3.1 Partial Derivatives

Overview: Partial derivatives are defined by differentiation in one variable, viewing all others as constant (frozen at some value). The reduction to one-dimensional calculus immediately provides a wealth of knowledge and technique, but as in our earlier discussion of limits, it has some drawbacks. In this section, we define partial derivatives, describe the multiple notations for them, compute them, and develop their basic properties. In later sections the full multidimensional perspective is developed.

3.1.1 Defining partial derivatives

The calculus of one variable derivatives used the limit of slopes of secant lines to find the derivative. Differentiating in one variable while freezing (or ignoring) the variation in the others is called **partial differentiation** and is defined as a similar limit. In two variables, there are two possible partial derivatives of order one, based on varying each variable separately, while in three variables, there are three such derivatives.

Definition 3.1 *The partial derivatives of a function $f(x, y)$ defined on an open domain containing (x, y) are denoted by $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ and are defined as the following limits:*

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y}(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

assuming the limits exist.

We usually say: "the partial derivative of f with respect to x" when referring to $\frac{\partial f}{\partial x}(x, y)$.

Example 3.2 *Consider $f(x, y) = 2x + 3y$, a linear function: $\frac{\partial f}{\partial x}(x, y) = 2$ and $\frac{\partial f}{\partial y}(x, y) = 3$. For a general linear function in two variables, $f(x, y) = Ax + By + C$ we have $\frac{\partial f}{\partial x}(x, y) = A$ and $\frac{\partial f}{\partial y}(x, y) = B$, which are both constant for all x and y values.*

Geometrically, the partial derivatives of a scalar function f measure the local slopes of the one-dimensional graphs obtained by slicing the two-dimensional graph of f along the coordinate directions. In the non-linear example of a paraboloid below, this is indicated in the figures. The first pair shows the slicing of the 3-D plot while the second pair show the restricted functions.

Example 3.3 *Consider the function $x^2 + y^2$ near the point $(1/2, 0)$. Find the partial derivatives at this point and explain their meaning.*

Figure 1: The graphs are both of $x^2 + y^2$, one sliced in x with $y = 0$ and the other sliced in y with $x = 1/2$.

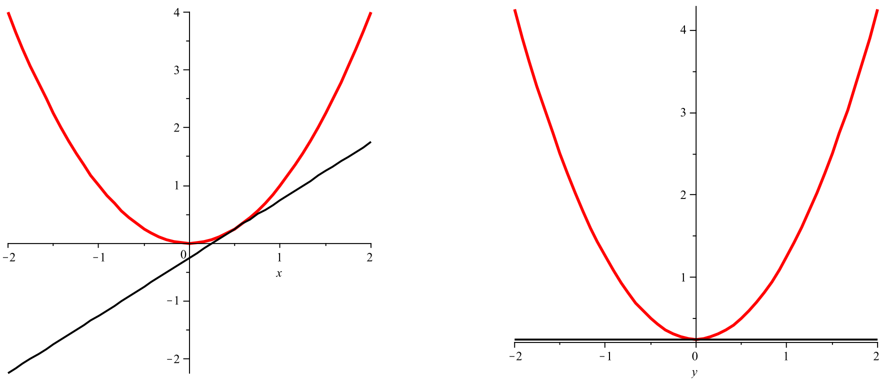

Figure 2: These graphs are the restricted functions x^2, varying in x with $y = 0$ frozen and the other varying in y with $x = 1/2$ frozen, along with the tangent lines.

To find the derivatives, we calculate with $g(x, y) = x^2 + y^2$ *that* $\frac{\partial g}{\partial x}(x, y) = 2x$ *and* $\frac{\partial g}{\partial y}(x, y) = 2y$ *so the partial derivatives are* $\frac{\partial g}{\partial x}(1/2, 0) = 2(1/2) = 1$ *and* $\frac{\partial g}{\partial y}(1/2, 0) = 2 \cdot 0 = 0$.

Meaning: The partial derivatives are the derivatives of the restricted function with the frozen values in the other variable. At this location, this means that the tangent lines to the restricted graphs have slopes of 1 and 0 respectively. For different values of x *and* y*, the slopes will typically be different.*

The two tangent lines in the two coordinate planes intersect and determine the **tangent plane**, to be discussed in the next section.

