SOLUTION

MATH 213, Section 002 Name DONNA WANDA CALCULATOR
Spring, 2007, Prof. Sachs
Exam 3 G number

- Show all work clearly and completely. *You may lose points for incorrect, incomplete, or unclear work, even if your final answer is correct.*
- Do all work in the space provided, or, if extra room is needed, continue your work on the back of the exam pages, being sure to indicate clearly where the work is. Put your final answer in the box. Extra paper will not be accepted.
- This exam has 8 problems, each of which is worth 10 or 15 points.
- GMU Honor Code is of course in effect. You may not use notes, books, or each other. A calculator is ok.

1. (10 pts.) For the vector function $\vec{v}(x,y,z) = e^x \cos y\,\vec{i} + e^x \sin y\,\vec{j} + (3z+7)\,\vec{k}$ compute $\vec{\nabla} \times \vec{v}$, the curl of $\vec{v}$. What does this say about finding a scalar $F(x,y,z)$ so that $\vec{\nabla} F = \vec{v}$?

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ e^x\cos y & e^x \sin y & 3z+7 \end{vmatrix} = \vec{i}\left[(3z+7)_y - (e^x\sin y)_z\right] + \vec{j}\left[(e^x\cos y)_z - (3z+7)_x\right] + \vec{k}\left[(e^x\sin y)_x - (e^x\cos y)_y\right]$$

$$= \vec{i}\,(0) + \vec{j}\,(0) + \vec{k}\left(e^x\sin y + e^x\sin y\right)$$

sign is +

NOTE: Derivative "operator" goes in row 2 NOT specific derivatives

Ans: $2e^x\sin y\,\vec{k}$

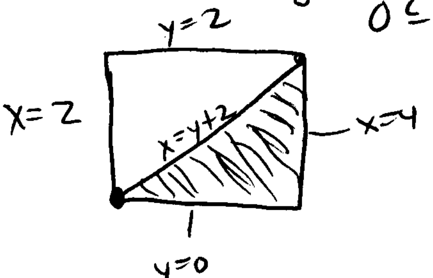
keep as vector

2. (10 pts.) Find an equivalent integral with the order of integration reversed:

$$\int_0^2 \int_{y+2}^4 f(x,y)\,dx\,dy$$

Domain $y+2 \le x \le 4$
$0 \le y \le 2$

So $x$ ranges fully from 2 to 4 and $y$ runs up at given $x$ from 0 to $y=x-2$




SOME OF YOU USED WRONG SIDE OF "BOX"

Ans: $\int_2^4 \int_0^{x-2} f(x,y)\,dy\,dx$

3. (15 pts.) Evaluate the following integral by changing to polar coordinates:

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2\,dy\,dx}{(1+x^2+y^2)^3}$$

$-2 \le x \le 2$
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ } disk, radius 2 – all quadrants [Some of you goofed here]

$0 \le \theta \le 2\pi$
$0 \le r \le 2$

radius is $\to 2$
[DON'T USE $-2 \le r \le 2$]

$$= \int_0^2 \int_0^{2\pi} \frac{2}{(1+r^2)^3}\, r\,d\theta\,dr$$

$u = 1+r^2$, $du = 2r\,dr$

$\int u^{-3}\,du = -\frac{u^{-2}}{2} + C$

$$= 2\pi \cdot \int_0^2 2r \cdot (1+r^2)^{-3}\,dr$$

$$= 2\pi \cdot \left.\frac{(1+r^2)^{-2}}{-2}\right|_{r=0}^{r=2} = \pi\left[-\frac{1}{25}+1\right]$$

Ans: $\frac{24\pi}{25}$

NOTE: IF $u$ substitution is used, transform endpoints too OR come back to $r$ at evaluation time

