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CHAPTER 5

Fundamental Theorems of Vector Calculus

This chapter explores the fundamental theorems of vector calculus. These theorems are often referred to by names such as Green's Theorem, Stokes' Theorem, and Gauss's Theorem, with Ostragradskii appended to Gauss in some cases. They are the generalizations of the one variable result, along with some new and interesting twists. Integration of a derivative is again linked to the change in some quantity. The multivariable setting forces a variety of the dimensions of the integration domains and the different vector derivatives used.

The main approach to understanding these theorems is again **"Think locally, act globally"**. In all situations we will use our "Linear eyes" to side-step algebraic issues and focus on what ought to be true. This helps us to follow how circulation leads to the integral of the curl while flux leads us to the integral of the divergence. "Acting globally" will be again the second component of the analysis, where we move from location to location and create integrals. Linear functions and vector fields are our building blocks. Constant derivatives are the model for continuously varying cases. In the final section, we look ahead at the unifying concept of differential forms.

Exciting picture goes here

5.1 Preliminary: several generalizations of the Fundamental Theorem are possible

Overview: This section discusses the generalizations of the fundamental theorem of calculus to the multivariable setting. The fundamental theorem in one variable is recalled in both of its forms. Dimension will matter, as will the basic feature of integrals as sums of integrals over non-overlapping subdomains. The three main versions of the fundamental theorem in 3-D are described.

The Fundamental Theorem in one variable revisited: In one-variable calculus, the fundamental theorem on a fixed interval, say $[a, b]$, says that $\int_a^b f(x)\,dx = F(b) - F(a)$ where f is continuous and $F'(x) = f(x)$ for all x in the interval $[a, b]$.

The other part of the fundamental theorem looks at the integral of f from a to a *variable* endpoint x and confirms that this defines a function

$$G(x) = \int_a^x f(t)\,dt$$

with G differentiable and $G'(x) = f(x)$. This establishes that every continuous function is the derivative of a continuously differentiable function. Many texts start here and use the variable endpoint result to establish the fixed domain case. This is not very convenient in our multivariable setting. To aid in following later discussions, a review of the Fundamental Theorem on a fixed interval is useful.

An approach to the one-variable Fundamental Theorem based on "thinking locally" divides the interval $[a, b]$ into non-overlapping subintervals which cover the interval. $F(b) - F(a)$ is the total change in F as x runs from a to b. Partition the interval as $a = x_0 < x_1 < \ldots < x_n = b$. Then

$$\begin{aligned} F(b) - F(a) &= F(x_1) - F(x_0) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1})) \\ &= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} F'(t_k)(x_k - x_{k-1}). \end{aligned}$$

The last equality came by thinking locally, i.e. assuming linear behavior of F, replacing $F(x_k) - F(x_{k-1})$ by $F'(t_k) \cdot (x_k - x_{k-1})$, where t_k is some point in the subinterval (of course it could be any location if truly linear since F' would be constant). This is exact for linear functions and modified to be exact for some choice of t_k by an appropriate mean value theorem for the general differentiable function F. This shows the total change to be always equal to a Riemann sum for the integral of F'. The limit is therefore both the total change in F and the integral of f, so they are equal.

The variable endpoint version involves a direct calculation of the derivative of G using the definition of the derivative as limit. Its main use (other than to establish the fixed endpoint case) is to show that every continuous function on a closed interval is the derivative of a differentiable function on that interval. As we saw in Chapter Three, that is not true in the multivariable setting for the gradient operation nor for the curl. Among our vector derivatives, only the divergence has a similar property: every continuous scalar function is the divergence of some differentiable vector function. But variable endpoints can yield an alternate integral form for the "anti-gradient" as will be seen in section 5.2.

There are several ways to generalize the fundamental theorem when we move to several variables. Let's explore what they might involve, at least in terms of integrals and dimensions. Justifying the results usually is described in a similar fashion to the way the Fundamental Theorem of Calculus was just discussed rather than the variable endpoint approach. The rest of the chapter will explore these ideas more fully, along with an optional section on complex functions and another on differential forms, which reveals the unifying idea in all these variations.

5.1.1 Change in a function F from one point to another

The first direct generalization of the fundamental theorem considers a function F and its total change as we move from point A to point B in two or more dimensions. Here the new twist is that in moving between two locations many paths are possible. Therefore the integral in one variable was on an interval but now we need an integral along a particular path. This is the **line integral**, which we introduced in Chapter Four(??). Interestingly, the total change from A to B for a nice function **does not depend on the path**, while we learned in Chapter Four that line integrals typically do depend on the path taken. So unlike the 1-D version, the integrands must be restricted to a special set of possible vector fields!

