## 3.8 Finding Antiderivatives; Divergence and Curl of a Vector Field

**Overview:** The antiderivative in one variable calculus is an important concept. For partial derivatives, a similar idea allows us to solve for a function whose partial derivative in one of the variables is given, as seen earlier. However, when several partial derivatives of an unknown function are given simultaneously, there may not be any function with those values! In particular, the gradient field for smooth functions must respect Clairaut's theorem, so not every vector field is the gradient of some scalar function! This section explores these questions and in answering them, introduces two very important additional vector derivatives, the Curl and Divergence.

### 3.8.1 Solving $\nabla f = \mathbf{v}$ for $f$ given $\mathbf{v}$

**Looking back:** Recall how antiderivatives were introduced in one variable calculus: to find $F$ with a given derivative $f$, you learned:

1. For $f$ given by a simple formula, guess and check led to an antiderivative $F$.
2. Any other anti derivative $G$ had the property $G = F + C$.
3. When all else fails, any continuous $f$ has anti-derivative $F(x) = \int_a^x f(t)\,dt$ for any fixed $a$.

Earlier in this chapter the corresponding result for a single partial derivative was found to be similar, except that instead of adding constants to get new solutions from old ones, functions independent of the single variable could be added.

The gradient vector has partial derivatives in components, so we might expect that this result would allow us to solve the system of equations $\nabla G = \mathbf{v}$. This consists of $n$ simultaneous equations in dimension $n$. This is partially correct, but a new twist emerges: not every vector field $\mathbf{v}$ is a gradient field!

The following simple example illustrates the difficulty:

**Example 3.44** *An example in 2D: solve for G so that $\frac{\partial G}{\partial x} = x + y$, $\frac{\partial G}{\partial y} = 2x + 3y$.*

*Our strategy would be to solve one of these for a general form, then impose the other condition. This is like solving two equations in 2 unknowns by elimination. So, starting with partial differential equation in x, we find a partial solution $G = G_1$:*

$$G_1(x, y) = \frac{1}{2}x^2 + xy + c(y)$$

*and then taking the y partial derivative of $G_1$ we must satisfy*

$$\frac{\partial G_1}{\partial y}(x, y) = x + c'(y) = 2x + 3y.$$

*This can't be solved for $c(y)$ since $c'(y) = x + 3y$ requires some $x$ dependence. Alternatively, (please check it) solving the $y$ partial differential equation first, we get $G_2(x, y) = 2xy + 3/2y^2 + c_2(x)$ and then the partial differential equation in $x$ cannot be satisfied also!*

What went wrong?

Mixed partial derivatives must be equal, so if $G$ solves both equations, then $\frac{\partial^2 G}{\partial y \partial x} = \frac{\partial^2 G}{\partial x \partial y}$ means $\frac{\partial}{\partial y}(\frac{\partial G}{\partial x}) = \frac{\partial}{\partial y}(x+y) = 1$ must equal $\frac{\partial}{\partial x}(\frac{\partial G}{\partial y}) = \frac{\partial}{\partial x}(2x+3y) = 2$ which is not true! Therefore no G exists!

In general, the equality of mixed partial derivatives in 2D means that if $\nabla \mathbf{G} = (\mathbf{v_1}, \mathbf{v_2})$ is continuously differentiable, it implies $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$ at every point.

For the **general linear homogeneous vector field** this becomes the following condition, interpreted as describing the range of the gradient operator on purely quadratic functions: Supposing $v_1 = Ax+By, v_2 = Cx+Dy$ we find $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = C - B$ so $\nabla \mathbf{G} = \mathbf{v}$ becomes the condition that the $2 \times 2$ matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symmetric! Sound familiar? We already know for purely quadratic functions the gradient is linked to the Hessian, which is symmetric.

