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The full orbifold K-theory of abelian symplectic quotients

by

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Abstract

In their 2007 paper, Jarvis, Kaufmann, and Kimura defined the **full orbifold K-theory** of an orbifold $\mathfrak{X}$, analogous to the Chen-Ruan orbifold cohomology of $\mathfrak{X}$ in that it uses the **obstruction bundle** as a quantum correction to the multiplicative structure. We give an explicit algorithm for the computation of this orbifold invariant in the case when $\mathfrak{X}$ arises as an abelian symplectic quotient. To this end, we introduce the **inertial K-theory** associated to a T-action on a stably complex manifold M, where T is a compact abelian Lie group. Our methods are integral K-theoretic analogues of those used in the orbifold cohomology case by Goldin, Holm, and Knutson in 2005. We rely on the K-theoretic Kirwan surjectivity methods developed by Harada and Landweber. As a worked class of examples, we compute the full orbifold K-theory of weighted projective spaces that occur as a symplectic quotient of a complex affine space by a circle. Our computations hold over the integers, and in the particular case of these weighted projective spaces, we show that the associated invariant is torsion-free.

Key Words: full orbifold K-theory, inertial K-theory, Hamiltonian T-space, symplectic quotient

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1. Introduction

Orbifolds and their invariants, including homotopy groups, cohomology rings, and K-theory rings, are an active area of current research. Much recent work concerns stringy versions of these invariants (which take into account the so-called twisted sectors), motivated by the seminal work of Dixon, Harvey, Vafa, and Witten [12]. Examples of such invariants are the **orbifold cohomology** of Chen and Ruan [10] and the **full orbifold K-theory** introduced by Jarvis, Kaufmann, and Kimura [22] and further developed by Becerra and Uribe [6].

The main result of this manuscript is the complete description of the full orbifold K-theory of abelian symplectic quotients, using techniques from equivariant symplectic geometry. The examples to which these methods apply include many orbifold toric varieties (or smooth toric Deligne-Mumford stacks) as discussed in e.g. [8, 9, 14, 21, 23], including weighted projective spaces, a topic of active current research [5, 7, 11, 16, 20, 30]. Another class of examples are the orbifold weight varieties of [25] and [14, Section 8].

We introduce a new ring, called **inertial K-theory**, associated to a T-action on a manifold M, where T is a compact abelian Lie group and M is a stably complex manifold. This ring generalizes the **stringy K-theory** defined by Becerra and Uribe [6, Definition 2.1], which applies to a locally free T-action on M.[1] In contrast, the definition of inertial K-theory does not require that T act locally freely. An important special case is when X is a Hamiltonian T-space. In this setting, the restriction map to the T-fixed set allows us to simplify the product for the purposes of computation. We then use an analogue of the **Kirwan surjectivity theorem** from equivariant symplectic geometry to prove that the inertial K-theory surjects onto (an integral lift of) the full orbifold K-theory of [22].

We take a moment to discuss other versions of K-theory for orbifolds discussed in the literature. In [22], the authors also introduce the **stringy K-theory** $\mathcal{K}(X,G)$ associated to a smooth projective variety X with an action by a finite group G.[2] In this case, the G-invariant part of $\mathcal{K}(X,G)$, which in [22] is called the **small orbifold K-theory**, is isomorphic as a vector space, but not as a ring, to the orbifold K-theory of the global quotient $\mathfrak{X} = [X/G]$ as defined by Adem and Ruan [1]. In the setting of a global quotient by a finite group, the full orbifold K-theory of [22] contains the small orbifold K-theory $\mathcal{K}(X,G)^G$ as a subring. However, the full orbifold K-theory is far more general; in particular, it may be defined for stacks which are not global quotients.

Our definition of inertial K-theory $NK_T^{\diamond}(M)$ follows ideas introduced by Goldin, Holm, and Knutson [14] in the setting of cohomology. The ring $NK_T^{\diamond}(M)$ is well defined for any stably complex T-space M. In the case when the T-action is locally free, $NK_T^{\diamond}(M)$ is the stringy K-theory of Becerra and Uribe and in particular is isomorphic to (an integral lift of) the full orbifold K-theory of the associated orbifold [6, Section 2].