Breaking up the path from A to B into a finite number of pieces leads us to an educated guess of one fundamental theorem:

$$F(\mathbf{B}) - F(\mathbf{A}) = \int_C \nabla F \cdot d\mathbf{r}$$

where C is a curve that runs from A to B. To make this guess, look at a small piece of the curve and assume the curve piece is straight and the function F is linear. Then the change in F is given by the dot product of the gradient vector with the change in position (recall directional derivative and gradient in Chapter 3). This gives us the form of the integrand, namely the gradient of F.

An interesting consequence of this result is that line integrals of the form given above are in fact independent of the given path, since the total change $F(\mathbf{B}) - F(\mathbf{A})$ is the same for all paths that start at $\mathbf{A}$ and end at $\mathbf{B}$. A fuller discussion is in the next section.

The two other main generalizations in 3-D will involve the other two derivatives and will occur in higher dimensional integrals. They are described here and developed over several sections of the chapter.

5.1.2 Second generalization: circulation integrals around closed loops linked to integrals of derivatives inside

From the first result above, when we integrate the gradient along curves, the change in the function results. Closed curves (loops) return to their starting point, so a line integral of the gradient around a loop must yield 0, since the function has no net change. Yet we know that some line integrals around loops are not zero. The second version of a fundamental theorem results when we look more closely at such loops.

If we have a loop in 3-D, it can be the boundary of some surface in 3-D (in fact many different ones), while in 2-D, loops will enclose a planar domain. Let's start in 2-D. For an oriented loop L enclosing a rectangular region R, consider for a given vector field $\mathbf{v}$ the integral $\int_L \mathbf{v} \cdot d\mathbf{r}$, which represents the **circulation** of $\mathbf{v}$ around the loop L. By splitting this up into subrectangles as in double integration theory, the circulations add to give the total circulation. [need picture here] Note that on the common edges the line integrals cancel (higher dimensional version of subinterval endpoints for usual FTC). This allows us to "think locally" and assume small rectangles with linear vector fields. On such a rectangle we find that the opposite edges almost cancel but the variation in $\mathbf{v}$ will contribute. A calculation, done in detail in a later section, shows that the two horizontal edges contribute based on the vertical variation of the first component of $\mathbf{v}$ while the vertical edges contribute based on the horizontal variation of the second component of $\mathbf{v}$, which suggests **curl of** $\mathbf{v}$ is the integrand. This is consistent with the situation when $\mathbf{v}$ is a gradient since then the curl is zero and the circulation is also zero.

This result is known as Green's theorem in 2-D and says:

$$\int_L \mathbf{v} \cdot d\mathbf{r} = \int_L P(x,y)\,dx + Q(x,y)\,dy = \int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where $\mathbf{v} = P\mathbf{i} + Q\mathbf{j}$ and L is the oriented loop that encloses the region R.

Moving into 3-D is more difficult, but guided by the linear approximation locally we will find the following to be true for the circulation around a loop which bounds a surface S with outer unit normal $\mathbf{n}$ and surface area element $d^2\sigma$:

$$\int_L \mathbf{v} \cdot d\mathbf{r} = \int_S\int (\nabla \times v) \cdot \mathbf{n}\, d^2\sigma.$$

This is known as Stokes' theorem.

This implies that surface integrals of the form given above are in fact independent of the surface used as long as the boundary loop is fixed.

For a general vector integrand, not necessarily the curl of some vector field, the surface integral as in Stokes' theorem measures the **flux** through the surface. Our third and final generalization will look at surfaces that enclose volumes and relate the flux through the surface to an integral of a derivative over the inside volume.

5.1.3 Third generalization: flux and volume integration

For a closed surface S with outer unit normal vector $\mathbf{n}$ and surface element $d^2\sigma$ in 3-D which is the boundary of a volume V, given any differentiable vector field $\mathbf{w}$ on an open set containing S and V, the follow identity (known as Gauss' theorem or the divergence theorem) holds:

$$\int_S\int (\mathbf{w}\cdot\mathbf{n})\, d^2\sigma = \int\int_V\int (\nabla\cdot w)\, dx\, dy\, dz.$$

Again, the way to see this should be correct is to observe that the flux is the sum of the fluxes on subsurfaces. There is cancellation of the common surfaces on neighboring pieces because the normal vectors are opposite while the vector function is the same. Then thinking locally, we can look at a linear vector field on a small box. Pairing the six faces of a box shows that the x- variation of the x- component of $\mathbf{w}$ is the only contribution to the flux across the faces normal to the x-axis, and similarly in y and z, which leads to the divergence of $\mathbf{w}$ as the integrand in the volume integral. Details are in section 5.6 below.

5.1.4 Exercises