March 3, 2004

DISCUSSION OF INFINITY FOR HONORS 225

Sets containing infinitely many elements come in at least two different sizes. In this brief handout, we will summarize the class discussion on counting the number of elements and also covering sets with intervals. These two concepts are not the same, but linked in a particular way.

The counting numbers $1, 2, 3, \ldots$ form the most basic infinite set. They are also known as the natural numbers. We will call a set **countably infinite** if there is a way to list its elements using the counting numbers. In fancy language, we would say that there is a one-to-one correspondence between the set and the natural numbers. If a set is neither a finite set of elements nor a countably infinite set, we call it (imaginatively) **uncountably infinite**.

Example: *The integers form a countably infinite set.* We count starting at 0, working out, as in $0, 1, -1, 2, -2, 3, -3, \ldots$. You might have a different listing in mind. We only need one that works.

Example: *The rational numbers between 0 and 1 form a countably infinite set.* Use the lowest term form for each fraction, by running through the fractions in some order, such as by increasing denominators, so that $1/2$ is first, $1/3, 2/3$ are next, then $1/4, 3/4$, and so on.

Example: *The positive rational numbers form a countably infinite set.* View pairs of positive integers (m, n) as the number m/n. Count the grid starting at bottom left corner and exhausting leftward diagonals, as in class.

The set of all real numbers between 0 and 1 contains all the rationals, plus all the irrationals. We want to figure out if this set is countably infinite. In class, at first most students believed the irrational numbers were less plentiful than the rational numbers. If so, then the real numbers would be countably infinite, since two countably infinite sets when joined together form a countably infinite set. To see why that is true, do the counting in alternation, one from subset 1, then one from subset 2, and continue forever. Each element will eventually be counted in order and this shows the desired result. [More advanced notion: a union of a countably infinite number of countably infinite subsets is still countably infinite – count first one from the first set, then one from the first two sets, then one from the first three, and so on. Subtlety is needed to make sure we get elements counted from each subset eventually.]

Most students are more comfortable with rational numbers and therefore guess that there are more of them. This turns out to be wrong. One intuitive description: imagine an infinite number of lottery power ball machines, which each generate one digit in the decimal expansion. Would you expect such decimals to repeat or end in all zeroes?

Amazing Fact: *The real numbers between 0 and 1 form an uncountably infinite set.*

To show that the real numbers between 0 and 1 are uncountably infinite, which is a famous result of **Georg Cantor** (1845-1918), we recall the decimal expansions of real numbers. Each real number has a decimal expansion, and if we do not consider a decimal that ends in an infinite repeating set of only 9's, the decimal expansion is unique. Cantor proved that the real numbers between 0 and 1 cannot be countably infinite by contradiction. IF we assume the real numbers are listed in some way, starting with a first one, then a second, and so forth, then we will show that this listing misses at least one real number. Indeed, such a list misses most of them, but we will be happy to find one.

Cantor's famous diagonal argument goes as follows: pick a first decimal digit other than 9 that differs with the first number, a second digit that differs from that of the second listed number, and so on forever. The decimal created in this way is different from each number on the list, so it is not on the list!

This is short, deep and fun!

Another important notion linked to exotic sets is the "length" of such sets. We will say that **a subset of the real numbers has length 0** if *for any positive length* we can chop up an interval of that positive length into intervals *of positive length* and use the pieces to cover the subset. Notice the careful language in this definition: we use intervals of positive length to cover the set and we must be able to use arbitrarily small intervals to do the covering.

Fact: *Any countably infinite subset of the real numbers has length 0.* To show this, for any given interval, use half of it to cover the first number, half of the remaining subinterval to cover the second number, then half of the remaining subinterval to cover the third number, and so on. This will give an infinite number of pieces with total length any positive number that covers the countably infinite set.

Fact: *There are uncountably infinite sets with length 0.* The most famous example is called the Cantor set (see text, page 75). We start with the interval of size 1 from 0 to 1 and throw out the middle third. Then we take the middle third of each of the two pieces and discard them. Then the middle thirds of each piece. Continue indefinitely. The resulting strange object is the Cantor set. The Cantor set has length 0, since the length of the pieces is always multiplied by a factor of 2/3 from the previous stage. To finish the discussion, we need to check that the Cantor set is not countably infinite. We do this by viewing it as having the same number of elements as the original interval(!!??). At each level of cutting up, we can view the two thirds as two halves by changing the scale. This is very much like viewing the symbols L, R as the binary "digits" $0, 1$ in a base 2 expansion. Thus the Cantor set is created by working base 3, with "digits" $0, 1, 2$ and throwing out the 1's. We can view the 0 as a base 2 0 and the 2 in base 3 as a 1 in base 2.

In our text in Chapter 14, we will assign a notion of dimension to exotic sets like the Cantor set. It will turn out to have *a fractional dimension*!