MATH 213
Fall, 2008, Prof. Sachs
Exam 3

Name Ivana Calculator

G number

- Show all work clearly and completely. *You may lose points for incorrect, incomplete, or unclear work, even if your final answer is correct.*

- Do all work in the space provided, or, if extra room is needed, continue your work on the back of the exam pages, being sure to indicate clearly where the work is. Put your final answer in the box. Extra paper will not be accepted.

- This exam has 8 problems, each of which is worth 10 or 15 points.

- GMU Honor Code is of course in effect. You may not use notes, books, or each other. A calculator is ok.

1. (10 pts.) Compute the curl of $\vec{v}$, written as $\vec{\nabla} \times \vec{v}$, for the given vector function

$$\vec{v}(x,y,z) = (3\,x^2 + 12\,x\,y + 12\,y^2)\,\vec{\imath} + (6\,x^2 + 12\,x\,y + 3\,y^2)\,\vec{\jmath} + (\cos(z+\pi/4)\,\vec{k}.$$

What does this say about finding a scalar $\phi(x,y,z)$ so that $\vec{\nabla}\phi = \vec{v}$? If it is possible, say so but **DO NOT FIND IT.**

$$\vec{\nabla}\times\vec{v} = \frac{\partial}{\partial x}(6x^2+12xy)\,\vec{k} - \frac{\partial}{\partial x}(\cos(z+\pi/4))\,\vec{j}$$
$$+ \frac{\partial}{\partial y}(\cos(z+\pi/4))\,\vec{i} - \frac{\partial}{\partial y}(3x^2+12xy+12y^2)\,\vec{k}$$
$$+ \frac{\partial}{\partial z}(3x^2+12xy+12y^2)\,\vec{j} - \frac{\partial}{\partial z}(6x^2+12xy+3y^2)\,\vec{i}$$
$$= 0\vec{i} + 0\vec{j} + ([12x+12y] - [12x+24y])\,\vec{k}$$

Ans: $-12y\vec{k}$ so no ϕ exists

2. (10 pts.) Solve the system of partial differential equations:

$$\vec{\nabla}\phi(x,y) = \frac{x}{\sqrt{x^2+y^2}}\,\vec{\imath} + \frac{y}{\sqrt{x^2+y^2}}\,\vec{\jmath}$$

(you need not test separately whether it is solvable, just find the most general solution).

$\phi_x = x\cdot(x^2+y^2)^{-1/2}$ so $\phi = (x^2+y^2)^{1/2} + g(y)$ → match

$\phi_y = y\,(x^2+y^2)^{-1/2}$ so $\phi = (x^2+y^2)^{1/2} + h(x)$

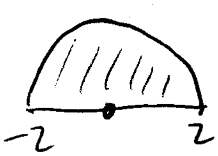
Ans: $\phi = (x^2+y^2)^{1/2} + C$

3. (15 pts.) Evaluate the following integral by changing to polar coordinates:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 2(1+x^2+y^2)^{\frac{3}{2}}\,dy\,dx$$

$0 \le y \le \sqrt{4-x^2}$

$-2 \le x \le -2$



$$\int_0^{\pi}\int_0^2 2(1+r^2)^{3/2}\; r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{2}{5}(1+r^2)^{5/2}\Big|_0^2 \cdot d\theta$$

Ans: $\frac{2\pi}{5}\left[5^{5/2} - 1\right]$

4. (15 pts.) Evaluate the triple integral:

$$\int_{-1}^{1}\int_{0}^{3}\int_{0}^{y}(x^2 y + y^2 + z)\,dz\,dy\,dx$$

$$\int_{-1}^{1}\int_{0}^{3}\left[x^2y^2 + y^3 + \tfrac{1}{2}y^2\right]dy\,dx = \int_{-1}^{1} x^2\cdot\tfrac{y^3}{3} + \tfrac{y^4}{4} + \tfrac{1}{6}y^3\Big|_0^3\,dx$$

$$= \int_{-1}^{1}\left[9x^2 + \tfrac{81}{4} + \tfrac{9}{2}\right]dx = 3x^3\Big|_{-1}^{1} + \left(\tfrac{81}{4}+\tfrac{9}{2}\right)2$$

$$= 6 + 9 + 81/2$$

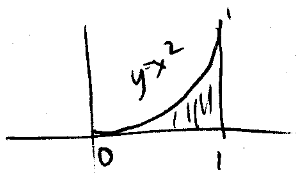
Ans: $\frac{111}{2} = 55.5$

5. (10 pts.) (a) Give a careful description of why it is legitimate to change the order of integration in a double or triple integral.

