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EULER'S FORMULA FOR COMPLEX EXPONENTIALS

According to Euler, we should regard the complex exponential e^{it} as related to the trigonometric functions $\cos(t)$ and $\sin(t)$ via the definition:

$$e^{i\,t} = \cos\,t + i\,\sin\,t \qquad \text{where as usual in complex numbers } i^2 = -1. \qquad (1)$$

The justification of this notation is based on the formal derivative of both sides, namely

$$\begin{aligned}
\frac{d}{dt}(e^{i\,t}) &= i\,(e^{i\,t}) = i\cos\,t + i^2\,\sin\,t \\
&= i\cos\,t - \sin\,t \quad \text{since } i^2 = -1 \\
\frac{d}{dt}(\cos\,t + i\,\sin\,t) &= -\,\sin\,t + i\,\cos\,t \quad \text{since } i \text{ is a constant.}
\end{aligned}$$

along with the initial value of 1 for both sides at $t = 0$, assuming $e^0 = 1$ holds for complex values too.

The motivation for looking at this combination comes from the link between point in the plane with coordinates (x, y) and complex numbers formed by the relation $z = x+iy$. Thus polar coordinates for the plane becomes the combination $r\cos\theta + ir\sin\theta$, which suggests that the combination may be interesting to look at.

This turns out to be a very important unification and simplification of many results in both trigonometry and calculus, in which the formula leads us to correct manipulations. This is illustrated first for some trig identities and then some diffentiation results which are otherwise hard to compute. ECE, physics and math students will use this later on!

1. Trig Identities: The notation suggests that the following formula ought to hold:

$$e^{i\,s} \cdot e^{i\,t} = e^{i\,(s+t)} \qquad (2)$$

which converts to the addition laws for cos and sin in components:

$$\cos\,(s+t) = \cos\,s\cos\,t - \sin\,s\sin\,t, \qquad \sin\,(s+t) = \sin\,s\cos\,t + \cos\,s\sin\,t. \qquad (3)$$

This codifies the addition laws in trig in a way you can always recover.

We can also express the trig functions in terms of the complex exponentials $e^{i\,t}$, $e^{-i\,t}$ since we know that cos(t) is even in t and sin(t) is odd in t. This reads as follows:

$$e^{i\,t} = \cos\,t + i\,\sin\,t, \;\; e^{-i\,t} = \cos\,t - i\,\sin\,t \qquad (4)$$

so adding (and dividing by 2) or subtracting (and dividing by 2 i) gives:

$$\cos\,t = \frac{e^{i\,t} + e^{-i\,t}}{2}, \;\; \sin\,t = \frac{e^{i\,t} - e^{-i\,t}}{2\,i}. \qquad (5)$$

Another manipulation suggested by the notation:

$$e^{it}e^{-it} = e^{it-it} = e^0 = 1 \tag{6}$$

which leads to

$$\begin{aligned} 1 \quad &= (\cos\, t + i\, \sin\, t)\,(\cos\,(-t) + i\, \sin\,(-t)) = (\cos\, t + i\, \sin\, t)\,(\cos\, t - i\, \sin\, t) \\ &= \cos^2\, t - i^2\, \sin^2\, t = \cos^2\, t + \sin^2\, t. \end{aligned}$$

There are many other uses and examples of this beautiful and useful formula. As a further example note that lots of identities can be derived.

$$\text{For any positive integer } n, e^{int} = (e^{it})^n = (\cos\, t + i\, \sin\, t)^n. \tag{7}$$

2. Calculus: The functions of the form $e^{at}\cos\, bt$ and $e^{at}\sin\, bt$ come up in applications often. To find their derivatives, we can either use the product rule or use Euler's formula

$$\begin{aligned} (\frac{d}{dt})(e^{at}\cos\, bt + i\, e^{at}\sin\, bt) &= (\frac{d}{dt})(e^{(a+ib)\,t} = (a+ib)e^{(a+ib)\,t} \\ &= (a+ib)(e^{at}\cos\, bt + i\, e^{at}\sin\, bt) \\ &= (ae^{at}\cos\, bt - b\, e^{at}\sin\, bt) \, + \, i\,(b\, e^{at}\cos\, bt \, + a\, e^{at}\sin\, bt). \end{aligned}$$

This finds both derivatives simultaneously and is especially nice for higher derivatives (try the second derivatives yourself both ways!).

3. Differential equations: This formula really comes into its own when we need to solve differential equations with constant coefficients. Then the goal is to find the right numbers a, b so that the above functions which we just differentiated solve a given equation. For example, electrical circuits lead to differential equations that relate current, charge and voltage based on the circuit elements. Circuit elements are described by certain parameters like inductance, resistance, and capacitance. These become coefficients in the differential equation.

Example The differential equation $a\, y'' + b\, y' + c\, y = 0$ can be solved by seeking exponential solutions with an unknown exponential factor. Substituting $y = e^{r\,x}$ into the equation gives a solution if the quadratic equation $a\, r^2 + b\, r + c = 0$ holds. For lots of values of a, b, c, the solutions are complex. Euler's formula allows us to interpret that easy algebra correctly.