5.8 Introduction to Differential Forms

Overview: The language of differential forms puts all the theorems of this Chapter along with several earlier topics in a handy single framework. The introduction here is brief. In differential forms, all the fundamental theorems are known as Stokes' theorem.

Differential forms are a useful way to summarize all the fundamental theorems in this Chapter and the discussion in Chapter 3 about the range of the gradient and curl operators, as well as the integration theory on manifolds of lower dimension. They were formalized by E. Cartan, based on earlier work of Poincare and others (see the article by V. Katz for some history). Clearly from the name, differential forms are linked to differentials, so let's recall our discussion from Chapter 3 on differentials.

When $f(x, y, z)$ is differentiable, we defined the differential df as the linear combination $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$ which means that the gradient vector ∇f is the vector whose dot product with the vector differential $d\mathbf{r}$ gives df. Objects involving linear combinations of single differentials dx, dy, dz are called *differential 1-forms* if the coefficients multiplying the differentials are themselves differentiable functions. More formally, the following defines a differentiable 1-form in 3-D, with coordinates x, y, z:

Definition 5.17 *A one-form ω is an expression $\omega = a(x, y, z)\,dx + b(x, y, z)\,dy + c(x, y, z)\,dz$. We will usually assume the coefficients a, b, c are themselves differentiable functions.*

The transformation from a differentiable function f to its differential df maps functions to 1-forms and is linear. This suggests that we consider "d" as a linear operator from scalar functions to 1-forms. A slightly strange convention will regard differentiable functions as "zero forms".

Since 1-forms can be integrated as line integrals over curves to obtain numbers, we can now reinterpret all the line integral computations earlier in this chapter as the integration of 1-forms along oriented curves.

Notation: The integral of a 1-form ω along an oriented curve γ is written $\int_\gamma \omega$.

The "product" of differentials that was used in two dimensional integrals involved objects like $dx\,dy$ and $dy\,dz$ will now be viewed as *differential 2-forms*. In double and triple integrals expressions, the order was reversible, but in the language of differential forms these products become skew-symmetric, like the vector cross product, so reversing order creates a minus sign.

For the usual coordinates x, y, z in three dimensions, the two forms $dx\,dy$, $dy\,dz$ and $dz\,dx$ are viewed as having positive orientation, the reversed orders are viewed as negative orientation, and the pairing of identical differentials, such as $dx\,dx$, which never arose in integrations, will be declared to be 0. Most users of differential forms record this by using a "wedge product" notation to convey the orientation: they write $dx \wedge dy$ and so on, with the property that $dy \wedge dx = -dx \wedge dy$. Some books leave out the wedge symbol. This should be viewed as representing our favorite oriented infinitesimal parallelogram.

More formally, the following defines a differentiable 2-form in 3-D, with coordinates x, y, z:

Definition 5.18 *A two-form is an expression* $a(x,y,z)\,dx \wedge dy + b(x,y,z)\,dy \wedge dz + c(x,y,z)\,dz \wedge dx$. *We will usually assume the coefficients* a, b, c *are themselves differentiable functions.*

Wedge product of one-forms: Algebraic rules then tell us how to form the **exterior or wedge product of any pair of 1-forms**:

$$(A\,dx + B\,dy + C\,dz) \wedge (P\,dx + Q\,dy + R\,dz)$$

using linearity and the above rules, along with the facts $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. This yields the product to be 2-form:

$$(AQ - BP)\,dx \wedge dy + (BR - CQ)\,dy \wedge dz + (CP - AR)\,dz \wedge dx.$$

The slightly skewed convention that arranges the three terms in our 2-form in the order of the missing third differential, so the order is $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$, lets the coefficients of the wedge product be the cross product of the vectors $< A, B, C >$ and $< P, Q, R >$! Our bookkeeping device is already handy, but there is more to come.

d **for 1-forms:** We define the (extended) exterior derivative d as mapping 1-forms to 2-forms as follows:

$$d(dx) = d(dy) = d(dz) = 0,$$

and in general,

$$d(P\,dx + Q\,dy + R\,dz) = dP \wedge dx + dQ \wedge dy + dR \wedge dz.$$

Now one might guess that this would be linked to the curl of the vector field $\mathbf{v}$, which would have components P, Q, R. The rule about $d(dx) = 0$ and similarly in y and z leads to "non-diagonal" terms. To confirm that claim, look at the term with P and then use a usual permutation argument:

$$dP \wedge dx = (\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz) \wedge dx$$

leading to terms $\frac{\partial P}{\partial z}\,dz \wedge dx - \frac{\partial P}{\partial y}\,dx \wedge dy$. After the cycling, arranging terms in the usual order of the three basic two-forms leads to

$$d(P\,dx+Q\,dy+R\,dz) = (\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\,dy\wedge dz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\,dz\wedge dx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\,dx\wedge dy$$

where the "rules" take care of the various minus signs.

