + \frac{1}{24}x^4$

or $\frac{1}{24}x^2(1-x)^2$

5. (20 points) For $P(u)$ given below, find the Euler-Lagrange equation (conditions for minimizer), explaining your work:

$$P(u) = \int_0^1 [\frac{1}{2}(\frac{du}{dx})^2 + \cos(x)u(x)]\,dx$$

Assume $u(0) = 0$ and $u(1) = 0$.

$$\frac{d}{dt} P(u+tv)\Big|_{t=0} = \left\langle \frac{\delta P}{\delta u}, v \right\rangle \leftarrow \text{"dot product"}$$

Direct calculation: $\frac{d}{dt} P(u+tv)\Big|_{t=0} = \int_0^1 \left[\frac{du}{dx}\frac{dv}{dx} + \cos x \cdot v(x)\right] dx$

$$= \int_0^1 \left(-\frac{d^2u}{dx^2} + \cos x\right) v(x)\, dx$$

after integration by parts. Thus Euler-Lagrange eqn is:

$$\frac{\delta P}{\delta u} = -\frac{d^2u}{dx^2} + \cos x = 0 \quad \text{with } u(0) = u(1) = 0$$

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Some of you solved this — nice but unnecessary.

6. (20 points) Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ satisfies Laplace's equation. Then from the gradient field $\vec{w} = u_x\vec{i} + u_y\vec{j}$, find the stream function $s(x,y)$, recalling that this means writing the components of $\vec{w}$ as $\frac{\partial s}{\partial y}$ and $-\frac{\partial s}{\partial x}$ respectively.

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3, \quad \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and then using s:

Laplace's equation

$$\frac{\partial s}{\partial y} = 4x^3 - 12xy^2 \qquad -\frac{\partial s}{\partial x} = -12x^2y + 4y^3$$

Integrating one or other, then substituting we find

$$s(x,y) = 4x^3y - 4xy^3 + C$$