THE METHOD OF ARCHIMEDES*

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The works of Archimedes have come down to us in two streams of tradition, one of them continuous, the other broken by a gap of a thousand years between the tenth century and the year 1906, when the discovery of a manuscript in Constantinople brought to light an important work called the *Method*, on the subject of integration.

Newton and his contemporaries in the seventeenth century were much puzzled by one aspect of the integrations to be found in the continuous tradition. In the books on the *Sphere and Cylinder*, for example, it is clear that the somewhat complicated method employed there for finding the volume of a sphere represents merely a rigorous proof of the correctness of the result and gives no indication how Archimedes was led to it originally. The discovery of 1906 removes the veil, at least to some extent.

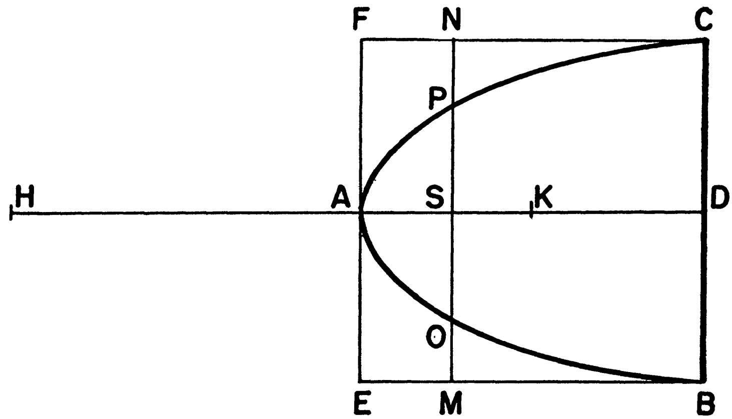

FIG. 1

The newly discovered *Method* consists of imagining the desired volume as cut up into a very large number of thin parallel slices or discs, which are then suspended at one end of an imaginary lever in such a way that they are in equilibrium with a solid whose volume and center of gravity are known. Thus, in Proposition 4 of the *Method*, Archimedes shows that the volume of a paraboloid of revolution is one-half of the volume of the circumscribing cylinder by slicing the two solids (see Figure 1 which represents a plane section through their common axis AD) at right angles to AD. For let us take HAD to be the bar of a balance with $HA = AD$ and with the fulcrum at A, and imagine the circle PO to be removed from the paraboloid and suspended at H. Since $AD/AS = DB^2/SO^2$ in the parabola BAC, we have

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$$\frac{HA}{AS} = \frac{AD}{AS} = \frac{MS^2}{SO^2} = \frac{\text{(circle in cylinder)}}{\text{(circle in paraboloid)}},$$

so that, by the law of the lever, the circle in the cylinder, remaining where it is, is in equilibrium with the circle from the paraboloid resting in its new position. If we deal in the same way with all the circles making up the paraboloid, we find that the cylinder, resting where it is with its center of gravity at the midpoint K of AD, is in equilibrium about A with the paraboloid placed with its center of gravity at H. Since $HA = AD = 2AK$, the volume of the paraboloid is therefore one-half of that of the cylinder, as desired.

Many accounts of the *Method* have been given since its discovery in 1906; for example, by T. L. Heath in his *Supplement to the Works of Archimedes*, Cambridge, 1912. In all of them, as in the original work of Archimedes himself, we are invited to *imagine* the lever and the objects suspended from it. But if we construct an *actual* lever and *actual* discs, the various figures, which may be spheres, cones, *etc.*, see below, will be observed to balance, slice by slice, as successive slices are added. The whole procedure then becomes a picturesque and effective illustration of the concept of an integral as the limit of a sum.

To find the volume of a sphere, a problem which Archimedes considered so important that he asked to have the result engraved on his tombstone, a cone and a sphere are together weighed against a cylinder (see Figure 2 and the accompanying sketch). Here the circle NM, resting where it is in the large cylinder $GLEF$ is in equilibrium about A with two circles placed at H, the one circle PO being taken from the given sphere and the other RQ from the cone FAE. For we have

$$OS^2 + QS^2 = OS^2 + AS^2 = AO^2 = CA \cdot AS = MS \cdot SQ$$

and therefore

$$\frac{HA}{AS} = \frac{MS}{SQ} = \frac{MS^2}{MS \cdot SQ} = \frac{MS^2}{OS^2 + QS^2}.$$

Thus, by the law of the lever as before,

one-half of cylinder equals cone plus sphere

from which, since the cone is one-third of the cylinder,

sphere equals one-sixth cylinder.

Thus the cylinder circumscribed about the sphere, being one-quarter as great as the large cylinder $GLEF$, is three-halves as great as the sphere, which is the result stated on the tombstone of Archimedes.

