2.3 Geometry of curves: arclength, curvature, torsion

Overview: The geometry of curves in space is described independently of how the curve is parameterized. The key notion of curvature measures how rapidly the curve is bending in space. In 3-D, an additional quantity, torsion, describes how much the curve is wobbling out of a plane. Alternative methods of calculation for curvature and torsion are developed.

2.3.1 Idea of curved vs. straight as independent of parametrization

The goal of this subsection is to create some measurement of curvedness that is a property of the road, not the driver. A first question: which of the two circles is "more curved"?

Figure 1: The two circles have different "curviness"

You most likely decided that the one with the smaller radius is more curved. How can we quantify that? How should we relate more general curves to circles? What could play the role of the radius? How curved is a line?

2.3.2 Circular model: tied to radius of circle

If a smaller radius leads to a more curved circle, it follows that the measurement of curvature should increase as the radius of a circle decreases. Curvature has a long history including work in ancient Greece, but the curvature of a general curve in the plane was solved by Newton, building on earlier work by Oresme, Huyghens, and others (see Lodder[TBA REF] and references therein). **Newton defined the curvature for a circle as the reciprocal of the radius**. Newton then went deeper and developed the general case in terms of the radius of the best circular approximation, made instantaneously at each point. For each such approximating circle, known now as the **osculating circle**, the **radius of curvature** is defined as the radius of the osculating circle and **curvature** is then the reciprocal of the radius of curvature. From our recent analysis of circular motion, the acceleration will play a role in this definition, particularly its normal component.

Newton's definition is not particularly easy to use for calculation, but it will be illustrated in a simple example and the extension to general planar curves is given in a sequence of problems at the end of the section, problems TBN.

Example 2.10 *Curvature at the vertex of a parabola: Let* $y = a\,x^2$ *for* $a > 0$ *define a parabola. Find the best instantaneous circle approximation at the vertex* $(0,0)$ *and use it to calculate the radius of curvature and the curvature at the vertex.*

By symmetry, we can suppose the circle to have center along the y*-axis. Since the parabolas bend up, the circles that vie for best approximation should lie above the* $x - axis$*. The circles of radius* R *of that form pass through* $(0,0)$ *with center at* $(0,R)$ *so they have equations:* $x^2 + (y-R)^2 = R^2$*. Now we can look for second derivatives to match up by choice of radius* R*. The circle splits into two semicircles when we express* y *as a function of* x *and we are focusing on the lower half, which goes through our origin. Thus* $y = R - \sqrt{R^2 - x^2}$ *near* $(0,0)$*. Using a Taylor expansion in powers of* x *to second order is best done by algebraic substitution* $t = x^2$ *from* $\sqrt{R^2 - t} \approx \sqrt{R^2} - \frac{1}{2}t/R$*, which now becomes:* $y \approx R - R + \frac{1}{2}x^2/R$*, which should equal* $y = a\,x^2$ *locally. This means we choose* $2R = 1/a$*. The radius of curvature at the vertex of the family of parabolas is* $R = 1/2a$ *and the curvature is* $1/R = 2a$*. Note that this is also the value of the second derivative at the vertex.*

A graphical illustration of the approximation to a parabola by circles is given in the figure below, where the value of a is 5, so the radius of curvature at the vertex is $R = 0.1$.

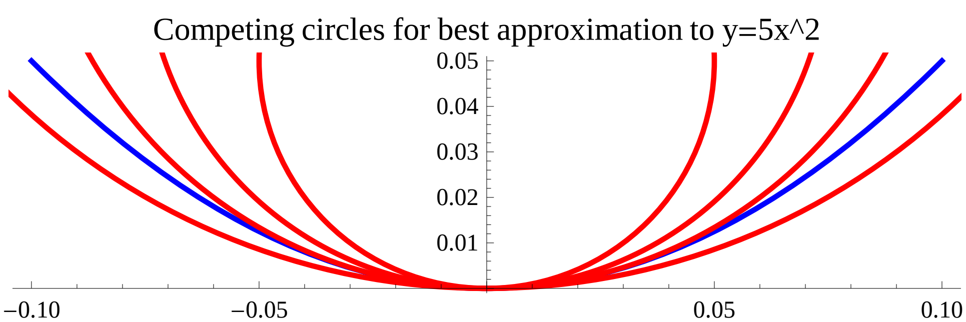

Figure 2: The parabola is the blue curve, while the red circles have radii: $0.05, 0.075, 0.1, 0.15$

A question to ponder: The parabola has a constant second derivative. Do you think the parabola has constant curvature? Why or why not? What does this suggest about the relation between curvature and second derivatives in general?