When the T-action is not locally free, then as far as we are aware, the inertial K-theory ring $NK_T^{\diamond}(M)$ is a new ring which has not appeared previously in the literature and, does not correspond to the K-theory of a stack. The point of introducing $NK_T^{\diamond}(M)$ is that we can use it along with integral K-theoretic analogues [18, 19] of Kirwan surjectivity to compute the full orbifold K-theory

[1] What the authors in [6] call 'stringy K-theory' is analogous to the full orbifold K-theory of [22] in the case that the orbifold is a global quotient by an abelian Lie group.

[2] The stringy K-theory of [22] is defined only when G is finite, and differs from the stringy K-theory of Becerra and Uribe [6].

of orbifolds $\mathfrak{X}$ arising as abelian symplectic quotients. For example, orbifold toric varieties (as studied e.g. in [26]) arise in this manner via the Delzant construction.

Our computations depend on an inertial K-theory analogue of standard localization results in equivariant symplectic geometry. Specifically, when M is a Hamiltonian T-space, the stringy product on $NK_T^\diamond(M)$ may be reformulated using the T-fixed point sets and their normal bundles, simplifying computations. This is called the $\star$-product, and mimics a similar product in orbifold cohomology as in [14]. Our main theorem, proven in Section 3, is the following.

Theorem 1.1 *Let T be a compact connected abelian Lie group, and (M,ω,Φ) a Hamiltonian T-space with proper moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ, so that T acts locally freely on $\Phi^{-1}(\alpha)$. Then the inclusion $\iota : \Phi^{-1}(\alpha) \hookrightarrow M$, induces a* **ring homomorphism** *in inertial K-theory:*

$$\kappa : NK_T^\diamond(M) \xrightarrow{\iota^*} NK_T^\diamond(\Phi^{-1}(\alpha)) \xrightarrow{\cong} \mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T]) =: \mathsf{K}_{\mathrm{orb}}([M/\!/_\alpha T]) \tag{1.1}$$

from the inertial K-theory $NK_T^\diamond(M)$ of M onto the integral full orbifold K-theory $\mathsf{K}_{\mathrm{orb}}([M/\!/_\alpha T])$ of the quotient orbifold $[M/\!/_\alpha T] := [\Phi^{-1}(\alpha)/T]$. Furthermore, this map is **surjective**.

We summarize the steps for the proof of this theorem and its use in effective computations. The key tool is the ring $NK_T^\diamond(M)$ of the original Hamiltonian T-space M. As a vector space $NK_T^\diamond(M) = \bigoplus_{t\in T} K_T(M^t)$, where M^t consists of fixed points of the t action on M, for $t \in T$. For each t, $K_T(M^t)$ may be computed using well-known methods in equivariant topology (see e.g. [15, 17]). We may also apply the ordinary K-theoretic Kirwan surjectivity theorem, which states that the map κ_t induced by inclusion from $K_T^*(M^t)$ to $K_T^*(\Phi^{-1}(0)\cap M^t)$ is a surjection [18]. However, the ring structure on $NK_T^\diamond(M)$ is not the obvious one on $\bigoplus_{t\in T} K_T(M^t)$. Thus the main technical challenge, as was the case in [14], is to prove that the Kirwan map κ given by (1.1) is indeed a ring homomorphism.

An additional benefit of our point of view is that $NK_T^\diamond(M)$ surjects onto the full orbifold K-theory of any of its orbifold symplectic quotients (at any regular value α). This is because the isotropy information for every orbifold symplectic quotient $[M/\!/_\alpha T]$ is contained in the ring $NK_T^\diamond(M)$ when M is a Hamiltonian T-space. As an illustration of our surjectivity theorem, in Section 4 we calculate the orbifold cohomology of those weighted projective spaces that occur as symplectic reductions of $\mathbb{C}^n$ by a linear S^1-action. We will discuss symplectic toric orbifolds in greater detail in a subsequent paper.

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2. The inertial K-theory of a stably complex T-space

Throughout, T will denote a compact connected abelian Lie group (i.e. a compact torus). In this section, we define a new ring, the **inertial K-theory**, associated to a stably complex T-manifold M. The definition is similar in spirit to that of the inertial cohomology associated to M as in [14]. When M is a Hamiltonian T-space, this inertial K-theory ring gives rise to a surjective ring homomorphism onto the full orbifold K-theory defined by [22] of the symplectic quotient $M//_\alpha T$.

