SOLUTION

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Fall, 2008, Prof. Sachs

Exam 2 G number

- Show all work clearly and completely. *You may lose points for incorrect, incomplete, or unclear work, even if your final answer is correct.*
- Do all work in the space provided, or, if extra room is needed, continue your work on the back of the exam pages, being sure to indicate clearly where the work is. Put your final answer in the box. Extra paper will not be accepted.
- This exam has 8 problems, each of which is worth 10 or 15 points.
- You may not use notes, books, or each other.

1. (10 pts.) Let $f(x, y, z) = (x + y)^{-2} + \sin(\pi z)$.

   Find the gradient vector $\vec{\nabla} f(x, y, z)$ at all points and the linear approximation to f near the point $x = 1, y = 2, z = -1$.

$f_x = -2(x+y)^{-3}$, $f_y = -2(x+y)^{-3}$, $f_z = \pi\cos(\pi z)$

so $\boxed{\vec{\nabla} f = -2(x+y)^{-3}\vec{i} - 2(x+y)^{-3}\vec{j} + (\pi\cos(\pi z))\vec{k}}$

At $(1,2,-1)$, $\vec{\nabla} f = -2(3)^{-3}\vec{i} - 2(3)^{-3}\vec{j} + (\pi\cdot(-1))\vec{k}$

$f(1,2,-1) = 3^{-2} + 0 = 1/9$

LINEAR APPROX

Answer: $\frac{1}{9} - \frac{2}{27}(x-1) - \frac{2}{27}(y-2) - \pi(z+1)$

PRODUCT RULE

2. (15 pts.) If $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$, find $u_t - u_{xx}$.

$u = (4\pi t)^{-1/2} e^{-x^2/4t}$ so $\frac{\partial u}{\partial t} = -\frac{1}{2}(4\pi t)^{-3/2}\cdot(4\pi)\cdot e^{-x^2/4t}$ ① $+ (4\pi t)^{-1/2}\cdot\left(\frac{x^2}{4t^2}\right)\cdot e^{-x^2/4t}$ ②

$\frac{\partial u}{\partial x} = (4\pi t)^{-1/2}\cdot\left(\frac{-2x}{4t}\right)e^{-x^2/4t}$ so $\frac{\partial^2 u}{\partial x^2} = (4\pi t)^{-1/2}\left[\frac{-2}{4t}e^{-x^2/4t} + \frac{4x^2}{16t^2}e^{-x^2/4t}\right]$

matches first above ①

matches second above ②

$u_t - u_{xx}$ has cancellation $\longrightarrow$

Ans: 0

3. (15 pts.) (a) Suppose h is a function of the variables u, v, and w, say $h = g(u,v,w)$ and that each of these three quantities u, v, w is a function of x, y. Express $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ in terms of the derivatives of g and those of u, v, w. **DO NOT PUT THIS ANSWER IN THE BOX, ~~JUST BOX IT IN.~~**

$$\boxed{\begin{aligned}\frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial x}\\ \frac{\partial h}{\partial y} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial y}\end{aligned}}$$

(b) If $h = ln(u^2+v^2+w^2)$ and $u = x+2y$, $v = 2x-y$, $w = 2xy$, find $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ at the point $x = y = 1$. **PUT THIS ANSWER IN BOX BELOW.**

Using above: $\frac{\partial h}{\partial u} = \frac{2u}{u^2+v^2+w^2}$, $\frac{\partial h}{\partial v} = \frac{2v}{u^2+v^2+w^2}$, $\frac{\partial h}{\partial w} = \frac{2w}{u^2+v^2+w^2}$

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 2$

$\frac{\partial v}{\partial x} = 2$, $\frac{\partial v}{\partial y} = -1$

$\frac{\partial w}{\partial x} = 2y = 2$, $\frac{\partial w}{\partial y} = 2x = 2$