Derivative as function: As in one-variable calculus, finding a partial derivative at various points leads to the fruitful viewpoint of regarding the derivative of a function as a new function. You will notice that we did this already above without comment! Thinking of $\frac{\partial f}{\partial x}$ as a function allows us to consider differentiation of that function, which is then a second derivative.

More variables: If there are three or more variables, partial derivatives in each variable are defined in a similar fashion by allowing change only in that variable and finding the instantaneous rate of change while freezing all other variables. Graphs are harder to draw but the slices are again graphs of one-variable functions and the partial derivative finds the slope of the tangent line in that restricted slice.

Computing the partial derivative at a point can be done two ways: (1) freeze the value of the remaining variables first and then find the derivative of the restricted function; or (2) compute the partial derivative at a general location, then restrict that new function to the given values. The mathematical issue of when these two approaches yield the same answer is one you dealt with already a bit in one-variable calculus: the **continuity of the derivative when viewed as a function**. In our examples so far, this issue has not really arisen since the derivatives have not involved the other variables. A simple function like $f(x, y) = xy$, with partial derivatives $\frac{\partial f}{\partial x}(x, y) = y$, $\frac{\partial f}{\partial y}(x, y) = x$, both clearly continuous, does generate such a question. Unless stated as hypothesis, you should not assume partial derivatives are continuous functions. In practice, most of the functions you will encounter will be continuous. Yet as we shall see in a later discussion in the project on shocks, discontinuities do happen and in certain situations are one of the most interesting features of the mathematical analysis.

3.1.2 Vector-valued functions

The partial derivative of a vector-valued function is similar to the one variable derivative of vector-valued functions in Chapter 2. The partial derivative is defined and computed componentwise, but may also be viewed in purely vector terms.

Example 3.4 *The vector function* $\mathbf{r}(x,y) = (2x+4y)\mathbf{i} + (3x-y)\mathbf{j}$ *has* $\frac{\partial \mathbf{r}}{\partial x} = 2\mathbf{i} + 3\mathbf{j}$.

Quick Reading Check: What is $\frac{\partial \mathbf{r}}{\partial y}$?

3.1.3 Transferred Results and Techniques from one-variable calculus

Algebra of derivatives: The algebra of differentiation in one variable, which is built on the limit definition of derivative, carries over to partial derivatives. Therefore the derivative formulas you learned earlier both for specific functions (exponential, log, trig, powers) and for general operations (product, quotient, composition) extend to this case. General composite functions (full chain rule), implicit functions, and inverse functions are more complicated and will be understood later in this chapter.

Example 3.5 *If* $g(x,y) = Cx^2 + Dxy + Ey^2$*, then* $\frac{\partial g}{\partial x}(x,y) = 2Cx + Dy$.

Example 3.6 *If* $F(x,y) = \frac{1}{x^2+y^2}$*, then* $\frac{\partial F}{\partial y}(x,y) = -\frac{1}{(x^2+y^2)^2} \cdot 2y = -\frac{2y}{(x^2+y^2)^2}$.

Mean Value Theorem: If the restricted function, which varies in only one variable, is differentiable, then the one variable mean value theorem for derivatives also applies. This tool bridges the gap between limit statements and their finite, non-limit counterparts and is often helpful in proofs. Favorite results such as L'Hopital's rule are based on the mean value theorem. In our setting, we can conclude that a change in f based on only one input variable changing, divided by the change in that input, can be expressed as the partial derivative somewhere in between. For example with two input variables x and y, assuming f is continuous on $[a,b]$ and f_x exists for $a < x < b$ with y frozen, then for some x_1 between a and x,

$$\frac{f(x,y) - f(a,y)}{x-a} = \frac{\partial f}{\partial x}(x_1, y).$$

In proofs, such an expression is often used before computing some limit, especially to eliminate denominators that tend to zero. Assuming continuity of the partial derivative in the above expression, squeezing will push x_1 to a as x tends to a. When we use this argument, as we will repeatedly in the next section and in this section when discussing Clairaut's theorem, we will say "by the mean value theorem" and this is what that means. This is what you do when you write $\frac{sin(t)-sin(a)}{t-a} = \cos(t_1)$ for example, where the division is not really possible algebraically and instead the mean value theorem creates that alternative expression with no denominator.

Technical fine point: As in the previous chapter, a vector-valued function that is differentiable will have an appropriate mean value theorem, which requires possibly distinctive values for each component separately. **Higher derivatives and**

approximation: If the restricted function has second derivatives, the one-variable information provided there can also be exploited. This includes convexity ideas, local extrema testing, and quadratic and higher order Taylor polynomial approximation in the one variable.