Two forms can be integrated over surfaces. As in regular surface integrals, it is best to view these as parameterized surfaces and then convert everything over to the parameter plane. This means writing differentials naturally and performing the wedge products back in the parameter plane. Mathematicians call this the "pull-back" of the differential form. We will not belabor this but illustrate one such substitution now and describe the general procedure once we consider 3-forms.

Example 5.19 *Use the usual parameters for the unit sphere in 3-D to express the two-form $dx \wedge dy$ in terms of ϕ and θ:*

Since $x = \sin\phi\,\cos\theta,\ y = \sin\phi\,\sin\theta,$ the algebra of differentials leads to a direct calculation:

$$\begin{aligned} dx \wedge dy &= (\cos\phi\,\cos\theta\,d\phi - \sin\phi\,\sin\theta\,d\theta) \wedge (\cos\phi\,\sin\theta\,d\phi + \sin\phi\,\cos\theta\,d\theta) \\ &= \cos\phi\,\sin\phi(\cos^2\theta + \sin^2\theta)\,d\phi \wedge d\theta = \cos\phi\,\sin\phi\,d\phi \wedge d\theta \end{aligned}$$

Another stage of extension defines d for 2-forms, with the result a 3-form. In 3-D, there is only one basic 3-form: $dx \wedge dy \wedge dz$, and all 3-forms are of the form: $g(x, y, z)dx \wedge dy \wedge dz$ for some function g. The minus sign convention for single swaps forces cyclic versions like $dy \wedge dz \wedge dx$ to be the same. There is also an extension of the wedge product for say a 1-form with a 2-form to create a 3-form or the wedge product of three 1-forms, which is done by using the basic differentials and their wedge multiplication along with linearity and ordinary multiplication of functions.

The exterior derivative of a two-form is defined similarly to the derivative of a 1-form, namely:

$$d(P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy) = dP\wedge dy\wedge dz+dQ\wedge dz\wedge dx+dR\wedge dx\wedge dy).$$

But now this simplifies to a single 3-form (can you guess which?):

$$d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})\,dx \wedge dy \wedge dz$$

so this captures the divergence!

Example 5.20 *Find the wedge product of a general 1-form $A\,dx + B\,dy + C\,dz$ with a general 2-form $P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$.*

$$(A\,dx+B\,dy+C\,dz)\wedge(P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy) = (AP+BQ+CR)dx\wedge dy\wedge dz$$

after noting that the six other terms have repeated differentials and are therefore each 0 and that these three generate the same basic 3-form after cycling the terms.

The above example captures the dot product back!

More fun comes from noting that our result $d(dx) = 0$ and similarly in y and z will extend to the result that for any function f (a 0-form), we have $d(df) = 0$ and for any 1-form ω, we also have $d(d\omega) = 0$, which are nifty encoding of two vector derivative identities:

$$\nabla\times(\nabla f) = \mathbf{0},\ \ \nabla\cdot(\nabla\times v) = 0.$$

This fact is true even in higher dimensions, namely in any number of variables and for any order form ω, $d(d(\omega)) = 0$.

Looking forward: In higher mathematics, forms ω with the property that $d\omega = 0$ are called **closed forms**, which those which satisfy $\omega = d\alpha$ are called **exact forms**, at which point math geeks would say: all exact forms are closed. They might then ask: are all closed forms exact? This leads into the topology topic known as (deRham) cohomology, which answers that "it depends", based on the domain in question.