2.1. The definition of inertial K-theory

We begin by defining the inertial K-theory additively as a K_T-module. Suppose M is a stably complex T-space in the sense of [14]. Since T is abelian, each M^t is a T-space for $t \in T$.

Definition 2.1 The **inertial K-theory $NK_T^\diamond(M)$ of a stably complex T-space** M is defined, as a $K_T(\mathrm{pt})$-module, as

$$NK_T^\diamond(M) := \bigoplus_{t\in T} K_T(M^t). \tag{2.1}$$

The grading $\diamond$ is with respect to group elements of T; as we will show in Definition 2.3, the product of a homogeneous t_1-class and a homogeneous t_2-class is a homogeneous $(t_1 \cdot t_2)$-class. Here $K_T(X)$ denotes the integral T-equivariant K-theory of Atiyah and Segal [28]. In the case when T acts with finite stabilizers, $NK_T^\diamond(M)$ coincides with [6, Definition 2.1] and with the full orbifold K-theory of [22] by [6, Section 2].

We now proceed with the definition of the product on $NK_T^\diamond(M)$ which follows that of [6] and [14]. We begin with some observations about the normal bundle to the fixed point set M^H for a fixed subgroup $H \subset T$; these lead to a simple description (using **logweights**) of the **obstruction bundle** with respect to which we twist the product. Let V be a connected component of M^H and let $p \in V$. The linear action by H on each fiber $T_pM \oplus \tau$ (where τ is the trivial stabilization in the stably complex structure) is the identity on precisely $T_pV \oplus \tau$, so there is an induced linear action of H on the normal bundle $\nu(V,M) := (TM|_V)/TV \cong (TM \oplus \tau)/(TV \oplus \tau)$ over $V \subset M^H$. Since H preserves the almost complex structure on $TM \oplus \tau$, it follows that $\nu(V,M)$ is a complex vector bundle. The (complex) rank of $\nu(V,M)$ may vary as V varies over the components of M^H.

We may write

$$\nu(V,M) \cong \bigoplus_{\mu \in \widehat{H}} \nu(V,M)_\mu, \tag{2.2}$$

where $\hat{H}$ denotes the character group of H and $\nu(V,M)_\mu$ is the subbundle of $\nu(V,M)$ on which H acts by weight μ. Each element $t \in H$ acts on a fiber $\nu_p(V,M)_\mu$ with eigenvalue $\exp(2\pi i a_\mu(t))$, for a constant $a_\mu(t) \in [0,1)$ called the **logweight** of t on $\nu(V,M)_\mu$.

Now let

$$\widetilde{M} := \bigsqcup_{(t_1,t_2) \in T \times T} M^{t_1,t_2}$$

where M^{t_1,t_2} is the submanifold of M consisting of points fixed simultaneously by t_1 and t_2. For each pair $(t_1,t_2) \in T \times T$, let $\mathcal{O}_{t_1,t_2}$ be the union over all connected components V of M^{t_1,t_2} of $\mathcal{O}_{t_1,t_2}|_V$, where

$$\mathcal{O}_{t_1,t_2}|_V := \bigoplus_{a_\mu(t_1)+a_\mu(t_2)+a_\mu((t_1t_2)^{-1})=2} \nu(V,M)_\mu \tag{2.3}$$

is the complex subbundle of $\nu(V,M)$ given by the component on which the sum of the logweights of t_1, t_2, and $(t_1t_2)^{-1}$ is 2.

Definition 2.2 The **obstruction bundle** $\mathcal{O} \to \widetilde{M}$ is the (disjoint) union of the bundles $\mathcal{O}_{t_1,t_2}$ in (2.3) for all pairs $t_1,t_2 \in T$, i.e.

$$\mathcal{O} := \bigsqcup_{\substack{(t_1,t_2)\in T\times T \\ V \text{ c.c. } M^{t_1,t_2}}} \mathcal{O}_{t_1,t_2}|_V = \bigsqcup_{(t_1,t_2)\in T\times T} \mathcal{O}_{t_1,t_2},$$

where "c.c." denotes "connected component of".