$\rightarrow$ $r=0 \iff u=1$
$r=2 \iff u=5$

4. (15 pts.) Evaluate the triple integral:

$$\int_{-1}^{1}\int_{0}^{3}\int_{0}^{3}(x^2 y + y^2 + z^2)\, dz\, dy\, dx$$

Some goofed here

$$\int_{-1}^{1}\int_{0}^{3}\left[x^2yz + y^2z + z^3/3\right]_{z=0}^{z=3} dy\, dx = \int_{-1}^{1}\int_{0}^{3}\left[3x^2y + 3y^2 + 9\right] dy\, dx$$

$$= \int_{-1}^{1} \tfrac{3}{2}x^2y^2 + y^3 + 9y \Big|_{y=0}^{y=3} dx = \int_{-1}^{1}\left[\tfrac{27}{2}x^2 + 27 + 27\right] dx$$

$$= 9/2\, x^3 \Big|_{-1}^{1} + 54 \cdot 2 = 9 + 108$$

Ans: 117

5. (10 pts.) Give a careful description of why it is legitimate to change the order of integration in a double or triple integral.

In a double or triple integral, we are finding the limit of the corresponding Riemann sums. IF we choose division points to form a grid — lined up in x and y (and z)

places where we evaluate f for each term in Riemann sum

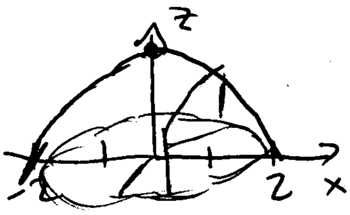
then each such sum is a nested sum over either order of summation. Therefore any limit must be the same in either order of summation.

6. (10 pts.) In converting a triple integral from rectangular coordinates $(x, y, z)$ to spherical coordinates $(\rho, \phi, \theta)$, Cal Clueless replaced the differentials $dx\,dy\,dz$ by $\rho^2\,d\rho\,d\phi\,d\theta$. Briefly explain why this is wrong.

Cal has the wrong scale factor → it should be $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. This comes from a 3x3 Jacobian or geometrically → the small steps in $\rho, \phi, \theta$ are orthogonal and the angles need a radial factor. For $d\phi$ it is $\rho$ BUT for $d\theta$, we move around latitudes – equator is longer than near N or S pole – so we get $\rho \sin\phi\, d\theta$ factor → combines to $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

7. (15 pts.) Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$ and below by the $xy$ plane.

Key word CYLINDER → suggests base in polar $(r, \theta)$
Keep $z$ since $z$ runs from 0 to $4-x^2$




Slices in x are rectangles in y-z

Rectangular $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2} dz\,dy\,dx$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2)\,dy\,dx = \int_{-2}^{2} 2\cdot(4-x^2)\sqrt{4-x^2}\,dx$$

use this substitution
[not many were OK here]

Cylindrical
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

SOME GOOFED HERE

$$\int_0^2\int_0^{2\pi}\int_0^{4-r^2\cos^2\theta} r\,dz\,d\theta\,dr = \int_0^2\int_0^{2\pi} r\,(4-r^2\cos^2\theta)\,d\theta\,dr = \int_0^2 [8\pi r - \pi r^3]\,dr$$

$$= 4\pi r^2 - \frac{\pi r^4}{4}\Big|_0^2$$
$$= 16\pi - 4\pi$$
$$= \boxed{12\pi}$$

NEED $\displaystyle\int_0^{2\pi}\cos^2\theta\,d\theta = \pi \Leftarrow \int_0^{2\pi}(\cos^2\theta + \sin^2\theta)\,d\theta = 2\pi$

or double angle $\displaystyle\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)\,d\theta = \int_0^{2\pi}\cos 2\theta\,d\theta = 0$

8. (15 pts.) (a) What is a Jacobian determinant for a coordinate transformation $x = g(u, v), y = h(u, v)$?

It is the determinant found using gradients as rows?

Ok to use $x, y$ also

$$\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$$

(b) Why does it enter in a change of variables in a double integral?

It measures the local scale of area → small rectangle $du\,dv$ in $u,v$ is parallelogram in $x,y$

(c) Find the Jacobian determinant of the given change of coordinates: $x = u^2 + v^2,\ y = 2u\,v.$

$\frac{\partial x}{\partial u} = 2u$, $\frac{\partial x}{\partial v} = 2v$, $\frac{\partial y}{\partial u} = 2v$, $\frac{\partial y}{\partial v} = 2u$

so $\begin{vmatrix} 2u & 2v \\ 2v & 2u \end{vmatrix} = 4u^2 - 4v^2$