We now illustrate these calculations further with a few examples:

Example 5.21 *Find $d(x\,dx + y\,dy)$ in 2-D:*

$$d(x\,dx + y\,dy) = dx \wedge dx + dy \wedge dy = 0.$$

Example 5.22 *Find $\alpha = d(1/r)$ in 3-D and show that $d\alpha = 0$, where r is as usual $\sqrt{x^2 + y^2 + z^2}$:*

$$\alpha = d(1/r) = -\frac{x}{r^3}\,dx - \frac{y}{r^3}\,dy - \frac{z}{r^3}\,dz = -\frac{1}{r^3}(x\,dx + y\,dy + z\,dz)$$

In the calculation of $d\alpha$, the 3-D version of our previous example appears and therefore gives 0 for the exterior derivative of the second factor. This means we can organize our calculation and exploit these cancellations as we use the product rule:

$$d\alpha = -d(\frac{1}{r^3}) \wedge (x\,dx + y\,dy + z\,dz) - \frac{1}{r^3} d(x\,dx + y\,dy + z\,dz).$$

Each of these two terms will be zero, the second part from the calculation alluded to and the first one since we obtain

$$-d(\frac{1}{r^3}) \wedge (x\,dx + y\,dy + z\,dz) = \frac{3}{r^5}\,(x\,dx + y\,dy + z\,dz) \wedge (x\,dx + y\,dy + z\,dz) = 0$$

since the wedge product of the 1-form $(x\,dx + y\,dy + z\,dz)$ with itself is 0.

Remark: You are free to calculate the above example out in full and generate many terms that cancel in the end.

Pull-backs described: For changes of variables since forms are objects to integrate, the behavior is like integrals under substitution. The algebra of forms encodes the various algebraic manipulations beautifully.

Lemma 5.23 *For any differential form in variables x, y, z, a change of variables giving a new form in variables u, v, w creates a new form of the same order by substitution for differentials.*

5.8.1 Stokes' theorem

The unifying feature of this formalism occurs when we look at integration linked with the exterior derivative d. For any $1-form$ which is in the range of d, we get path independence and therefore we can write:

$$\int_\gamma df = f(b) - f(a)$$

where γ is any path from a to b which admits a good integration theory (for example piecewise smooth). For any $2-form$ in the range of d in 3-D, say $\omega = d\phi$ integrated over a surface, we find by the traditional Stokes' theorem that the integral reduces to the integration of the 1-form ϕ over the oriented boundary of the surface. A similar result holds for $3-forms$ integrated over volumes when the $3-form$ is in the image of d, namely the integration over the volume becomes the integration of the corresponding $2-form$ over the bounding surface. Thus the unifying feature of the language introduced in this section is the following, which subsumes Green's theorem, the fundamental theorem for line integrals, Stokes' theorem, and the divergence theorem, along with others in higher dimensions which we have never considered:

Theorem 5.24 Stokes' theorem, general statement: *On any piecewise smooth, closed, oriented $k+1$ dimensional subdomain B with oriented boundary denoted ∂B, for every smooth $k - form\,\omega$, it follows that*

$$\int_B d\omega = \int_{\partial B} \omega.$$

For further reading on this, see the references at the end of the chapter.

5.8.2 EXERCISES

For problems 1 - 5, find $d\omega$ for each of the following differential forms:

1. $\omega = 2\,dx + 3\,dy$.
2. $\omega = 2\,y\,dx + 3\,x\,dy$.
3. $\omega = y\,dx \wedge dz$
4. $\omega = x^2\,y\,z\,dx + x\,y^2\,z\,dy + x\,y\,z^2\,dz$
5. Check that $d(d\omega) = 0$ for the previous examples.
6. If the symbolism of the "del operator" is transferred to differential forms, it becomes the "operator form" related to the exterior derivative d by the formal expression $d = \frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy + \frac{\partial}{\partial z}\,dz$. With this convention, show formally why $d \wedge d = 0$ reduces to the equality of mixed partial derivatives.

7. Consider the wedge product of three 1-forms to create a 3-form. Relate this to the algebra of the vector triple product of three vectors (which was the determinant of a matrix).

8. Physicists use space-time, with four coordinates x, y, z, t. What are the dimensions of the basic 1-forms, 2-forms, 3-forms, and 4-forms in this four-dimensional setting?

9. Cal Clueless believes that $dx\, dy$ is the surface area on the sphere, and that it equals $d\phi\, d\theta$ anyway. Clear up his confusion by redoing Example 5.19 with the proper expressions in x, y and then ϕ, θ.

10. Ivana Calculator doesn't like slick calculations, so she grinds out the example $d(d(1/r))$ done in this section. Do this and show her how to manage the resulting mess effectively.

11. Professor Boris Tudeth dislikes a purely formal calculation. How would he state the result of our "pull-back" calculation on the surface of the unit sphere in more formal language? He will give extra credit if you state it in a general situation.