1.2 Dot Product of Two Vectors (aka Scalar Product)

Overview: The dot product of two vectors is an algebraic operation that is important to understand both geometrically and algebraically. It forms the basic tool for orthogonal projection, which describes how we view 3-D objects on 2-D screens. Projection onto 1-D objects is also important since it describes how much one vector points in a second vector direction.

A more complicated vector operation arises from the following question: how much does the wind push me along the direction of a straight path? This requires the calculation of the component of one of the vectors (the wind) in the direction of a second vector (the direction of my path). This problem is already familiar to us when we think about triangle trigonometry. The component of one vector in the direction of the second vector is given by the magnitude of the first multiplied by the cosine of the angle between the two vectors. This description is independent of a choice of coordinates but is it not particularly easy to calculate if our coordinate choice is already made. Therefore we will derive that expression in terms of coordinates.

To make the algebra easier (by being more symmetrical in the vectors), the object we will consider is the product of the length of **each vector** multiplied by the cosine of the angle between them. This object will turn out to be a special kind of product with the property that it is additive in each factor. To see this it is easier to use the coordinate description! This illustrates the power of having multiple representations for the same object.

1.2.1 Dot product defined geometrically

Definition 1.17 *The* **dot product** *of the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is defined to be the* **scalar**

$$|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta, \quad \text{where } \theta \text{ is the angle between the vectors}$$

and it usually denoted $\mathbf{a}\cdot\mathbf{b}$, *which explains the name of dot product.*

Consequences of the geometric formula:

- The dot product is symmetric in the vectors: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.
- If either vector is scaled, the dot product scales in the same way. So if $\mathbf{a}\cdot\mathbf{b} = 2$, it follows that $(3\mathbf{a})\cdot\mathbf{b} = 6$.
- The dot product of the zero vector with any other vector is zero: $\mathbf{a}\cdot\mathbf{0} = 0$.
- The dot product of any vector with itself is the length squared: $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.
- The dot products of two distinct standard unit coordinate vectors is zero, therefore $\mathbf{i}\cdot\mathbf{j} = 0$ in 2-D and in 3-D $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.

- The dot product of any two vectors $\mathbf{a}$ and $\mathbf{b}$ satisfies the double inequality: $-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$
- In rotated coordinates, the dot product is unchanged. Also if we rotate both vectors simultaneously in time, the dot product is constant no matter how the rotation happens in time.

Important remark: Unlike multiplication of numbers, it is possible for the dot product to be zero without either 'factor' being the zero vector. This occurs when the cosine factor is zero rather than the magnitudes and therefore happens when the vectors make a right angle with each other.

Definition 1.18 *Two vectors are said to be* **orthogonal** *when the angle between them is a right angle, or equivalently when their dot product is zero.*

Shortcomings of the geometric formula: Finding the dot product of vectors especially with given coordinates may be somewhat lengthy. As well, if we wish to understand how the dot product relates to vector addition, the geometric version is not particularly easy.

1.2.2 Algebraic version in coordinates

Coordinate version in a special case: To create the coordinate version in general, first consider the special choice of coordinates where the first vector is lined up with the x coordinate vector and both vectors live in a 2-D plane. We record this result in a lemma:

Lemma 1.19 *The dot product of the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *with components* $< a_1, 0 >$ *and* $< b_1, b_2 >$ *equals* $a_1\, b_1$.

The lemma follows directly by writing $|\mathbf{a}| = a_1$, $|\mathbf{b}| \cos\theta = b_1$ (from basic trigonometry) and then calculating $a_1\, b_1$.

Law of cosines recalled and general coordinates in 3-D: The lemma is very useful since it allows us to recall the law of cosines from trigonometry (and its proof!), namely that if the vectors $\mathbf{a}$ and $\mathbf{b}$ with lengths a and b and angle between them θ form the sides of a triangle lying in the usual coordinate plane, then the third side ($\mathbf{b} - \mathbf{a}$) has its length (c) given by the following generalization of the Pythagorean theorem:

$$c^2 = a^2 + b^2 - 2ab\, \cos\theta$$

which is shown by calculating the square of the distance between the points with coordinates $(a, 0)$ and $(b\, \cos\theta, b\, \sin\theta)$:

$$c^2 = (b\, \cos\theta - a)^2 + (b\, \sin\theta - 0)^2$$

and using our favorite trig identity. The calculation is left as an exercise.