**Example 3.45** *Find $G$ so that $\nabla G = (2x+3y)\mathbf{i} + (3x+4y)\mathbf{j}$.*

**Solution:** *Solving in order of $x$ then $y$, we find: $G(x,y) = x^2 + 3xy + g_1(y)$, and then using the $y$ equation, we find: $G(x,y) = x^2 + 3xy + 2y^2 + C$, for any constant $C$.*

Here are two examples of non gradient linear fields in 2D that are of some importance in applications:

1. rigid rotation generator with constant angular velocity: $v_1 = -\omega y, v_2 = \omega x$ then $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = \omega + \omega = 2\omega$.

2. Linear shear $v_1 = ay, v_2 = 0$, where $a$ is a constant. Then $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial x} = -a$.

You should recall plotting these examples in Chapter 1 when vector fields were introduced.

**Non-linear examples:** For non-linear vector fields, a similar situation holds: Not every vector field is a gradient, since gradients must have linked components when Clairaut's theorem is considered; to find a function that has the given gradient, use elimination; at the end, the solution is unique up to a constant.

**Example 3.46** *Find $G(x,y)$ so that $\nabla G = \mathrm{e}^x \cos y\, \mathbf{i} - \mathrm{e}^x \sin y\, \mathbf{j}$.*

**Solution:** *Solving in order of $x$ then $y$, we find: $G(x,y) = \mathrm{e}^x \cos y + g_1(y)$, and then using the $y$ equation, we find: $G(x,y) = \mathrm{e}^x \cos y + C$, for any constant $C$.*

What happens in 3D? Now there are three equations if we write out $\nabla \mathbf{G} = \mathbf{v}$ in components:

$$\frac{\partial G}{\partial x} = v_1, \ \frac{\partial G}{\partial y} = v_2, \ \frac{\partial G}{\partial z} = v_3$$

and these lead to 3 equations involving equality of mixed partial derivatives of $G$. These pairs are the original 2D condition $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$ along with its cyclic versions sending simultaneously $x \to y \to z \to x$ and subscripts $1 \to 2 \to 3 \to 1$. These are $\frac{\partial v_3}{\partial y} - \frac{\partial v_1}{\partial z} = 0$ and $\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0$. These get organized into a vector by the unusual but useful rule of making each component be the one that is neither a subscript nor a derivative. In particular, this makes the original 2D term $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$ appear in the **third component**. Note that each of the components $v_1, v_2, v_3$ of $\mathbf{v}$ and each of the partial derivative variables $x, y, z$ appears twice, once with a plus sign and once with a minus sign.

This twisted vector derivative of a vector function is called the **curl of the vector field** $v$. It is often denoted using the vector operator $\nabla$ as follows:

$$\mathbf{curl}(\mathbf{v}) = \nabla \times \mathbf{v}.$$

So using the mnemonic device for cross products, we find

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = (\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})\,\mathbf{i} + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})\,\mathbf{j} + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\,\mathbf{k}.$$

As the name suggests, the curl of a vector field measures its local rotation in some sense. It plays an important part in several application areas. In fluid dynamics, the curl of the velocity vector field is called the vorticity. In electromagnetic theory, the curl occurs when magnetic and electric effects are linked.

The interpretation of the curl will be developed in Chapter 5, where a fundamental theorem (Stokes' theorem) ties its integral with another quantity. For now, we regard the above conditions as a concise and clever formulation of conditions which gradient vector fields must satisfy, summarized in the following:

**Theorem 3.47** *If* $\mathbf{v} = \nabla G$ *where* $G$ *has continuous second derivatives, then* $\nabla \times \mathbf{v} = \mathbf{0}$.

**Alternative organization:** For general vector functions, the condition on being a gradient can be expressed using the Jacobian matrix:

$$\text{If } \nabla G = \mathbf{v}, \text{ then } \frac{\partial \mathbf{v}}{\partial \mathbf{r}} = \frac{\partial(v_1, v_2, v_3)}{\partial(x, y, z)} \quad \text{should be a symmetric matrix.}$$

Notice that the curl has components equal to the entries in the matrix $J - J^T$, where $J$ is the Jacobian matrix. The diagonal elements of such a matrix are always zero, while the off-diagonals come in pairs (with opposite signs). Thus a 2-D Jacobian ends up with one component for curl, while a 3-D Jacobian has three independent entries.

