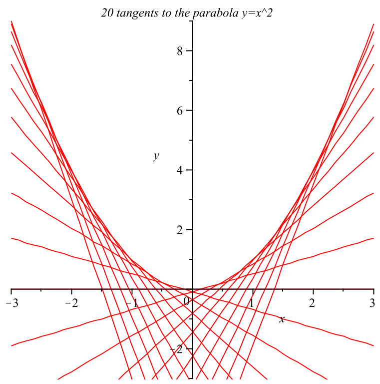
INTEGRATION – Chapter 3 – Envelopes of tangent lines and tangent planes

An interesting use of partial derivatives is to describe the graph of a function of one variable in terms of the family of all its tangent lines. We know that knowing the derivative everywhere symbolically describes the function up to a constant, but here there is a corresponding picture of the family of tangent lines. This generalizes in higher dimensions to a function of two variables in terms of the family of all tangent planes to the graph. The mathematical term is **envelope** and the mathematical lineage goes back at least to Clairaut.

Here is a picture of some of the tangent lines to the parabola $y = x^2$, in which you can visualize the parabola graph as being at the edge of the 'shading' coming from the family of lines.



To find the parabola from the family of tangent lines algebraically, view the point of tangency as a new variable, say a. We know the tangent line at $x = a$ has equation $y = a^2 + 2a(x - a)$. The parabola lies next to all the tangent lines, so at some value of a, the points on the parabola are generated by the variation in a. Algebraically, we require the simultaneous solution of the equations $y = F(x, a)$ and $0 = \frac{\partial}{\partial a}F(x, a)$ to express the envelope as a function of x alone, by eliminating the parameter a. For the parabola, this becomes:

$$y = a^2 + 2a(x - a), \; 0 = 2a + 2(x - a) - 2a$$

so we find $a = x$ and therefore $y = x^2$ is recovered!

Project: First consider the case of the cubic $y = x^3$ and perform a similar analysis: create the family of tangent lines, then use partial derivatives to recover the function itself. Try plotting the tangent lines to see how the graph emerges. Also plot in 3-D the family and try to visualize the projection into the $x - y$ plane.

Secondly, do the general case for a differentiable function f and describe what condition you needed to eliminate the tangency point $x = a$. Link that to the geometry of the tangent lines.

Third do a corresponding analysis for the two parameter family of tangent planes to the paraboloid:

$$z = x^2 + y^2, \text{ has tangent planes } z = a^2 + b^2 + 2a(x - a) + 2b(y - b).$$

Setting the a and b partial derivatives to zero, eliminate a and b to find the paraboloid itself.

Extensions: Consider the second order Taylor approximations, to find a function as the envelope of parabolas (don't do the parabola itself here!).

Another extension is to view a space curve as the envelope of the osculating circles (refer back to section on curvature).

Further reading: See various sources on the geometry of ruled surfaces (surface swept out by lines, as our 3-D graph of all tangent lines including the parameter as variable), and also for Clairaut equations (ordinary and partial differential equations).