Having mentioned second derivatives, we now consider the full set of higher derivatives, not just those found by restricting to one variable slices.

3.1.4 Defining Higher Order Derivatives

Repeated differentiation is extremely useful. Most of the important partial differential equations that describe nature involve second derivatives and some involve higher order derivatives also. High order approximation of functions by polynomials matches not only the function and its first derivatives, but higher derivatives also.

An Example: Consider the quadratic function of three variables,

$$g(x, y, z) = x^2 + 3xy + 4xz + 6y^2 + 8yz + 5z^2.$$

Then there are three partial (first) derivatives, $\frac{\partial g}{\partial x}(x, y, z) = 2x+3y+4z$, $\frac{\partial g}{\partial y}(x, y, z) = 3x+12y+8z$, $\frac{\partial g}{\partial z}(x, y, z) = 4x+8y+10z$, which are all purely linear and can be differentiated again. This leads to NINE possible second derivatives by taking any partial derivative of the three first derivatives. For this example the three partial derivatives of each partial derivative can be read off from the linear form of each first derivative: The x, y, and z derivatives of $\frac{\partial g}{\partial x}$ are 2, 3 and 4 respectively; The x, y, and z derivatives of $\frac{\partial g}{\partial y}$ are 3, 12 and 8 respectively; the x, y, and z derivatives of $\frac{\partial g}{\partial z}$ are 4, 8 and 10 respectively.

All but three of these are called "mixed" second partial derivatives since they involves two different variables. Higher order derivatives which involve at least two variables are also called "mixed". In our example, it turned out that the mixed derivatives had the same value when the order of differentiation was reversed. This might have been accidental, but in fact with continuity assumed for the various derivative functions, such agreement must occur. The iterated limits involved in the definitions can be done in different orders but must yield the same answer.

Other notations: Several alternative notations are used for partial derivatives. One common notation uses an upper case D with subscript for the variable to differentiate, either with its name (x, y, t, etc) or its location as a number (1, 2, 3, etc). In such notation the previous example would read $\frac{\partial F}{\partial y}(x, y) = D_y F(x, y) = D_2 F(x, y) = -\frac{2y}{(x^2+y^2)^2}$. Another notation is the variable name or position used as a subscript directly on the function $F(x, y)$, as in either $F_y(x, y)$ or $F_1(x, y)$. This is often useful, but some other authors will use this notation for components of a vector **F**. Careful reading is always a good idea!

Notations for Higher Derivatives: Writing second derivatives as we did above is a bit bulky, so more compact notation is used. Recall that in one variable calculus the Leibniz notation for second derivatives was $\frac{d^2f}{dx^2}$, where the exponent 2 on the x in the denominator is not really referring to x^2 but rather to the differential $(dx)^2$. The two most common notations for second partial derivatives are a condensed ∂-notation and a subscript notation. The ∂-notation for an expression like $\frac{\partial}{\partial z}\frac{\partial g}{\partial y}(x,y,z)$ is $\frac{\partial^2 g}{\partial z \partial y}(x,y,z)$ while the notation for $\frac{\partial}{\partial y}\frac{\partial g}{\partial y}(x,y,z)$ is $\frac{\partial^2}{\partial y^2}g(x,y,z)$.

For higher-order derivatives, the ∂ notation extends as in:

$$\frac{\partial^4}{\partial z^4}f(x,y,z) \text{ and } \frac{\partial^3}{\partial z^2 \partial y}g(x,y,z).$$

Other notations: The D and subscript notations get extended to higher-order partial derivatives as well. For these, it is customary for the subscript order when reading from left to right to refer to the sequencing of differentiation working from inside out (meaning first to last). So $f_{yx}(x,y,z,t) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(x,y,z,t))$.

3.1.5 Equality of Mixed Partial Derivatives

An important result about mixed partial derivatives states informally: "mixed partial derivatives are equal to one another". More precisely, every mixed partial derivative of order n or less of an n times continuously differentiable function f has the same values as every other mixed partial derivative of f that involves the same differentiations done in a rearranged sequence of differentiation. In particular, if $f(x,y)$ is twice continuously differentiable on an open set S, then $f_{xy}(x,y) = f_{yx}(x,y)$ for every point $(x,y) \in S$. This is called Clairaut's Theorem. In the chapter on integration a proof of Clairaut's theorem is given by assuming that it is false for some function at some point and deriving a contradiction. For now, we content ourselves with checking some examples and presenting the basic question as a limit exchange problem, leaving the key ideas of a more traditional proof as a challenging exercise.