If squares are substituted for the circles of cross-section in these figures, the argument remains unchanged and we have the solution of another famous problem (Proposition 15 in the *Method*), namely to find the volume common to

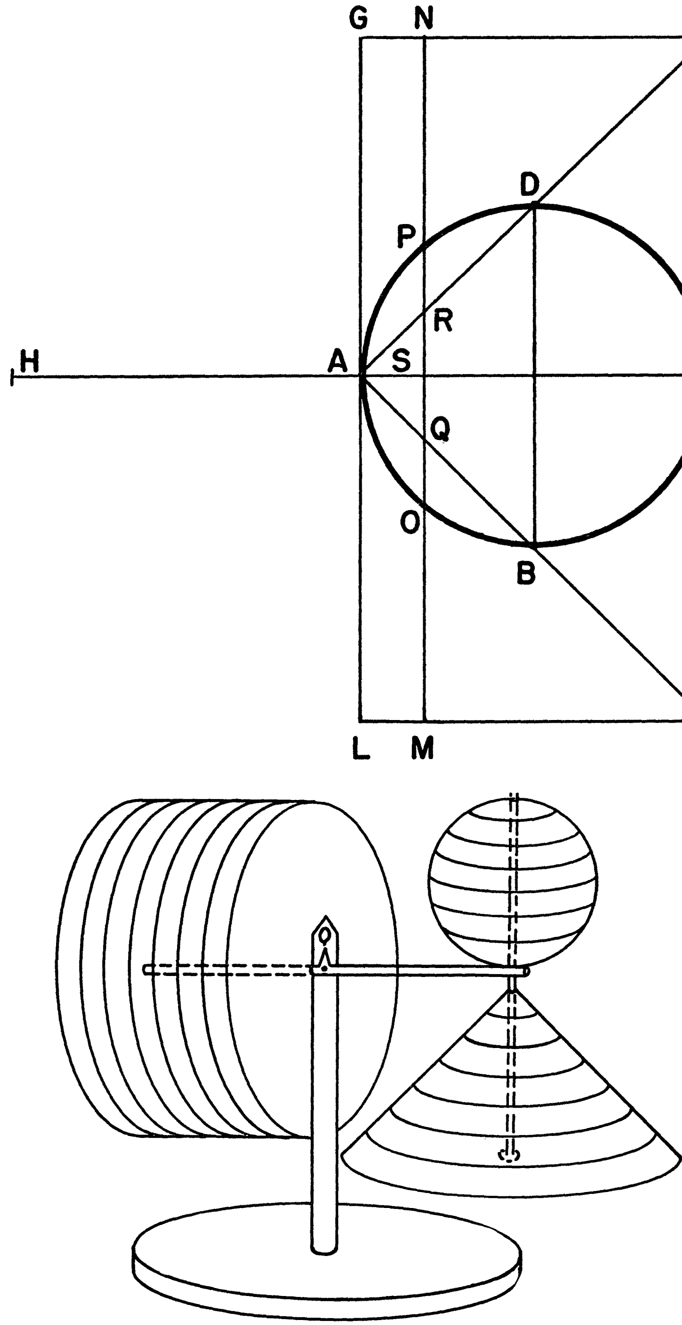

FIG. 2

two right circular cylinders intersecting at right angles.

The actual models were constructed by D. A. Eberle of the Psychology Workshop at Purdue University. The various slices were cut from a piece of white pine 1/2″ thick and 7″ wide. Thus the cylinder *GLEF* is composed of seven slices, each with a diameter of 7″. The seven slices for the cone, being first cut as stepwise increasing cylindrical discs with easily calculated radii, were placed all together on a mandrel passing through a 3/16″ hole through their centers and were then shaped down on a lathe, a procedure found to be especially necessary for the square cross-sections in the problem of the intersecting cylinders. The lever itself is a piece of steel 9″ by 1/2″ by 1/32″, placed so that its 1/2″ face is vertical. In each disc a thin slit was cut with a fine hacksaw from edge to center so that the disc could be slipped onto the lever.

SIMPLE DEVICES FOR EFFICIENCY IN THE ELEMENTARY THEORY OF EQUATIONS

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1. Introduction. The instructor who attempts to teach his students to find the "best" method for solving each problem frequently finds that the process of producing several solutions, from which the "best" is to be chosen, tends to stimulate ingenuity and produce a habit of efficiency in most students. Not infrequently, it is necessary to consider whether the author has indicated the "best" procedure in the directions given, or whether the ultimate aim—such as finding all of the roots of a given equation—can be more readily attained by disregarding the author's directions involving some specific method—such as testing for rational roots. In such cases shall we disregard the directions and obtain the results more efficiently; shall we consider the problem satisfactorily worked only when we have followed the author's directions; or shall we work the problem efficiently and then follow the author's instructions? The answer probably depends upon how frequently such directions are given, and whether or not there are enough problems remaining to teach the students the desired standard techniques.

We shall consider here the "best" methods for working some of the college algebra problems ordinarily contained in the chapter on theory of equations, and the frequency with which these methods can be applied to problems in certain text books.*

Before entering upon the main points of discussion the author would like to say that no special method was used in selecting the books referred to. Those at hand were merely used to decide whether or not the devices to be given here

* These books are listed at the end of this article and referred to by number.