Let $\epsilon(\mathcal{O}_{t_1,t_2}|_V) := \lambda_{-1}(\mathcal{O}_{t_1,t_2}|_V^*)$ denote the T-equivariant K-theoretic Euler class of this bundle $\mathcal{O}_{t_1,t_2}|_V \to V$. We define the **virtual fundamental class of** $\widetilde{M}$ to be the sum

$$\epsilon(\mathcal{O}) := \sum_{\substack{(t_1,t_2)\in T\times T \\ V \text{ c.c. } M^{t_1,t_2}}} \epsilon(\mathcal{O}_{t_1,t_2}|_V).$$

We note that this virtual fundamental class is also sometimes called a 'quantum correction' in the literature. (The sense in which it is a 'correction' can be seen in Definition 2.3 below.) We now define the product on $NK_T^\diamond(M)$. By extending linearly, it clearly suffices to define the product $b_1 \odot b_2$ of two homogeneous classes $b_1 \in NK_T^{t_1}(M) = K_T(M^{t_1})$ and $b_2 \in NK_T^{t_2}(M) = K_T(M^{t_2})$. Let $e_j : M^{t_1,t_2} \hookrightarrow M^{t_j}$ denote the canonical T-equivariant inclusion map for any $t_1,t_2 \in T$ and $j = 1,2$, and let $\overline{e}_3 : M^{t_1,t_2} \hookrightarrow M^{t_1t_2}$ denote the inclusion map into points fixed by the product t_1t_2.

Definition 2.3 Let M be a stably complex T-manifold and $NK_T^{\diamond}(M)$ its inertial K-theory. Let $t_1,t_2 \in T$. The $\odot$ **product on the inertial K-theory** $NK_T^{\diamond}(M)$ is defined, for $b_1 \in NK_T^{t_1}(M)$ and $b_2 \in NK_T^{t_2}(M)$, by

$$b_1 \odot b_2 = (\overline{e}_3)_!(e_1^* b_1 \cdot e_2^* b_2 \cdot \epsilon(\mathcal{O})), \tag{2.4}$$

where $\cdot$ denotes the usual product in the T-equivariant K-theory $K_T(M^{t_1,t_2})$, and the subscript ! denotes the K-theoretic push-forward along an inclusion. By extending linearly, $\odot$ is defined on all of $NK_T^{\diamond}(M)$.

Note that for $b_1 \in NK_T^{t_1}(M)$ and $b_2 \in NK_T^{t_2}(M)$, the product $b_1 \odot b_2 \in NK_T^{t_1t_2}(M)$ is by definition a homogeneous class in the t_1t_2-summand of $NK_T^{\diamond}(M)$. If the Euler class were not included in the definition, the product would not be associative.

It is straightforward to show that $NK_T^{\diamond}(M)$ is a $K_T(\mathrm{pt})$-algebra.

Proposition 2.4 *Let M be a stably complex T-manifold. Then $NK_T^{\diamond}(M)$ is a commutative, associative, unital algebra over the ground ring $K_T(\mathrm{pt})$ with the multiplication $\odot$ of Definition 2.3.*

Remark 2.5 As observed in [6, Section 2], if T acts on M locally freely then $NK_T^{\diamond}(M) \otimes \mathbb{Q}$ is isomorphic as an algebra to the full orbifold K-theory $\mathsf{K}_{\mathrm{orb}}([M/T])$ [22] of the quotient M/T. More specifically, the natural map $NK_T^{\diamond}(M) \to NK_T^{\diamond}(M) \otimes \mathbb{Q}$ is an integral lift of $\mathsf{K}_{\mathrm{orb}}([M/T])$ as rings.

In view of the proposition and remark above, we will henceforth occasionally abuse language and refer to $NK_T^{\diamond}(M)$ as being isomorphic to $\mathsf{K}_{\mathrm{orb}}([M/T])$. The precise statement is that when T acts locally freely on M, the ring $NK_T^{\diamond}(M)$ is an integral lift of $\mathsf{K}_{\mathrm{orb}}([M/T])$.