Theorem 3.7 (Clairaut's Theorem) *If $f(x,y)$ is twice continuously differentiable on an open set S, then $f_{xy}(x,y) = f_{yx}(x,y)$ for every point $(x,y) \in S$. More generally, in higher order derivatives and higher dimensions, all equivalent mixed partial derivatives are equal at all points in an open set when the partial derivatives are all continuous.*

The Issue in Proving Clairaut's Theorem: For a function of two variables x and y, let (x,y) be a point in S. For h and k sufficiently small and non-zero, the rectangle determined by the points $(x,y), (x+h,y), (x,y+k), (x+h,y+k)$ lies in S. The mixed derivative $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(x,y))$ is defined as the limit of a difference of limits:

$$\lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(x+h,y) - \frac{\partial f}{\partial y}(x,y)}{h}$$

$$= \lim_{h\to 0}(\lim_{k\to 0} \frac{(f(x+h,y+k) - f(x+h,y)) - (f(x,y+k) - f(x,y))}{hk}.$$

The corresponding reversed order of limits is the mixed partial derivative in reversed order and equality requires a proof that these double limits always have the same values if the partial derivatives are assumed to be continuous. The mean value theorem used twice expresses the fractions without denominators so the limits can be compared. This is an exercise below.

A heuristic argument in favor of Clairaut's theorem will be given in a later section of this chapter when the quadratic and higher-order approximations are discussed.

3.1.6 Finding "partial antiderivatives"

Knowing one partial derivative of a function in an open set near a given point, what do we know about the function itself?

Looking back: In one variable calculus, knowing the derivative meant knowing a lot about the function itself – we knew the function up to a constant. You also learned to create a reasonable sketch of the graph of a function from the graph of its derivative. We are now exploring whether the partial derivative in several variables has this valuable notion of having nearly all the information about the function.

Let us look at some examples which will show how much one partial derivative determines the function.

Example 3.8 *Solve for $F(x, y)$, given that $\frac{\partial F}{\partial x} = 1$.*

You probably said $F(x, y) = x + C$ where C is a constant. Which is partially correct! $x + C$ is indeed a family of functions that solves the partial differential equation, but **this is not the most general partial anti-derivative!** Note that each of the functions $x + y, x + y^2 + 13, x + \cos y,\ x + \sqrt{1 + y\sqrt{1 + y}}$ still have $\frac{\partial F}{\partial x} = 1$. This suggests that the most general form is $F = x + g(y)$ where $g(y)$ is **any suitable function** of y and surely includes all constants. It is clear that such a form will solve the equation and that any term added that also varies in x will lead to a different partial derivative.

Try this: solve for $W(x, y, z)$ so that $\frac{\partial W}{\partial x} = \mathrm{e}^{2x}+y+4xz$. You should have decided $W(x, y, z) = (1/2)\mathrm{e}^{2x} + xy + 2x^2z + c(y, z)$ where $c(y, z)$ is any function of y and z.

Theorem 3.9 *Suppose $f(x, y)$ is continuous on some domain in D. Then there exist functions $F_1(x, y)$ and $F_2(x, y)$ with the properties: $F_1, F_2, \frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}$ are continuous in D and $\frac{\partial F_1}{\partial x} = f$ for all points in D, and $\frac{\partial F_2}{\partial y} = f$ for all points in D. Any other functions G_1 and G_2 which satisfy the same partial differential equation as F_1 and F_2 and the same continuity requirements differ from F_1 and F_2 respectively by an arbitrary continuous function in the other variable. The generalization* *to more variables works similarly: a partial derivative in one variable and continuity determines a function up to an arbitrary continuous function of the remaining variables.*

For the proof, we use the Fundamental Theorem in one variable to find solutions in each coordinate slice. This can often be done symbolically and in general can be done using one-dimensional integrals as in the Fundamental Theorem of Calculus.

Analyzing Graphically: Another indication that one partial derivative is insufficient knowledge to nearly identify the original function comes from graphical considerations.