$u = 1+2 = 3$ so $u^2+v^2+w^2 = 9+1+4 = 14$

$v = 2-1 = 1$

$w = 2\cdot1\cdot1 = 2$

Ans: $h_x = \frac{6+4+8}{14} = \boxed{9/7}$

$h_y = \frac{12-2+8}{14} = \boxed{9/7}$

4. (10 pts.) (a) Find the derivative of the function $f(x, y, z) = x^2 + y^2 + z^4$ at $P_0(-1, 1/2, 1)$ in the direction of the unit vector $\vec{u} = \frac{-\vec{i}+2\vec{j}-2\vec{k}}{3}$.

$$\vec{\nabla f} \cdot \vec{u} = \left(2x\vec{i} + 2y\vec{j} + 4z^3\vec{k}\right) \cdot \left(\frac{-\vec{i} + 2\vec{j} \mp 2\vec{k}}{3}\right)$$

$$= 2/3 + 2/3 - 8/3 = \boxed{-4/3}$$

(b) Find the direction(s) in which the same function f is increasing most rapidly at the same P_0 and find that rate of increase.

$|\nabla f|$ is magnitude, $\left|\frac{\vec{\nabla f}}{|\nabla f|}\right|$ = direction (unit vector)

Ans: rate $|\vec{\nabla f}| = \sqrt{4+1+16} = \sqrt{21}$
direction $\frac{-2\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{21}}$

5. (10 pts.) If $r = \sqrt{x^2+y^2}$, $\theta = arctan(y/x)$, express the differentials dr and $d\theta$ in terms of dx and dy.

$$dr = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x \cdot dx + \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y\, dy$$

$$= \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$$

$$d\theta = \frac{1}{1+(y/x)^2} \cdot \left(-y/x^2\right) dx + \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}\,dy$$

Ans:

$$= \frac{-y}{x^2+y^2}\,dx + \frac{x\,dy}{x^2+y^2}$$

6. (15 pts.) (a) What are critical points of a function $f(x, y)$?

Pts where $\vec{\nabla} f = 0$ or does not exist

(b) Describe briefly why critical points are the only candidates for local maxima or minima (Hint: what is excluded and why?)

If $\vec{\nabla} f$ exists and is non-zero, at a point there is an uphill ($\vec{\nabla} f$) direction and a downhill ($-\vec{\nabla} f$) locally, so it can't be a local max / min.

(c) How do you decide if a critical point is a local maximum or minimum?

Second derivative test: Look at $f_{xx} f_{yy} - f_{xy}^2$ at the pt. (and at f_{xx} also)

$f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$: Local min

$f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$: Local max

$f_{xx} f_{yy} - f_{xy}^2 < 0$: Saddle point

7. (15 pts.) Find all the local maxima, local minima, and saddle points of the function $f(x, y) = y^2/2 + \cos(x)$ for $-\pi/2 \leq x \leq 3\pi/2$.

$f_x = -\sin x$

$f_y = 2y$

critical pts: $y = 0$

$\sin x = 0 \rightarrow x = 0$ or $x = \pi$

Second deriv. test:

$f_{xx} = -\cos x$: $f_{xx}(0,0) = -1$

$f_{yy} = 2$ $f_{yy}(0,0) = 2$

$f_{xy} = 0$ So SADDLE at $(0,0)$

$f_{xx}(\pi, 0) = 1$

$f_{yy}(\pi, 0) = 2$

$f_{xy}(\pi, 0) = 0$

$>$ $\boxed{\text{local min at } (\pi,0)}$

no local maxima

Ans:

8. (15 pts.) (a) How are level sets related to the gradient vector field?

∇f is orthogonal to level sets [direction]

AND size ∇f is related to spacing of level sets

— closer levels $\Leftrightarrow$ bigger gradient. [magnitude]

(b) Given the level sets picture below, sketch the gradient field on it paying attention to where the gradient is large and where it is small and to its direction. Include at least points marked by dots