SOLUTION

MATH 213, Section 001 Name Prof. Boris Tudeth
Fall, 2007, Prof. Sachs
Exam 2

G number

- Show all work clearly and completely. *You may lose points for incorrect, incomplete, or unclear work, even if your final answer is correct.*

- Do all work in the space provided, or, if extra room is needed, continue your work on the back of the exam pages, being sure to indicate clearly where the work is. **Put your final answer in the box.** Extra paper will not be accepted.

- This exam has 8 problems, each of which is worth 10 or 15 points.

- **GMU Honor Code is in effect.** No notes, books, or each other's work.

1. (10 pts.) Let $f(x,y,z) = (x+y)^2 + (y+3z)^3$.

   Find the gradient vector $\vec{\nabla} f(x,y,z)$ and the linear approximation to f near the point $x=1, y=2, z=-1$.

$$\vec{\nabla} f = f_x \vec{\imath} + f_y \vec{\jmath} + f_z \vec{k} = 2(x+y)\vec{\imath} + \left[2(x+y) + 3(y+3z)^2\right]\vec{\jmath} + 3(y+3z)^2 \cdot 3\,\vec{k}$$ by CHAIN RULE

[easier than expanding]

At $x=1$, $y=2$, $z=-1$, we have $x+y=3$, $y+3z=-1$ so

$$\vec{\nabla} f \Big|_{(1,2,-1)} = 6\vec{\imath} + 9\vec{\jmath} + 9\vec{k}$$ also $f(1,2,-1) = 3^2 + (-1)^3 = 8$

ok as functions

Ans: $\vec{\nabla} f = 6\vec{\imath} + 9\vec{\jmath} + 9\vec{k}$

$L = 8 + 6(x-1) + 9(y-2) + 9(z+1)$

Remarks Expanding is OK but harder

Don't waste time with rewriting L

2. (10 pts.) If $f(x,y,z) = \frac{1}{2\pi} e^{-x^2-y^2}$, find $f_{xx} + f_{yy}$.

$$f_x = \frac{1}{2\pi}\left(e^{-x^2-y^2}\right)\cdot(-2x), \quad f_y = \frac{1}{2\pi} e^{-x^2-y^2}(-2y)$$

so $$f_{xx} = \frac{1}{2\pi} e^{-x^2-y^2}\left[(-2x)(-2x) + (-2)\right] = \frac{1}{2\pi} e^{-x^2-y^2}\left[4x^2-2\right]$$

$$f_{yy} = \frac{1}{2\pi} e^{-x^2-y^2}\left[(-2y)(-2y)+(-2)\right] = \frac{1}{2\pi} e^{-x^2-y^2}\left[4y^2-2\right]$$

Ans: $\frac{1}{2\pi} e^{-x^2-y^2}\left[4x^2+4y^2-4\right]$

or equivalent

Remark PRODUCT RULE FOR f_{xx}, f_{yy}

3. (15 pts.) (a) Suppose w is a function of the variables P, V, and T, say $w = g(P,V,T)$ and that each of these three quantities P, V, T is a function of x, y, z. Express $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ in terms of the derivatives of g and those of P, T, V. **DO NOT PUT THIS ANSWER IN THE BOX, JUST BOX IT IN.**

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial P}\frac{\partial P}{\partial x} + \frac{\partial w}{\partial V}\frac{\partial V}{\partial x} + \frac{\partial w}{\partial T}\frac{\partial T}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial P}\frac{\partial P}{\partial y} + \frac{\partial w}{\partial V}\frac{\partial V}{\partial y} + \frac{\partial w}{\partial T}\frac{\partial T}{\partial y}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial P}\frac{\partial P}{\partial z} + \frac{\partial w}{\partial V}\frac{\partial V}{\partial z} + \frac{\partial w}{\partial T}\frac{\partial T}{\partial z}$$

(b) If the derivatives have the following values at $x=1, y=0, z=2$, what is $\frac{\partial w}{\partial x}$ at that location? $\frac{\partial P}{\partial x} = 2, \frac{\partial P}{\partial y} = 5, \frac{\partial P}{\partial z} = 1, \frac{\partial V}{\partial x} = -1, \frac{\partial V}{\partial y} = 4, \frac{\partial V}{\partial z} = 10, \frac{\partial T}{\partial x} = 3, \frac{\partial T}{\partial y} = 6, \frac{\partial T}{\partial z} = 0, \frac{\partial w}{\partial P} = 1, \frac{\partial w}{\partial V} = -4, \frac{\partial w}{\partial T} = 2$. **PUT THIS ANSWER IN BOX BELOW.**

