SOLUTION

MATH 213, Section 001 Name REED D. BOOK

Fall, 2008, Prof. Sachs

Exam 1

G number

- Show all work clearly and completely. *You may lose points for incorrect, incomplete, or unclear work, even if your final answer is correct.*
- Do all work in the space provided, or, if extra room is needed, continue your work on the back of the exam pages, being sure to indicate clearly where the work is. Put your final answer in the box. **Extra paper will not be accepted.**
- This exam has 8 problems, each of which is worth 10 or 15 points.
- You may not use notes, books, or each other.

1. (10 pts.) Find the point in the triangle (vector form is fine) determined by the points $P(1,2,2)$, $Q(1,0,3)$, and $R(0,2,1)$ that is 1/3 of the way from the midpoint of PQ towards the point R.

MIDPOINT $\vec{PQ}$ has coordinates $(1, 1, 5/2)$ – average P, Q coordinates

$$\vec{OM} + \frac{1}{3}\vec{MR} = \langle 1+\tfrac{1}{3}(-1), 1+\tfrac{1}{3}(1), \tfrac{5}{2}+\tfrac{1}{3}(-\tfrac{3}{2})\rangle = \langle 2/3, 4/3, 2\rangle$$

Note sum is $\frac{1}{3}(\vec{OP}+\vec{OQ}+\vec{OR})$

Answer: $(2/3, 4/3, 2)$ or equivalent

2. (15 pts.) Let $\vec{u} = \langle 1, 1, -1\rangle$, $\vec{v} = \langle 4, 1, -5\rangle$, and $\vec{w} = \langle 0, 0, 1\rangle$.
Find the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ and describe its geometric significance.

Either: $\vec{u}\cdot(\vec{v}\times\vec{w}) = \det \begin{vmatrix} 1 & 1 & -1 \\ 4 & 1 & -5 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \det \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}$ (expand by row 3)

$= \boxed{-3}$

or $\vec{v}\times\vec{w}$ first, then $\vec{u}\cdot(\vec{v}\times\vec{w})$

Answer: -3 ; left-hand, volume system $=3$

3. (10 pts.) Find parametric equations for the line that is tangent to the given curve $\vec{r}(t) = (1/t^2)\,\vec{i} + 3\sin(3\pi\, t)\,\vec{j} + 4\sqrt{5+t^2}\,\vec{k}$ at the parameter value $t = 2$.

$\vec{r}(2) = 1/4\,\vec{i} + 0\vec{j} + 4\sqrt{9}\,\vec{k}$

$\vec{r}\,'(t) = -2/t^3\,\vec{i} + 9\pi\cos(3\pi t)\vec{j} + 4\cdot\frac{1}{2}(5+t^2)^{-1/2}\cdot 2t\,\vec{k}$

so $\vec{r}\,'(2) = -1/4\,\vec{i} + 9\pi\vec{j} + 8/3\,\vec{k}$

Line $\vec{r}(2) + \vec{r}\,'(2)\cdot(t-2)$

Answer: $+1/4\,\vec{i} + 12\vec{k} + (t-2)\cdot\left[-1/4\,\vec{i} + 9\pi\vec{j} + 8/3\,\vec{k}\right]$

or $x = 1/4 - 1/4(t-2)$
$y = 0 + 9\pi\cdot(t-2)$
$z = 12 + 8/3\,(t-2)$

4. (10 pts.) Write down **but do not evaluate** the integral that would find the length of the curve: $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^4\,\vec{k}$ for $0 \leq t \leq 2$.

$\frac{d\vec{r}}{dt} = \vec{i} + 2t\,\vec{j} + 4t^3\,\vec{k}$, $\left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt} = \sqrt{1+4t^2+16t^6}$

So

Answer:
$\int_0^2 \sqrt{1+4t^2+16t^6}\, dt$

5. (10 pts.) If $\vec{r}(t) = e^t\cos(t)\vec{i} + e^t\sin(t)\vec{j}$ is the position of a particle in space at time t, find **both** of the following at time $t=0$: its velocity vector and its speed.