In our basic example, $\frac{\partial F}{\partial x} = 1$, the graph of the x-derivative is the flat graph of height 1. For each fixed y value this gives a family of possible lines in the x-direction, but we can glue these lines together as y varies in many ways, indicated by all the different formulas above. This means that we can line up "frames" as y varies in many ways by lifting or dropping them as y varies. Think of a stack of sliding tilted playing cards, sliding in the y-direction.

Another way to see what one partial derivative doesn't describe: consider that the two well-known quadratic functions $x^2 + y^2$ and $x^2 - y^2$ have very different graphs with very different level sets, yet **their** x **partial derivatives are the SAME function** $2x$.

Clearly one partial derivative is insufficient information about a function to pin it down very much. While a natural extension of the derivative in one variable and an extremely useful concept, a single partial derivative is not the proper generalization of the derivative of a function. The next section introduces what some authors call the total derivative, which does capture all the information about how the function is changing if we use it globally.

3.1.7 EXERCISES

Writing: Explaining, Reacting, Questioning

1. Consider the linear function $z = f(x, y) = 2x + 3y$. Thinking about the graph and the level sets, how does f change when x increases by h? When y increases by k? When both change by 3 simultaneously? What do the level sets look like? How do the tangent lines combine to create the graph?

2. Describe in a brief paragraph the meaning both graphically and algebraically of the two partial derivatives of a function g described in polar coordinates (r, θ) In other words, what do g_r and g_θ describe?

3. What do we know by applying the mean value theorem for derivatives to the restricted functions, like $f_1(x) = g(x, 0) = x^2$ and $f_2(y) = g(1/2, y) = 1/4 + y^2$ in our first example?

4. If we know that $\frac{\partial f}{\partial x}(x, y, z) = 2$, what can we conclude? Why?

5. Of the various notations for partial derivatives, which do you prefer and which do you like the least? Why? If you claim they are equally good or bad for you, then pick two and describe their pros and cons.

6. The volume of a cylinder of radius r and height h is $V(r, h) = \pi r^2 h$. Find the partial derivatives of $V(r, h)$ and link them to quantities you integrated when doing volumes of revolution by either disks or shells.

7. What other information do you think is needed to determine a function from its x partial derivative, in the sense that the value of a derivative in one variable calculus, plus the value of the function at one point determines the function? Justify your conjecture.

8. (a) If a point $(x_0, y_0, f(x_0, y_0))$ is the highest point along the x slice of the graph of f through the point, what can you conclude about any of the partial derivatives of f at (x_0, y_0)?

   (b) If the point $(x_0, y_0, f(x_0, y_0))$ is the highest point on both the x and y slices of the graph of f through the point, what can you conclude about the partial derivatives of f at (x_0, y_0)?

Calculational Exercises

For problems 1 to 20, find the first-order partial derivatives in all variables at the points given, or when no point is specified, at all points in the domain:

1. $f(x, y) = \cos x \, \sin y$ at $(\pi/4, \pi/3)$.

2. $g(x, y) = x^2 + 4xy + y^2$ at $(2, -1)$.

3. $h(x, y, z) = 3 + 4x + 5y + 6z + xyz$ at $(-1, 1, 2)$.

4. $F(x,y) = x\,(x+y)^5$ at $(\frac{1}{3}, \frac{2}{3})$.
5. $p(x,y) = A\,x^2 + B\,xy + C\,y^2$ where A, B, C are constants.
6. $q(x,y,z) = a_{11}\,x^2 + 2\,a_{12}\,xy + 2\,a_{13}\,xz + a_{22}\,y^2 + 2\,a_{23}yz + a_{33}\,z^2$ where all factors a_{ij} are constants.
7. $u(x,y) = \sqrt{1 - x^2 - y^2}$
8. $v(x,y) = \mathrm{e}^{2x+3y+3}$
9. $\phi(x,y) = \frac{1}{\sqrt{x^2+y^2}}$
10. $U(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$
11. $w(x,y) = \frac{x}{x^2+y^2}$
12. $f(x,y) = \frac{xy}{x^2+y^2}$
13. $F(x,y) = x\,\mathrm{e}^{-x^2-y^2}$
14. $\theta(x,y) = \tan^{-1}(\frac{y}{x}) + C$ where C is a constant.
15. $u(x,t) = \sin(k\,x)\,\mathrm{e}^{-k^2\,t}$ where k is a constant.
16. $v(x,y) = \ln(x^2+y^2)$
17. $u(r,\theta) = r\cos\theta$
18. $v(r,\theta) = r\sin\theta$
19. $u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\,\mathrm{e}^{-\frac{x^2}{4Dt}}$ where D is a constant.
20. $r(a,b,c) = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$