For the **general linear homogeneous vector field** this becomes the following conditions, interpreted as describing the range of the gradient operator on purely quadratic functions: Supposing $v_j = A_{j1}x + A_{j2}y + A_{j3}z$, for $j = 1, 2, 3$. We find $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = A_{21} - A_{12}$ so $\nabla \mathbf{G} = \mathbf{v}$ becomes the condition that the $3 \times 3$ matrix $A$ that generates $\mathbf{v}$ is symmetric! In 3-D as well as 2-D, for purely quadratic functions the gradient is linked to the Hessian, which is symmetric.

### 3.8.2 Solvability of $curl(\mathbf{A}) = \mathbf{v}$

Once we have created the curl, the corresponding question arises:

What is the range of the curl operator? In more concrete terms, is every vector field $\mathbf{v}$ a solution of $\nabla \times \mathbf{A} = \mathbf{v}$? In electromagnetism, if $\mathbf{v}$ is the magnetic field, then such a vector function $\mathbf{A}$ is called the vector potential.

The answer is again NO, based again on the equality of mixed partial derivatives for such a solution $\mathbf{A}$.

The calculation of the appropriate condition on $\mathbf{v}$ comes by a judicious use of partial derivatives applied to the components of the vector system $\nabla \times \mathbf{A} = \mathbf{v}$:

$$\begin{aligned}
\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} &= v_1 \\
\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} &= v_2 \\
\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} &= v_3
\end{aligned}$$

As mentioned earlier, the first component on the left side involves the $y$ and $z$ derivatives and components only. Its $x$ partial derivative therefore involves x derivatives of terms with y and z each coming in once per term (either as derivative or component location) and likewise the y partial derivative of the second component of the curl and the z partial derivative of the third component have similar properties. Combining these six terms by adding, it turns out that each mixed partial derivative of each component of $\mathbf{A}$ appears twice, always involving the other variables in the derivatives and with opposite signs. This makes the total sum to be the scalar quantity $0$. Without grouping the calculation, it is less clearly the case but nonetheless true that

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

as follows by simplifying the expression in $\mathbf{A}$ from the terms:

$$\begin{aligned}\frac{\partial}{\partial x}(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}) &= \\ \frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_1}{\partial z\partial y}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_3}{\partial y\partial x} &= 0\end{aligned}$$

The divergence of a vector field $\mathbf{v}$ in 3D is defined to be the scalar quantity $\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$, and a necessary condition for a vector field $\mathbf{v}$ to be given as the curl of another vector field $A$ is that the divergence of $v$ must be $0$ at all points.

Vector operator notation for the divergence is $\nabla\cdot\mathbf{v}$.

The divergence is a measurement of outward flow rate (flow per area in 2D, per volume in 3D) and is important in fluid and solid dynamics as measuring deformation of volumes. Another major theorem that relates the integral of the divergence to another quantity is known as either the divergence theorem or Gauss' theorem and is discussed in Chapter 5.

**Example 3.48** *Divergence of linear vector field: For a 2D linear vector field of the form $v_1 = Ax+By$, $v_2 = Cx+Dy$, we find $\nabla\cdot\mathbf{v} = A+C$ which is the trace of the matrix (sum of diagonal elements). Note that it is constant in $x$ and $y$.*

**Example 3.49** *The position vector field $\mathbf{r}$ has components $v_1 = x$ and $v_2 = y$, so its divergence is $\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y} = 1+1 = 2$. This is of course a special case of the previous example.*

**Example 3.50** *Unit radial field $\mathbf{v} = \mathbf{e}_r$ in 2D has components $v_1 = \frac{x}{\sqrt{x^2+y^2}}$, $v_2 = \frac{y}{\sqrt{x^2+y^2}}$, which leads to a divergence of $\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}$, which after some calculation (exercise below), becomes $1/r$.*

### 3.8.3 Finding vector potentials

From our work so far, we know that a vector potential, if it exists, will not be unique. To any potential $\mathbf{A}$, an arbitrary gradient field can be added to get another vector potential with the same curl everywhere. Physicists sometimes call such additions a choice of gauge, and in electromagnetic field theory, special choices are made to simplify Maxwell's equations. For now, we will be happy to describe how to find one solution.