See WERQ problem TBA for some exploration of the curvature of a general curve in the plane using osculating circles and local approximation by parabolas.

2.3.3 Definitions as bending of tangent in arclength; alternate forms.

Eventually Newton's definition was refined to become the geometric version used today, which says: Along a curve, measure the instantaneous rate at which the tangent vector changes direction by differentiating with respect to arclength. This will define the curvature and a bending direction (in 3D especially) if the curvature is non-zero. The precise definition is:

Definition 2.11 *Let a parametric curve be given as $\mathbf{r}(t)$, with continuous first and second derivatives in t. Denote the arclength function as $s(t)$ and let $\mathbf{T}(t)$ be the* **unit** *tangent vector in parametric form. Then the curvature, usually denoted by the Greek letter kappa (κ) at parametric value t is defined to be the magnitude of derivative of $\mathbf{T}$ with respect to s at parameter value t, where as usual s denotes the arclength. The formula is then:*

$$\kappa = |\frac{d\mathbf{T}}{ds}|.$$

The formula looks cleaner than it really is. Both $\mathbf{T}$ and s involve square roots of sums of squares and the chain rule may be needed to differentiate in s, since s is not usually given explicitly and even if it is, to reparameterize requires t solved as a function of s. As written, to compute the curvature would require either reparameterizing in terms of the arclength, then differentiating, or computing in t and using the chain rule. Therefore after one example, the cycloid, a more concise approach will be developed that gives (almost always) an easier calculation of the curvature.

You are hopefully thinking right now: how does the geometric definition using radius of a circle relate to this? For circles, the two are the same! It is easiest to calculate conceptually: in one full circle trip of radius r, how much does the unit tangent turn? How far was such a trip? What is the derivative of the change in angle with respect to arclength? Wasn't that fun?

A more pedestrian calculation would say:one parametric version of motion around a circle of constant angular speed is $x = r \cos t$, $y = r \sin t$ with r constant. Arclength s is rt. The velocity vector is $< -r \sin t, r \cos t >$, so the unit tangent vector in terms of arclength on the given circle is $\mathbf{T}(s) =< -\sin(s/r), \cos(s/r) >$ so finally $|\frac{d\mathbf{T}}{ds}| = | < -\cos(s/r)1/r, -\sin(s/r)1/r > | = 1/r$. Not as much fun!

Example 2.12 *The* **cycloid** *has parametric form: $x = t - \sin t, y = 1 - \cos t$. We find $\mathbf{r}'(t) =< 1 - \cos t, \sin t >$ and $\mathbf{r}"(t) =< \sin t, \cos t >$. Therefore $\frac{ds}{dt} = |\mathbf{r}'(\mathbf{t})| = \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{2 - 2\cos t}$ after some algebra. $\mathbf{T} = \frac{\mathbf{r}'(t)}{\frac{ds}{dt}}$ While it turns out that $s(t)$ can be given explicitly in this case, see TBN above, that isn't very helpful. Using the chain rule, $|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}| = |\frac{d\mathbf{T}(t)}{dt}|$ so we find MORE GOES HERE*

2.3.4 Planar case: a useful formula

When a parametric curve lies in the $x - y$ plane, a formula for the angle the unit tangent makes with the positive x-axis, call it ϕ, can be found fairly cleanly. By definition, the derivative $\frac{dy}{dx}$ is the slope of the tangent line, so

$$\tan\phi = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Here the parametrization is general, so it includes the arclength parametrization. The chain rule used in several spots then leads to the formula:

$$\begin{aligned}\frac{d}{ds}(\tan\phi) &= \sec^2\phi\,\frac{d\phi}{ds} = \frac{d}{ds}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\ &= \frac{dt}{ds}\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\end{aligned}$$

which then leads to the following formula for parametric curves in the plane (using some algebra and the quotient rule):

$$\kappa = |\frac{d\phi}{ds}| = \cos^2\phi|\frac{dt}{ds}|\frac{|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}|}{(\frac{dx}{dt})^2}$$

which cleans up a bit after some algebra relating ϕ back to the parametric derivative form, to become:

$$\kappa = |\frac{d\phi}{ds}| = |\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}|\left((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2\right)^{-\frac{3}{2}}$$

which is pretty convenient.