    For problems 21 - 30 find all the second derivatives for problems 7-16 above:
21. $u(x,y) = \sqrt{1 - x^2 - y^2}$
22. $v(x,y) = \mathrm{e}^{2x+3y+3}$
23. $\phi(x,y) = \frac{1}{\sqrt{x^2+y^2}}$
24. $U(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$
25. $w(x,y) = \frac{x}{x^2+y^2}$
26. $f(x,y) = \frac{xy}{x^2+y^2}$
27. $F(x,y) = x\,\mathrm{e}^{-x^2-y^2}$

28. $\theta(x, y) = \tan^{-1}(\frac{y}{x}) + C$ where C is a constant.

29. $u(x, t) = \sin(k\,x)\,\mathrm{e}^{-k^2\,t}$ where k is a constant.

30. $v(x, y) = \ln(x^2 + y^2)$

31. The total energy of a simple conservative mechanical system is a function H which depends on three momentum coordinates $p_1,\ p_2,\ p_3$ and three position coordinates $q_1,\ q_2,\ q_3$. In many cases the energy is the sum of kinetic energy plus potential energy, and the potential energy is independent of the momentum values. This leads to the following form for the total energy: $H(p_1,\ p_2,\ p_3,\ q_1,\ q_2,\ q_3) = \frac{1}{2\,m}(p_1^2 + p_2^2 + p_3^2) + U(q_1,\ q_2,\ q_3)$ where m is a constant and U is a differentiable function. Find all six partial derivatives of H for any U.

32. For the Cobb-Douglas production function, $P(L, K) = bL^\alpha K^\beta$, find the partial derivatives and show that $P_L(L, K) = \alpha\frac{P}{L}$ and $P_K(L, k) = \beta\frac{P}{K}$.

    For problems 33 to 36, find one solution to each partial differential equation:

33. $\frac{\partial F}{\partial x}(x, y, z) = x^2 + y\,z$

34. $\frac{\partial G}{\partial y}(x, y) = \frac{y}{(x^2+y^2)^2}$

35. $\frac{\partial U}{\partial z}(x, y, z) = z\,\mathrm{e}^{-(x^2+y^2+z^2)}$

36. $\frac{\partial \phi}{\partial x}(x, y) = \frac{1}{(x+y+4)^3}$

    For problems 37 to 40, find the most general form of solution to each partial differential equation:

37. $\frac{\partial F}{\partial x}(x, y) = x^3 + x\,y^2$

38. $\frac{\partial G}{\partial y}(x, y, z) = y\,\sqrt{x^2 + y^2 + z^2}$

39. $\frac{\partial U}{\partial z}(x, y, z) = (x + 3\,y + 2\,z)\,\mathrm{e}^{2\,x+3\,z}$

40. $\frac{\partial \phi}{\partial x}(x, y, z, t) = \cos(2\,x + 2\,y)\,\sin(5\,t)$

    For problems 41 to 44, show that the given function is a solution to the partial differential equation $u_t = u_{xx}$:

41. $u(x, t) = \cos(x)\,\mathrm{e}^{-t}$ for all x and t.

42. $u(x, t) = \sin(kx)\,\mathrm{e}^{-k^2\,t}$ for any constant k and all x and t.

43. $u(x, t) = \frac{1}{\sqrt{4\pi\,t}}\,\mathrm{e}^{-\frac{x^2}{4t}}$ for all x and all $t > 0$.

44. $u(x, t) = \frac{1}{\sqrt{4\pi\,\sigma^2\,t}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{4\sigma^2\,t}}$ for any constants σ and μ, all x and all $t > 0$.

    For problems 45 to 48, show that the given function is a solution to the partial differential equation $u_{tt} = u_{xx}$:

45. $u(x,t) = \cos x\ \sin t$.

46. $u(x,t) = x^4 - 4\,x^3\,t + 6\,x^2\,t^2 - 4\,x\,t^3 + t^4$.

47. $u(x,t) = G(x+t)$ where G as a function of one variable has two continuous derivatives.

48. $u(x,t) = H(x-t)$ where H as a function of one variable has two continuous derivatives.