EULER'S FORMULA FOR COMPLEX EXPONENTIALS

According to Euler, we should regard the complex exponential e^{it} as related to the trigonometric functions $\cos(t)$ and $\sin(t)$ via the following inspired definition:

$$\mathrm{e}^{i\,t} = \cos\, t + i\, \sin\, t \qquad \text{where as usual in complex numbers } i^2 = -1. \tag{1}$$

The justification of this notation is based on the formal derivative of both sides, namely

$$\begin{aligned}
\frac{d}{dt}(\mathrm{e}^{i\,t}) &= i\,(\mathrm{e}^{i\,t}) = i\cos\, t + i^2\, \sin\, t \\
&= i\cos\, t - \, \sin\, t \quad \text{ since } i^2 = -1 \\
\frac{d}{dt}(\cos\, t + i\, \sin\, t) &= -\, \sin\, t + i\, \cos\, t \quad \text{ since } i \text{ is a constant.}
\end{aligned}$$

along with the initial value of 1 for both sides at $t = 0$, assuming $\mathrm{e}^0 = 1$ holds for complex values too.

The motivation for looking at this combination comes from the link between point in the plane with coordinates (x, y) and complex numbers formed by the relation $z = x + iy$, since z becomes the combination $r\cos\theta + ir\sin\theta$, which suggests that the combination may be interesting to look at (unit circle has $r = 1$).

This turns out to be a very important unification and simplification of many results in both trigonometry and calculus, in which the formula leads us to correct manipulations. As well, people use it in algebra and also in signal processing. This is illustrated first for some trig identities and then some differentiation and integration results which are otherwise hard to compute.

1. An Amusing Equation: From Euler's formula with angle π, it follows that the equation:

$$\mathrm{e}^{i\,\pi} + 1 = 0 \tag{2}$$

which involves five interesting math values in one short equation.

2. Trig Identities: The notation suggests that the following formula ought to hold:

$$\mathrm{e}^{i\,s} \cdot \mathrm{e}^{i\,t} = \mathrm{e}^{i\,(s+t)} \tag{3}$$

which converts to the addition laws for cos and sin in components:

$$\begin{aligned}
\cos\,(s + t) &= \cos\, s\cos\, t - \sin\, s\sin\, t, \\
\sin\,(s + t) &= \sin\, s\cos\, t + \cos\, s\sin\, t.
\end{aligned}$$

This codifies the addition laws in trig in a way you can always recover.

We can also express the trig functions in terms of the complex exponentials $e^{it},\ e^{-it}$ since we know that cos(t) is even in t and sin(t) is odd in t. This reads as follows:

$$e^{it} = \cos\, t + i\, \sin\, t,\ \ e^{-it} = \cos\, t - i\, \sin\, t \tag{4}$$

so adding (and dividing by 2) or subtracting (and dividing by 2 i) gives:

$$\cos\, t = \frac{e^{it} + e^{-it}}{2},\ \ \sin\, t = \frac{e^{it} - e^{-it}}{2\, i}. \tag{5}$$

Another manipulation suggested by the notation:

$$e^{it}e^{-it} = e^{it-it} = e^0 = 1 \tag{6}$$

which leads to

$$\begin{aligned} 1 \quad &= (\cos\, t + i\, \sin\, t)\,(\cos\,(-t) + i\, \sin\,(-t)) = (\cos\, t + i\, \sin\, t)\,(\cos\, t - i\, \sin\, t) \\ &= \cos^2\, t - i^2\, \sin^2\, t = \cos^2\, t + \sin^2\, t. \end{aligned}$$

There are many other uses and examples of this beautiful and useful formula. As a further example note that lots of identities can be derived. The following is known as DeMoivre's Theorem:

$$\text{For any positive integer } n, e^{int} = (e^{it})^n = (\cos\, t + i\, \sin\, t)^n. \tag{7}$$

This allows us to find solutions to algebra equations like $z^3 = 1$ by viewing 1 via Euler as having angle $0, 2\pi, 4\pi$. Then the three solutions are found to be

$$1, \qquad e^{\frac{2\pi i}{3}} = \frac{-1 + \sqrt{3}\, i}{2}, \qquad e^{\frac{4\pi i}{3}} = \frac{-1 - \sqrt{3}\, i}{2} \tag{8}$$

Note that you could find the solutions by factoring, from

$$z^3 - 1 = (z-1)(z^2 + z + 1), \tag{9}$$

but factoring won't be easy for $z^7 - 1$, while Euler's formula works beautifully.

Application: Signal Processing Using the solutions to $z^n = 1$ which form n equally spaced points around the circle, approximations for periodic functions (signals) are found using the finite Fourier transform. For powers of 2 (and other composite numbers), a fast algorithm exists to compute these (fast FT).