Remark:
Should have been
pretty direct

Ans: $w_x = 1\cdot 2 + (-4)(-1) + 2\cdot 3$
$= 12$

4. (10 pts.) (a) Find the derivative of the function $f(x,y,z) = x^2+y^2+z^4$ at $P_0(-1, 1/2, 1)$ in the direction of the unit vector $\vec{u} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$.

$$\vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 4z^3\vec{k} = -2\vec{i} + \vec{j} + 4\vec{k}$$

$$\vec{\nabla} f \cdot \vec{u} = \frac{1}{3}(-2+2+8) = \boxed{8/3}$$

(b) Find the direction(s) in which the same function f is increasing most rapidly at the same P_0 and find that rate of increase.

Direction is $\frac{\vec{\nabla} f}{|\nabla f|}$, magnitude $|\vec{\nabla} f| = \sqrt{(-2)^2+1^2+4^2}$
$= \sqrt{21}$

Ans: Rate $\sqrt{21}$
Direction $\frac{-2\vec{i}+\vec{j}+4\vec{k}}{\sqrt{21}}$

5. (10 pts.) If $x = r\cos\theta$, $y = r\sin\theta$, express the differentials dx and dy in terms of dr and $d\theta$.

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

REMARK Need differentials on both sides or else equation is non-sensical

Ans:

6. (15 pts.) (a) Find the most general solution to the equation

$$\vec{\nabla} f(x,y,z) = (x^2 + 2y + 2xz)\,\vec{\imath} + (2x + z^2 + y)\,\vec{\jmath} + (x^2 + 2yz + 3z^2)\,\vec{k}.$$

$f_x = x^2 + 2y + 2xz \Rightarrow f = \frac{1}{3}x^3 + 2xy + x^2 z + g(y,z)$

now $f_y = 2x + g_y = 2x + z^2 + y$ so $g = yz^2 + \frac{1}{2}y^2 + h(z)$

now $f_z = x^2 + 2yz + h'(z)$ so $h'(z) = 3z^2 \Rightarrow h(z) = z^3 + C$

COMBINING $\quad f(x,y,z) = \frac{1}{3}x^3 + 2xy + x^2 z + yz^2 + \frac{1}{2}y^2 + z^3 + C$

(b) Find the divergence of the gradient of f as given above, or in symbols $\vec{\nabla} \cdot \vec{\nabla} f(x,y)$. Use the given vector function for $\vec{\nabla} f$, don't recompute it from f.

$\vec{\nabla} \cdot (\vec{\nabla} f) = (x^2 + 2y + 2xz)_x + (2x + z^2 + y)_y + (x^2 + 2yz + 3z^2)_z$

$= 2x + 2z + 1 + 2y + 6z = 1 + 2x + 2y + 8z$

(equals $f_{xx} + f_{yy} + f_{zz}$ - Laplacian!)

(c) Can the equation $\vec{\nabla} f = M(x,y,z)\vec{\imath} + N(x,y,z)\vec{\jmath} + P(x,y,z)\vec{k}$ always be solved? Justify and explain your answer in either case.

NO — Since $f_{xy} = f_{yx}$ and $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$,

we require $\vec{\nabla} \times (M\vec{\imath} + N\vec{\jmath} + P\vec{k}) = \vec{0}$.

7. (15 pts.) Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 3x^2 + 6xy - 2y^3 + 3y^2$.

crit pts.

$f_x = 6x + 6y = 0$

$f_y = 6x - 6y^2 + 6y = 0$ $\Big\rangle$ so $y^2 = 0 \Rightarrow y = 0 \Rightarrow x = 0$

$f_{xx} = 6$, $f_{xy} = 6$, $f_{yy} = -12y + 6 = 6$ at $(0,0)$

Discriminant $= 0$ so CAN'T SAY

Ans: $(0,0)$ is critical pt. — CAN'T SAY WHAT TYPE

8. (15 pts.) (a) What is a Lagrange multiplier?

It is a number (to be decided later) so that a constrained minimization problem becomes $\vec{\nabla} f = \lambda \vec{\nabla} g$

(b) Find the equations used to find the point(s) that minimize $f(x, y, z) = x^2 + y^2 + z^2$ when $xyz = 1$, but do not solve them.

$\vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\vec{\nabla} g = yz\vec{i} + xz\vec{j} + xy\vec{k}$

So in vectors $\vec{\nabla} f = \lambda \vec{\nabla} g$

or

$2x = \lambda yz$

$2y = \lambda xz$

$2z = \lambda xy$

plus $xyz = 1$

NOTE Didn't ask to solve

[does happen at $x = y = z = 1$]