Proof of Proposition 2.4: The facts that $NK_T^{\diamond}(M)$ is commutative and that $1 \in K_T(M^{id})$ acts as the unit element is immediate from the definition (2.4). The $K_T(\mathrm{pt})$-algebra structure is also immediate from the corresponding structure on each summand. It remains to show associativity. Although we are not assuming that the T-action on M is locally free, the proof of associativity in our T-equivariant situation nevertheless follows precisely that of [6] (and [22]), so we do not reproduce it here. □

2.2. *The product on the fixed point set*

Localization is a standard technique in equivariant topology. Given M a stably complex T-manifold and a T-equivariant algebraic/topological invariant, it is natural to ask whether the invariant is encoded in terms of the T-fixed point set M^T and local T-isotropy data near the fixed points. The purpose of this section is to develop some inertial K-theoretic analogues of standard localization theory in equivariant topology. The presence of the quantum correction complicates matters, so we begin by defining a new ring structure, denoted by $\star$, on $K_T(M^T) \otimes \mathbb{Z}[T]$, where $\mathbb{Z}[T]$ is the group ring on T. This is a K-theoretic version of the $\star$ product on $H_T^*(M^T) \otimes \mathbb{Z}[T]$ introduced in [14]. When M is a Hamiltonian T-space, we show in Section 2.3 that the inertial K-theory injects into $K_T(M^T) \otimes \mathbb{Z}[T]$ as a ring, much as ordinary equivariant K-theory $K_T(M)$ injects into $K_T(M^T)$ in such a case. This is the main motivation for the product $\star$: the new product provides a different means of computing the product given in (2.4).

For simplicity, we assume throughout that M^T has finitely many connected components. In this case

$$K_T(M^T) \otimes \mathbb{Z}[T] = \bigoplus_{W \text{ c.c. } M^T} (K_T(W) \otimes \mathbb{Z}[T]),$$

where the direct sum is taken over connected components of M^T. When we refer to the restriction of a class in $K_T(M^T) \otimes \mathbb{Z}[T]$ to a connected component W, we mean the summand corresponding to W. As in the $\odot$ case, it suffices to define the $\star$ product of two homogeneous classes $\sigma_1 \otimes t_1$ and $\sigma_2 \otimes t_2$ in $K_T(M^T) \otimes \mathbb{Z}[T]$, where $t_1, t_2 \in T$ and $\sigma_1, \sigma_2 \in K_T(M^T)$. Moreover, it also suffices to specify the value of the product restricted to each connected component W of M^T.

Definition 2.6 Let $\sigma_j \otimes t_j \in K_T(M^T) \otimes \mathbb{Z}[T]$ for $j = 1,2$, where $t_j \in T$, be two homogeneous classes. The $\star$ **product on** $K_T(M^T) \otimes \mathbb{Z}[T]$ is defined by

$$(\sigma_1 \otimes t_1) \star (\sigma_2 \otimes t_2)|_W := \left[\sigma_1|_W \cdot \sigma_2|_W \cdot \prod_{I_\mu \subset \nu(W,M)} \epsilon(I_\mu)^{a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1 t_2)} \right] \otimes t_1 \cdot t_2, \tag{2.5}$$

for each connected component W of M^T. Here $\epsilon(I_\mu) \in K_T(W)$ denotes the T-equivariant K-theoretic Euler class of I_μ, and $\nu(W,M)$ denotes the normal bundle of W in M. By extending linearly over the group algebra $\mathbb{Z}[T]$, the $\star$ product is defined on all of $K_T(M^T) \otimes \mathbb{Z}[T]$.

The proof that the $\star$ product is associative is straightforward and identical to that of [14, Theorem 2.3], so we do not repeat the argument, but record the result here.