Example 2.13 *Suppose $y = a\,x^2$ and we use x as parameter. More formally we can write $x = t$ and $y = a\,t^2$. Then the above formula has $\frac{dx}{dt} = 1$ and $\frac{d^x}{dt^2} = 0$ while $\frac{dy}{dt} = 2a\,t$ and $\frac{d^2y}{dt^2} = 2a$, so our general formula in the plane yields the curvature for parabolas written in parametric form:*

$$\kappa = \frac{2a}{(1 + 4a^2\,t^2)^{\frac{3}{2}}}$$

which is often rewritten with x instead of t.

The parabola example extends to a general graph in the plane of the form $y = f(x)$ where f is a C^2 function of x. The details are left as a problem TBN to find:

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{\frac{3}{2}}}$$

written with x as parameter.

We will rework our cycloid example in this format soon, but first we wish to recast this to a general form for curves in 3-D by using vector algebra.

2.3.5 Vector algebra extension to 3-D

Looking at the formula derived above, the mixture of first and second derivatives is the $\mathbf{k}$-component of a cross product of the form: $< \frac{dx}{dt}, \frac{dy}{dt}, 0 > \times < \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, 0 >$, which has all other components 0. The general vector analogue of that is the cross product $\mathbf{r}' \times \mathbf{r}''$, leading to the vector rewrite of the above (to be justified):

The **curvature** of a curve is given by either of the following formulas:

$$\kappa = |\frac{d\mathbf{T}}{ds}| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

which is nice and tidy, yet reasonable to calculate! It is also not expressed in terms of coordinates directly.

The justification of the general formula above comes from our previous consideration of circular motion and the splitting of acceleration into tangential and normal components. Moving along a curved path (non-zero curvature) requires a normal component which is nonzero. The tangential acceleration is the component tied to acceleration in the current instantaneous direction, which has no effect on the turning of the unit tangent vector (recall: it changes the magnitude of the velocity vector only). Since the definition of osculating circle followed in constant angular speed has matched the velocity vector MORE GOES HERE

Example 2.14 *The* **cycloid** *still has parametric form:* $x = t - \sin t, y = 1 - \cos t$. $\mathbf{r}'(t) =< 1 - \cos t, \sin t >$ *and* $\mathbf{r}''(t) =< \sin t, \cos t >$. *As before,* $|\mathbf{r}'(\mathbf{t})| = \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{2 - 2\cos t}$. *Now* $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 1 - \cos t$ *so the curvature at* $\mathbf{r}(t)$ *is equal to* $\frac{1-\cos t}{(\sqrt{2(1-\cos t)})^3}$ *which becomes* $\frac{1}{2\sqrt{2(1-\cos t)}}$ *after some algebra.*

2.3.6 Planar vs. 3-D issue: torsion and TNB frame

The question of whether a path lies in some plane or not leads to another concept: torsion. Cars have torsion bars that seek to maintain a planar orientation of the passenger cabin even when the road is not planar, as when driving over railroad ties or cobblestones or a pot-holed road. In the geometry of curves in 3-D, the **binormal vector** is defined as the remaining unit vector to form a right-handed coordinate system with the unit tangent and unit normal. Then **torsion** is defined as the rate of change of the unit **binormal** vector, which is $\mathbf{T} \times \mathbf{N}$, with respect to arclength. If the motion takes place in a plane, then the binormal vector is constant (orthogonal to the plane of motion) and the torsion is zero. In general, the torsion is usually denoted by a Greek 't', which is τ, and the formula is:

$$\tau = |\frac{d\mathbf{B}}{ds}|$$

2.3.7 Brief sidebar on the helix and DNA coiling / supercoiling

Another important model of a curve in space is a the **helix**, which can be thought of as a circle in two directions paired with a constant velocity in the transverse (third) direction. It has the same acceleration as the traditional circular motion and physicists might think of it as arising DNA in cells is often very tightly wound

up since the length of a strand may be thousands of times longer than the cell dimensions. This is known as **supercoiling** and the mechanics of twisting and writhing may be familiar to you from playing with wound-up rubber bands or the nearly defunct phone cord on some landline phones. Modeled as a curve in space such supercoiled strands will have extremely large curvature in some places! Other models might view the DNA as a ribbon rather than a curve. The helix typically does one winding in about 10.4 base pairs, but the supercoiling refers to how the double helix is wound up about itself (biologists call this tertiary winding). As real objects, they must be extremely thin in the transverse directions (across the double helix) so that their volume remains feasible within the cell.

2.3.8 EXERCISES

Writing: Explaining, Reacting, Questioning

1. Consider the familiar sine graph. How do you think the curvature varies qualitatively as different points along the sine graph are considered? Where do you believe the curvature is largest? Where do you believe the curvature is smallest? Is it ever 0?