The geometry of the law of cosines is unchanged when using general locations in 3-D, so the law of cosines must be true in the general case. Note how special coordinate choices were exploited to deduce a general fact.

If the law of cosines is manipulated, an expression for the dot product is created that can be evaluated in general rectangular coordinates in 3-D.

Theorem 1.20 *For vectors in standard component form:* $\mathbf{a} =< a_1, a_2, a_3 >$, $\mathbf{b} =< b_1, b_2, b_3 >$ *the following result always holds:*

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2) = a_1\, b_1 + a_2\, b_2 + a_3\, b_3.$$

This is the easiest way to calculate a dot product when the components of the vectors are known. Moreover it leads to some other important properties that are more difficult to establish using the geometric (coordinate free) definition.

Consequences of the coordinate formula: The following facts are easier to show in coordinates:

- For any vectors $\mathbf{a}$,$\mathbf{b}$, and $\mathbf{c}$: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.
- For any vector $\mathbf{a}$, each of the vectors with components given below are orthogonal to $\mathbf{a}$: $< 0, a_3, -a_2 >, < -a_3, 0, a_1 >, < a_2, -a_1, 0 >$ and therefore so is their sum: $< a_2 - a_3, a_3 - a_1, a_1 - a_2 >$. Note how these relate when applying the cyclic rotation.
- The equation $Ax + By + Cz = 0$ for unknown $< x, y, z >$ components of a vector for given values of A, B, C represents the set of all vectors orthogonal to the given vector $< A, B, C >$. This defines a plane orthogonal to the given vector.

Some examples:

Example 1.21

Remark: If the dot products of the standard coordinate vectors with each other are used and linearity is assumed, then

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1\, b_1 + a_2\, b_2 + a_3\, b_3.$$

Direction cosines: For a given **unit** vector $\mathbf{u}$, the dot products with the usual coordinate unit vectors are called the **direction cosines** of the vector. From the dot product definition, the angles α, β, γ that the vector makes with the x, y and z axes satisfy the relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Picture goes here

1.2.3 Projections

The dot product can be used to project orthogonally. This follows from the appearance of the cosine in the geometric formula. While in 2-D there is only one scenario, which is projection onto a 1-D space, in 3-D we could in principle project onto 1-D and 2-D spaces like the z axis or the $x - y$ plane. Those examples can be written down in coordinates, but the general case is best described in vector algebra form.

Example 1.22 *The projection of the vector* $\mathbf{r} =< x, y, z >$ *to the corresponding vector in the* $x - y$ *plane is* $< x, y, 0 >$ *and to the vector along the* z *axis is* $< 0, 0, z >$*. Thus* $< 1, 2, 3 >$ *projects to* $< 1, 2, 0 >$ *in the* $x - y$ *plane.*

Note from this example, which was likely very comfortable for you, that the two parts add up to the original vector. So for a general situation in 3-D, the 1-D part generated in the direction of a given fixed vector and the 2-D part orthogonal to it will add up to the original vector. Finding one piece will automatically create the other by subtraction.

For a given fixed non-zero vector $\mathbf{a}$, every vector $\mathbf{v}$ can be decomposed into a piece in the direction of $\mathbf{a}$ and a piece orthogonal to $\mathbf{a}$, each projected orthogonally. For the part in the direction of $\mathbf{a}$, the answer should have size given by $|\mathbf{v}| \cos\theta = (\mathbf{a} \cdot \mathbf{v})/|\mathbf{a}$ so we have using the dot product and some notation which should be clear:

$$Proj_{\mathbf{a}}(\mathbf{v}) = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{a}|} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{a}|^2} \mathbf{a}$$

Lengths and areas under projection: The length of a vector when projected is shortened by the factor $\cos\theta$ as noted above. In 3-D, there can also be 2-D objects that are projected onto a plane, in which case the area is also decreased by multiplying by the cosine of the angle between the planes. This can be seen by considering the axes in the direction of the common line of intersection of two planes and in an orthogonal direction in the target plane. One direction is not rescaled at all while the other is shortened by a cosine factor.

Using the direction cosines by considering either the projection onto coordinate axes or the projection onto coordinate planes, there are then two versions of Pythagorean theorems, one for lengths and one for planar areas, each of which involves the sum of squares.

