

1.5 Lines and Planes

Overview: Lines and planes are linear objects that are very familiar but also key to understanding the local behavior of more general objects. The main features and ways of viewing lines and planes are reviewed. Using vectors and matrices simplifies the algebra.

In one variable calculus, lines play an important part as local approximations to general curves. In multi-variable calculus, lines and planes are the extensions of those ideas as local linear approximation. The description of lines and planes in two or three dimensions can be done in several ways, either built from the bottom up or top down, as will be developed in this section.

1.5.1 Line in plane revisited

Let's recall how many ways lines were described in a two dimensional setting as in calculus. First was the traditional slope intercept form, $y = mx + b$ followed by the extension to point slope form: $y - y_0 = m(x - x_0)$. As well, sometimes lines were viewed as coming from a symmetric form where x and y coordinates were treated similarly, as in $x + 3y = 7$. In calculus terms, the symmetric form describes an implicit function. Finally, there were also some situations where a line was described parametrically, using a parameter such as t .

1.5.2 Plane in 3-D as linear equation

In three dimensions, one linear equation will describe a plane. This may be in the explicit form solved for z or in implicit form where x , y and z are treated similarly. Planes will play a key role in the calculus of functions of two variables where a plane will take the role of the tangent line in the calculus of functions of one variable.

Example 1.34 *The function $z = f(x, y) = x + 3y$ has as its graph all points of the form $(x, y, x + 3y)$, which form a plane. This plane can be viewed as being generated by the slices where y is frozen, which are lines in x, z each with slope 1 and slices where x is frozen, which are lines in y, z with slope 3. Slices in values where z is frozen result in lines in x, y with slope $-1/3$. An implicit form would be $x + 3y - z = 0$.*

The general form of a plane as the graph of a function of x and y would be $z = Ax + By + D$ for some constants A, B, D . In first discussion coordinates, the planes given by $z = D$ were described. In some sense the general form is a tilted version of the coordinate plane.

Vector form: As noted before, an expression like $Ax + By + D$ can be viewed as the dot product of a vector \mathbf{v} with components $\langle A, B \rangle$ with the 2-D position

vector $\mathbf{r} = \langle x, y \rangle$ and then an addition of D . If some $x - y$ position on the plane is denoted by \mathbf{r}_0 with corresponding z value z_0 , then $z_0 = \mathbf{v} \cdot \mathbf{r}_0 + D$ and after subtraction we obtain a “point-slope” form of the plane: $z - z_0 = \mathbf{v} \cdot (\mathbf{r} - \mathbf{r}_0)$.

In high school, you also learned that three points determine a plane (unless they all lie on a line, which is the exceptional case). In the equation description, given the three points a normal vector must be found. Then the normal vector gets resized so that the z component can be isolated and the function form obtained. This is usually done by using the cross product since the three points will lead to two vectors connecting each of the other two points to one of the points. These vectors are actually **tangent to the plane**.

Example 1.35 *Given the points $A(1, 1, 1)$, $B(3, 2, 2)$ and $C(-1, -2, 1)$, find an equation of the plane containing them and check your answer. **Solution:** the vectors from A to B and from A to C are $\langle 2, 1, 1 \rangle$ and $\langle -2, -3, 0 \rangle$ and their cross product is calculated to yield the normal vector: $\langle 3, -2, -4 \rangle$ and thus the symmetric equation: $3(x - 1) - 2(y - 1) - 4(z - 1) = 0$ and the function form in “point-slope” form: $z - 1 = \frac{3}{4}(x - 1) - \frac{1}{2}(y - 1)$. We check easily that all three points are solutions.*

As described previously, the plane $Ax + By + Cz = D$ can be written in vector form as $\mathbf{n} \cdot \mathbf{r} = D$ where now \mathbf{r} is the 3-D position vector! This highlights the role of the vector $\mathbf{n} = \langle A, B, C \rangle$ which is normal (perpendicular) to the plane with the direction of \mathbf{n} pointing towards higher values of the linear function $Ax + By + Cz$ and the magnitude showing the rate of growth. Changing the value of the constant D shifts the plane to a new plane parallel to the original one, where the gap is linked to the magnitude of \mathbf{n} .