2. The computer demonstration TBN shows the osculating circles along a parabola. How do you expect the circles to vary as points along the parabola are chosen? After looking at the demonstration, were you right? What surprised you?

3. Consider the familiar tangent graph: $y = \tan x$. How do you think the curvature varies qualitatively as different points along the graph are considered? Where do you believe the curvature is largest? Where do you believe the curvature is smallest? Is it ever 0?

4. Consider the general ellipse given parametrically as $x = a\cos t$, $y = b\sin t$ which has $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. First assuming $a > b$, where do you think the curvature is the largest? The smallest? Second find the curvature and write a paragraph explaining the result if you were intuitively correct or describing your error if your intuition was not correct.

5. Using the geometric definition of curvature in the plane (osculating circles to curves), describe how the curvature of the graph of a function is linked to the curvature of the graph of its inverse function. Do this geometrically. Write out an example also algebraically. If you want to do a general algebraic (messy) calculation instead, see exercise ?? below.

6. Check algebraically that the parametric form for curvature of a curve agrees with the graph form: ($y = f(x)$) by using the parametric form: $x = t, y = f(t)$. Did you use the unit tangent vector form or the parametric version? Why did you do it the way you did?

7. Write a paragraph comparing calculation of curvature using the parametric form versus the unit tangent derivative for the helices ($x = \cos t, y = \sin t, z = at$ where $a > 0$ is a constant. Explain the good and bad aspects of each version in your case.

8. Write a paragraph comparing calculation of curvature using the parametric form versus the unit tangent derivative for a problem not in the book and not a helix. Try to make your example both instructive and clear. Explain the good and bad aspects of each version in your case.

9. Check that the curvature formulas above always have the proper dimension, assuming length for x, y, and z and time for t. Also compare with arclength parametrization.

10. The cycloid has a lot of calculus history attached to it. Read part of a history of mathematics text (for example, Katz) or find information online and write a paragraph or two on the cycloid.

11. (circular matching property for graphs – special case) Consider the graph of a function $y = f(x)$ near a point (x_0, y_0) where the derivative $f'(x_0) = 0$. The osculating circle must have its center at a point directly above or below (x_0, y_0). To simplify the algebra, pick new coordinates if needed so the point is at $(0, 0)$. The osculating circle is $x^2+(y-b)^2 = b^2$ for some value of b. At the same time, by Taylor polynomial approximation, $y \approx \frac{1}{2}f''(0)x^2$. Write y for the circle as a function of x and show that to match the Taylor polynomial there is only one choice of b, namely $b = 1/f''(0)$. Hint: Taylor polynomial of order 2 for the circle is not that hard, especially with an algebra trick.

12. (challenging, but doable, for the geometrically inclined) Use the previous result to derive the general formula by rotation and a change of coordinates: consider your point (x_0, y_0) as the origin, then using the tangent line, rotate and use new coordinates, of your own naming, for the change in the direction in the tangent and the change perpendicular to it. It will turn out that in those new coordinates you are back to the previous problem, so the curvature will be the reciprocal of the second derivative in the new variables. Then show (by Taylor approximation or otherwise) that the second derivative in these new variables is the expression $f''(x_0)(\cos\phi_0)^3$ where $f''(x_0)$ is the second derivative without rotation (meaning in (x, y) world) and the cosine term uses the angle of rotation, which means $\tan\phi_0 = f'(x_0)$.

13. (circular matching property in 2-D general case directly) Consider a curve in the plane given as $x(t)\imath + y(t)\jmath$. Remind yourself what the tangent vector of such a curve is and therefore find a convenient vector perpendicular to

the tangent direction. The best fit circle has its center, say (a, b), on the normal line (the perpendicular to the tangent line at the given point), so the distance from the center to the curve at $t = t_0$ will determine the circle. You can parameterize the family of circles tangent to the curve at some location as $(x(t) - a)^2 + (y(t) - b)^2 = R^2$ (perhaps an overuse of t) and then seek to match the circle with the graph as closely as possible. This requires (a, b) to lie on the normal line and the square of the radius to be $R^2 = (x(t_0) - a)^2 + (y(t_0) - b)^2$. Show that taking first and second derivatives in t at $t = t_0$ yields the center and radius and that the radius as determined in this way is the radius of curvature.

14. Challenging (limit version like tangent as limit of secant lines): Consider a curve in the plane (for simplicity). Pick three distinct points on it, say (x_0, y_0), (x_1, y_1), and (x_2, y_2). These points determine a circle through all three. Take the limit as the points (x_1, y_1) and (x_2, y_2) approach the fixed location (x_0, y_0) to find the equation of the osculating circle and the radius of curvature.