Example 1.23 *Consider the triangle with vertices* $(1, 0, 0), (0, 1, 0)$ *and* $(0, 0, 1)$ *and show that the projected areas onto the coordinate planes, when squared, add up to the square of the triangle's area.*

Solution: The projected triangles are all congruent since the vertices are at symmetric points. The projected triangles are right triangles with side lengths of 1, so each has area $\frac{1}{2}$. The sum of the squares is then $\frac{3}{4}$. The original triangular region is an equilateral triangle of side lengths $\sqrt{2}$ so it has area $\frac{\sqrt{3}}{2}$ which squares to $\frac{3}{4}$ as claimed.

Dot products will be employed many times in the remainder of this course. The exercises below will develop your conceptual and computational understanding of this important topic. You have likely already seen this idea in your precalculus experience. As well, it is often used to describe how we multiply matrices.

1.2.4 EXERCISES

Writing: Explaining, Reacting, Questioning

1. What does the relation $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ say about the relation of projecting the vector $\mathbf{a}$ in the direction of $\mathbf{b}$ compared to the projection of $\mathbf{b}$ in the direction of $\mathbf{a}$?

2. Many textbooks define the dot product directly in coordinates. List the main advantages and disadvantages of doing things that way. Find a book that defines the dot product in coordinates and read how the author(s) show that it has the geometric property used as the definition above. Did you find that easy or hard to follow? Why?

3. In the geometric definition of the dot product, it follows that rotating the coordinate system about the origin doesn't change the dot product. How would you show that fact for the coordinate formula in 2-D? In 3-D?

4. Using polar coordinates in 2-D and the coordinate formula for the dot product, show that the dot product in two variables depends only on the angle between the two vectors and is in fact the geometric version used as the definition.

5. Using spherical coordinates (ϕ, θ) for two points on the unit sphere, with coordinates (ϕ_1, θ_1) and (ϕ_2, θ_2) respectively, find the cosine of the angle between them expressed in terms of the spherical coordinates. Discuss the special cases when $\theta_1 = \theta_2$ or $\phi_1 = \phi_2$. Explain why the results are different. Also note that the angle between the two vectors is the length of the arc between them on the sphere, so you have found a formula for the distance along the surface of the sphere for spherical coordinates.

6. Showing that the dot product of vectors in 3-D is unchanged when we rotate the coordinate system is not easy if we try a direct assault. Anita Solushun notices that once you show it for the 2-D case, you could do any number of rotations in the three coordinate planes (which freeze one coordinate for both vectors while changing the other two). Assuming Euler's result that three such rotations will take any coordinate frame into another in 3-D, describe how that could show that the coordinate formula is the same in the rotated coordinates.

7. In physics, the force is tied to the acceleration via Newton's famous law. When a body moves down an inclined plane under gravity, can gravity be the only force present? As the plane becomes more inclined, what changes? What happens in the limit of a vertical incline? Galileo used the limit of inclined planes to conclude that the classic falling body under the influence of a constant force of gravity will fall a distance proportional to t^2 where t is the elapsed time. He did this by listening to the rhythm of a ball rolling down inclined planes over bumps spaced like t^2 as he varied the incline angle.

8. Does the dot product have all the features you would expect when using the word “product”? If not, what features are different? If so, what features did you check? (You missed some.) Do you think it should be called a product or should it have another name?

9. Given three vectors in 3-D, how could you use the dot product to check whether the third vector lies in the plane generated by the first two (or the line if they are proportional)?

10. Given two vectors in 2-D, how could you use the dot product to modify the second vector by subtracting a scalar times the first vector to create a vector orthogonal to the first one? When will this new vector be non-zero?

11. (Extension of previous problem) Answer the same question when the two vectors are in 3-D. Then consider a third vector as well and describe how to subtract multiples of the first two (or the first two after modification) to make a new third one orthogonal to the first two.

Calculational Exercises

For problems 1 to 12, find the indicated dot products:

1. (Pythagorean theorem for projected areas)

2. (coordinate axes and cyclic rotation revisited) Consider the usual coordinate unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and project them on the plane given by the normal vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$ by subtracting from each the component in the direction of the normal. Show that these projected vectors form angles of $\pm 2\pi/3$ with each other using the dot product.