1.5.3 Plane in 3-D: parametric version as generated by two lines

An alternative description of a plane in three dimensions is given in terms of two parameters: this corresponds to the familiar statement that two intersecting but distinct lines in space determine a plane. This is also stated in the form: three non-collinear points determine a plane.

Consider our previous example of three points and the two vectors that connect points. The set of all multiples of either one of those vectors passing through point A is a line through A . Using a combination of both will generate the plane in the form of a two-parameter expression, say in (s, t) , which is easily written in vector form using 3-D vectors: $\mathbf{r} = \mathbf{r}_0 + s\mathbf{v} + t\mathbf{w}$ which in our example becomes $\mathbf{r} = \langle 1, 1, 1 \rangle + s \langle 2, 1, 1 \rangle + t \langle -2, -3, 0 \rangle$. In coordinates this says:

$$x = 1 + 2s - 2t, \quad y = 1 + s - 3t, \quad z = 1 + s$$

which can be seen to always satisfy the equation of the plane no matter what values are used for s and t . The values of s and t describe coordinates that correspond to

using the grid generated by the two vectors instead of the usual coordinate axes. This is a plane given in parametric form.

To move from the parametric version to a linear equation for the plane, use the cross product as before. You could then regard our first description in terms of a linear function of x and y as a special case of the general parametric version where we are using as parameters x and y and therefore our three equations are: $x = x, y = y$, and $z = Ax + By + D$, so not writing the first two leads us back to the first formulation.

1.5.4 Line in 3-D: parametric form

How do we describe a line in three dimensional space? There are two alternative descriptions, one involving the parametric form used already in discussing planes and the other invoking the geometric description of a line as an intersection of two non-parallel planes.

The parametric form is as above only with one parameter: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ which goes from the location \mathbf{r}_0 and moves as t increases in the direction of \mathbf{v} .

This description also naturally aligns itself with the other geometric statement: two points determine a line. Use the first point and the vector between the two points to create the parametric version.

Example 1.36 Find the line through the points $(2, 3, 4)$ and $(7, 8, 9)$. **Solution:** $\mathbf{r} = \langle 2 + 5t, 3 + 5t, 4 + 5t \rangle$.

1.5.5 Line in 3-D: intersection of planes

Intersecting two planes given by equations means seeking simultaneous solutions of both equations. In 3-D, this means each plane is given either as an explicit or a symmetric form. In either case, simple algebra of elimination will lead to a parametric form. In algebra class, the parameter was called a “free” variable. Two equations in three variables leads to one such variable and the other two coordinates are then expressed in terms of that one.

Example 1.37 Find a parametric description of the line given by the intersection of $x + y - z = 0$ and $2x - y + 2z = 4$. **Solution:** Setting $y = 0$ (somewhat arbitrarily) in a hunt for a point yields $x = 1, y = 0, z = 1$. Using elimination of x from the second equation by subtracting twice the first equation from it leaves: $-3y + 4z = 4$, so if we write $z = 1 + t$ for the parameter (so $t = 0$ is at our point where $z = 1$, we get also $y = \frac{4}{3}t$ and finally $x = 1 - \frac{1}{3}t$. **Alternative solution:** Use the two normal vectors and the cross product to get a tangent vector for the line: $\langle 1, 1, -1 \rangle \times \langle 2, -1, 2 \rangle = \langle 1, -4, -3 \rangle$ which gives an alternative parametric version which moves in the opposite direction at a faster speed.

1.5.6 Remarks on dimension, parameters, equations

These geometric and algebraic discussions are the simplest examples of some important themes that we will highlight throughout the remainder of this course: dimension, parameters, equations, and symmetry. Note that it requires one parameter to map out a line and two to map out a plane. When using equations to restrict the variables, one linear equation in 3-D cuts the dimension down to 2, while two linear equations cuts the dimension down to 1. Our last example showed that alternative parametrization is possible also.