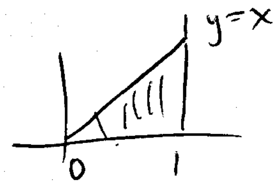
For Riemann sums it is alway legitimate so assuming the integral makes sense, this must be true.

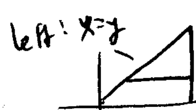
(b) Give an example of a double integral where the inside limit depends on the outside variable and explain how you would switch the order of integration for your example.

Easiest?



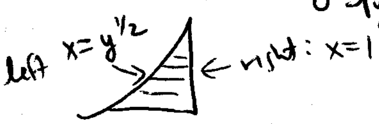
or





$$\int_0^1\int_0^{x^2} dy\,dx$$

becomes $\int_0^1\int_{\sqrt{y}}^{1} dx\,dy$



$\int_0^1\int_0^x dy\,dx$ becomes

$$\int_0^1\int_y^1 dx\,dy$$

6. (10 pts.) Find the Jacobian determinant $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|$ of the transformation:

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial x}{\partial z} = 0$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$, $\frac{\partial y}{\partial z} = 0$ so

$\frac{\partial z}{\partial r} = 0$, $\frac{\partial z}{\partial \theta} = 0$, $\frac{\partial z}{\partial z} = 1$

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

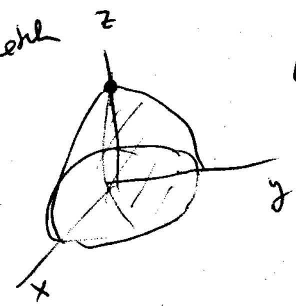
$= \boxed{r}$

What does this say about converting triple integrals from cylindrical to rectangular coordinates?

the volume element converts from $r\,dr\,d\theta\,dz$ to $dx\,dy\,dz$

7. (15 pts.) Find the volume of the solid that is bounded above by the surface $z = \sqrt{4 - x^2 - y^2}$, on the sides by the cylinder $x^2 + y^2 = 4$ and below by the xy plane.

Sketch



[Hemisphere, radius 2]

$$V = \int_0^2 \int_0^{2\pi} \sqrt{4 - r^2}\, r\, d\theta\, dr$$

$$= 2\pi \cdot \int_0^2 (4-r^2)^{1/2} \cdot r\, dr$$

$$= \pi \cdot \left(-\frac{2}{3}\right)(4-r^2)^{3/2} \Big|_{r=0}^{r=2} = \frac{2}{3}\pi \cdot 2^3$$

$$= \boxed{\frac{16\pi}{3}}$$

8. (15 pts.) (a) What is a Lagrange multiplier?

A Lagrange multiplier λ appears in the process of solving a constrained max-min problem: extremes of f with $g =$ constant means $\vec{\nabla} f = \lambda \vec{\nabla} g$ at such locations.

(b) Why is Lagrange's condition the correct one for a constrained minimizer to satisfy?

If $\vec{\nabla} f \neq \lambda \vec{\nabla} g$ ~~that~~ some point, the level sets cross so it can't be a constrained extreme point.

OR all vectors tangent to $g =$ constant are orthogonal to $\vec{\nabla} g$ and the directional derivative must vanish so these vectors are orthogonal to $\vec{\nabla} f$ also.

(c) Write down the condition(s) for the minimizer of the function $f(x, y) = x^2 + 7x\,y + 3y^2$ with the constraint: $x^2 + xy + 2y^2 = 4$
BUT DO NOT SOLVE IT.

$\vec{\nabla} f = \lambda \vec{\nabla} g$ becomes

$$\boxed{\begin{aligned} 2x + 7y &= \lambda \cdot (2x + y) \\ 7x + 6y &= \lambda (x + 4y) \end{aligned}}$$