For simplicity, the third component of $\mathbf{A}$ can be taken to be zero, since a gradient field can take care of that if needed. This means that some equations simplify:

$$\begin{aligned}-\frac{\partial A_2}{\partial z} &= v_1 \\ \frac{\partial A_1}{\partial z} &= v_2 \\ \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y} &= v_3\end{aligned}$$

These can be solved sequentially, namely $A_2$ is determined using the first equation up to a function of $x$ and $z$, while $A_1$ is determined by the second equation, up to a function of $y$ and $z$. The third equation can then be solved provided our solvability conditions holds.

**Example 3.51** *Find* $\mathbf{A}$ *so that* $curl(\mathbf{A}) = y\,\mathbf{i} + x\,\mathbf{j}$ *Note that the solvability condition holds.* **Solution:** *From the first equation,* $A_2 = -yz + c_2(x, y)$*, while the second yields:* $A_1 = xz + c_1(x, y)$ *and finally the third equation holds as long as* $(c_2)_x - (c_1)_y = 0$*, so if the goal is to find some solution, we can pick* $c_1 = c_2 = 0$ *for this example.*

**Remark on solving for divergence:** The condition of a fixed divergence function does not create any solvability issues. To find a vector function with given divergence is easy: solve with zero components in two entries and solve the single simple partial differential equation in the third. For example, to solve $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = k(x, y, z)$ for any known function $k(x, y, z)$, solve for $v_1$ with $v_2 = v_3 = 0$.

### 3.8.4 EXERCISES

**Writing: Explaining, Reacting, Questioning**

1. Find the “antigradient” of the constant vector field in 2D if one exists: given $u = A, v = B$, find $F$ so that $\nabla\mathbf{F} =< u, v >$.
2. For homogeneous linear vector fields in 2D, the general form has four parameters. Explain which linear vector fields are gradients and do a careful counting of how many conditions are generated (i.e. count dimensions) in solving $\nabla F =< Ax + By, Cx + Dy >$.
3. Cal Clueless regards the “del operator” $\nabla$ as applying to functions rather than being an operator. Therefore when he tries to calculate the curl of the vector field using the mnemonic device of the determinant he gets strange answers, such as:
4. Rewrite the divergence operation and the curl operation as matrix operations acting on functions (the components of $\mathbf{v}$ in columns. What matrices do you get?
5. From the form of the gradient in polar coordinates $r, \theta$, what is the polar coordinate condition in terms of the components for being a gradient?

**Calculational Exercises**
For problems 3-10 solve the equation or explain why there is no solution:

1. $\nabla\mathbf{F} =< x^2 - y^2, 2xy >$ .

2. $\nabla\mathbf{G} =< \sin 2\,x, \cos 2\,y >$

3. $\nabla\mathbf{F} =< \sin(2x+3y), 2\cos(2x+3y) >$

4. $\nabla\mathbf{G} =< \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} >$

5. $\nabla\mathbf{U} =< x+y, y+z, x+z >$

6. $\nabla\phi =< \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$

7. $\nabla\mathbf{F} =< \frac{1}{\sqrt{16-x^2-y^2}}$ at $(-1,2)$.

8. $\nabla\mathbf{F} =< f(x), g(y), h(z) >$ for any continuous functions $f, g, h$.

   For problems 12-15 solve the equation or explain why there is no solution:

9. Solve the following equations or show that there is no solution

10. For homogeneous linear vector fields in 2D, the general form has four parameters. Explain which linear vector fields are **curls** and do a careful counting of how many conditions are generated (i.e. count dimensions) in solving $\nabla\times\mathbf{u} =< \mathbf{Ax}+\mathbf{By}, \mathbf{Cx}+\mathbf{Dy} >$.

11. $\nabla\times\mathbf{v} =< \mathbf{2x}+\mathbf{3y}, \mathbf{3x}-\mathbf{2y} >$

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