3. Calculus: The functions of the form $e^{at}\cos bt$ and $e^{at}\sin bt$ come up in applications often. To find their derivatives, we can either use the product rule or use Euler's formula

$$\begin{aligned} (\frac{d}{dt})(e^{at}\cos\, bt + i\, e^{at}\sin\, bt) &= (\frac{d}{dt})e^{(a+ib)\,t} = (a+ib)e^{(a+ib)\,t} \\ &= (a+ib)(e^{at}\cos\, bt + i\, e^{at}\sin\, bt) \\ &= (ae^{at}\cos\, bt - b\, e^{at}\sin\, bt) \\ &\quad + i\,(b\, e^{at}\cos\, bt\, + a\, e^{at}\sin\, bt). \end{aligned}$$

This finds both derivatives simultaneously and is especially nice for higher derivatives (try the second derivatives yourself both ways!).

Integration: Even better is the integral aspect: To integrate $e^{a\,t}\cos\, bt$ and $e^{a\,t}\sin\, bt$ simultaneously, **integrate the complex exponential instead**!

$$
\begin{aligned}
\int (e^{a\,t}\cos\, bt + i\, e^{a\,t}\sin\, bt)\, dt \quad &= \quad \int e^{(a+ib)\,t}\, dt = \frac{1}{a+ib} e^{(a+ib)\,t} + C \\
&= \quad \frac{a-i\,b}{a^2+b^2}(e^{a\,t}\cos\, bt + i\, e^{a\,t}\sin\, bt) + C \\
&= \quad \frac{a}{a^2+b^2} e^{a\,t}\cos\, bt + \frac{b}{a^2+b^2}\, e^{a\,t}\sin\, bt) + C_1 \\
&+ \quad i\,(-\frac{b}{a^2+b^2}\, e^{a\,t}\cos\, bt\, + \frac{a}{a^2+b^2}\, e^{a\,t}\sin\, bt + C_2).
\end{aligned}
$$

Another integration result is that any product of positive powers of cosine and sine can be integrated explicitly. From Euler's formula this becomes an algebra problem with an easy calculus part, as illustrated in the following example:

$$
\int \cos^2 t\, dt = \int (\frac{e^{i\,t} + e^{-i\,t}}{2})^2\, dt = \int (\frac{e^{2\,i\,t} + 2 + e^{-2\,i\,t}}{4})\, dt \qquad (10)
$$

which can be done term-by-term.

There is clearly nothing special about the power 2 or cosine alone, so any positive power of sine and cosine can be expanded and then integrated.

The complex logarithm Using polar coordinates and Euler's formula allows us to define the complex exponential as

$$
e^{x+i\,y} = e^{x}\, e^{i\,y} \qquad (11)
$$

which can be reversed for any non-zero complex number written in polar form as $\rho\, e^{i\,\phi}$ by inspection: $x = \ln(\rho),\ y = \phi$ to which we can also add any integer multiplying 2π to y for another solution!

4. Differential equations: This formula really comes into its own when we need to solve differential equations with constant coefficients. Then the goal is to find the right numbers a, b so that the above functions which we just differentiated solve a given equation. For example, electrical circuits lead to differential equations that relate current, charge and voltage based on the circuit elements. Circuit elements are described by certain parameters like inductance, resistance, and capacitance. These become coefficients in the differential equation.

Example The differential equation $a\, y'' + b\, y' + c\, y = 0$ can be solved by seeking exponential solutions with an unknown exponential factor. Substituting $y = e^{r\,t}$ into the equation gives a solution if the quadratic equation $a\, r^2 + b\, r + c = 0$ holds. For lots of values of a, b, c, namely those where $b^2 - 4\, a\, c\, < 0$, the solutions are complex. Euler's formula allows us to interpret that easy algebra correctly.

Some Problems Involving Euler's Formula

1. Consider the equation $z^6 - 1 = 0$. Solve it in the two ways described below and then write a brief paragraph conveying your thoughts on each and your preference.

A. Euler's formula

B. View $z^6 - 1$ as a difference of squares, factor it that way, then factor each factor again. This identifies two quadratics that you can use to find the four roots besides 1 and -1. (Fun bonus: factor as difference of cubes originally and you get a degree four polynomial with those four roots as a product of quadratics)

2. Use Euler's formula to find the two complex square roots of i by writing i as a complex exponential. Do it also for $-i$ and check that $\sqrt{-i} = \sqrt{-1}\sqrt{i}$.

3. A crazy notion: find i^i by writing i as a complex exponential.

4. (Challenging) Factoring $z^2 + 1 = (z + i)(z - i)$ and using partial fractions, integrate (formally)

$$\int \frac{1}{z^2 + 1}\, dz$$

and try to get back to the arctan you know and love by using the complex log.