PRODUCT RULE

$\vec{r}\,'(t) = [e^t\cos t - e^t\sin t]\vec{i} + [e^t\sin t + e^t\cos t]\vec{j}$

Find $|\vec{r}\,'(0)|$ = speed from $\vec{r}\,'(0) = \vec{i} + \vec{j}$

Answer:
$\vec{r}\,'(0)$ = velocity vect $= \vec{i} + \vec{j}$
speed $= |\vec{r}\,'(0)| = \sqrt{2}$

6. (15 pts.) For the force $\vec{F} = -y\,\vec{i} + x\,\vec{j}$, find the work done ($\int \vec{F} \cdot d\vec{r}$) over the circular path: $\vec{r}(t) = 2\,\cos(t)\,\vec{i} + 2\,\sin(t)\,\vec{j}$ for $0 \le t \le 2\pi$.

$$\frac{d\vec{r}}{dt} = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}\;,\quad \begin{aligned} x &= 2\cos t \\ y &= 2\sin t \end{aligned}$$

so $\vec{F} \cdot \frac{d\vec{r}}{dt} = -y(-2\sin t) + x\cdot(2\cos t) = 4\sin^2 t + 4\cos^2 t$

$= 4$

so

Answer: $\int_0^{2\pi} 4\,dt = 8\pi$

7. (15 pts.) Solve the initial value problem for $\vec{r}$ as a vector function of t:
Differential equation: $\frac{d^2\vec{r}}{dt^2} = (t^3 + t^2)\,\vec{i} + \sqrt{t}\,\vec{j} + \vec{k}$
Initial conditions: $\vec{r}(0) = -\vec{i} + \vec{j} + \vec{k}$, $\vec{r}\,'(0) = \vec{j}$.

$$\vec{r}\,'(t) = \left(t^4/4 + t^3/3\right)\vec{i} + 2/3\, t^{3/2}\,\vec{j} + t\,\vec{k} + \vec{C}_1$$

using $t=0$ condition, $\vec{C}_1 = \vec{j}$

Integrate again $\quad \vec{r}(t) = \left(t^5/20 + t^4/12\right)\vec{i} + \left(\frac{4}{15}t^{5/2} + t\right)\vec{j} + t^2/2\,\vec{k} + \vec{C}_2$

Use $t=0$ condition to get:

Answer: $\left(t^5/20 + t^4/12 - 1\right)\vec{i} + \left(4/15\, t^{5/2} + t + 1\right)\vec{j} + \left(t^2/2 + 1\right)\vec{k}$

8. (15 pts.) Recall that the curvature κ is defined via either of the following formulas:

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}.$$

Find the curvature of the helix $\vec{r}(t) = a\cos(t)\,\vec{\imath} + a\sin(t)\,\vec{\jmath} + a\,t\,\vec{k}$ for any positive constant a.

$$\vec{r}\,'(t) = -a\sin t\,\vec{\imath} + a\cos t\,\vec{\jmath} + a\vec{k}$$

using last version of definition:

$$\vec{r}\,''(t) = -a\cos t\,\vec{\imath} - a\sin t\,\vec{\jmath}$$

$$\text{so } \vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -a\sin t & a\cos t & a \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = +a^2\sin t\,\vec{\imath} + a^2\cos t\,\vec{\jmath} + a^2\vec{k}$$

$$|\vec{r}\,' \times \vec{r}\,''| = a^2\sqrt{\sin^2 t + \cos^2 t + 1} = a^2\sqrt{2}$$

$$\text{also } |\vec{r}\,'(t)| = a\sqrt{\sin^2 t + \cos^2 t + 1} = a\sqrt{2}$$

$$\text{so } \kappa = \frac{a^2\sqrt{2}}{(a\sqrt{2})^3} = \frac{a^2\sqrt{2}}{a^3\sqrt{2}\cdot 2} = \boxed{\frac{1}{2a}}$$

Alternative

$$\vec{T}(t) = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{-\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + \vec{k}}{\sqrt{2}}$$

$$\frac{dT}{dt} = \frac{-\cos t\,\vec{\imath} - \sin t\,\vec{\jmath}}{\sqrt{2}} \quad \text{etc.}$$