Theorem 2.7 *Let M be a stably complex T-manifold. The multiplication $\star$ of Definition 2.6 makes $K_T(M^T) \otimes \mathbb{Z}[T]$ into a commutative, associative, unital algebra over the ground ring $K_T(\mathrm{pt})$.*

As stated above, the motivation for introducing $K_T(M^T) \otimes \mathbb{Z}[T]$ with the $\star$ product is to allow for a kind of "localization" theorem in inertial K-theory. Consider the inclusion $i_t : M^T \hookrightarrow M^t$ for each $t \in T$. This induces a map $i_t^* : K_T(M^t) \to K_T(M^T) \otimes t \subset K_T(M^T) \otimes \mathbb{Z}[T]$. Combining the i_t^* for each $t \in T$, we obtain a map of K_T-modules

$$i_{NK}^* : NK_T(M) \longrightarrow K_T(M^T) \otimes \mathbb{Z}[T], \tag{2.6}$$

which we refer to as a "restriction map," since it is induced by the geometric inclusion $M^T \hookrightarrow M^t$ for each $t \in T$. This morphism i_{NK}^* is in fact a ring homomorphism with respect to the $\odot$ and the $\star$ products.

Theorem 2.8 *Let M be a stably complex T-manifold. Let $K_T(M^T) \otimes \mathbb{Z}[T]$ be endowed with the product $\star$ of Definition 2.6. Then the restriction map i_{NK}^* : $(NK_T^{\diamond}(M), \odot) \to (K_T(M^T) \otimes \mathbb{Z}[T], \star)$ is a $K_T(\mathrm{pt})$-algebra homomorphism.*

Proof: This proof follows that of Theorem 3.6 of [14], though the explanation below is self-contained and works directly in terms of bundles instead of Euler classes.

We begin by noting that for any $t_1, t_2 \in T$, the exponent $a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1t_2)$ appearing in the definition of the $\star$ product is always either 0 or 1. Using the defining properties of Euler classes, we may deduce

$$\prod_{I_\mu \subset \nu(W,M)} \epsilon(I_\mu)^{a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1t_2)} = \epsilon \left(\bigoplus_{\substack{I_\mu \subset \nu(W,M) \\ a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1t_2)=1}} I_\mu \right),$$

so the expression appearing in the definition of the $\star$ product is the Euler class of a certain sub-bundle of $\nu(W,M)$. We will now show that this sub-bundle has a different description. Given μ such that $a_\mu(t_1) + a_\mu(t_2) - a_\mu(t_1t_2) = 1$, then either $a_\mu(t_1t_2) \neq 0$, in which case

$$\begin{aligned} a_\mu(t_1) + a_\mu(t_2) - a_\mu(t_1t_2) &= a_\mu(t_1) + a_\mu(t_2) - (1 - a_\mu((t_1t_2)^{-1}) = 1 \\ &\Longrightarrow a_\mu(t_1) + a_\mu(t_2) + a_\mu((t_1t_2)^{-1}) = 2 \\ &\Longrightarrow I_\mu \subset \mathcal{O}_{t_1,t_2}|_W, \end{aligned}$$

or else $a_\mu(t_1t_2) = 0$, in which case $I_\mu \subset \nu(M^{t_1,t_2}, M^{t_1t_2})|_W$. Conversely, if I_μ is an isotypic component of $\mathcal{O}_{t_1,t_2}|_W \oplus \nu(M^{t_1,t_2}, M^{t_1t_2})|_W$, then a similar simple argument shows that $a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1t_2)=1$. Thus we have shown that

$$\bigoplus_{\substack{I_\mu\subset\nu(W,M)\\ a_\mu(t_1)+a_\mu(t_2)-a_\mu(t_1t_2)=1}} I_\mu \;=\; \mathcal{O}_{t_1,t_2}|_W\oplus\nu(M^{t_1,t_2},M^{t_1t_2})|_W. \qquad (2.7)$$

This makes it clear that the obstruction bundle overM^{t_1,t_2} enters into the $\star$ product given by (2.3) as well.

To prove that i^*_{NK} is a ring homomorphism, it suffices to check the statement for homogeneous elements. Let $b_1\in K_T(M^{t_1})$ and $b_2\in K_T(M^{t_2})$. Then for each fixed point component W of M^T, we have

$$\begin{aligned} i^*_{NK}(b_1\odot b_2)|_W &= i^*_{NK}[(\overline{e}_3)_!(e_1^*b_1\cdot e_2^*b_2\cdot\epsilon(\mathcal{O}_{t_1,t_2}))]|_W\\ &= b_1|_W\cdot b_2|_W\cdot i^*_{NK}[(\overline{e}_3)_!(\epsilon(\mathcal{O}_{t_1,t_2}))]|_W\\ &= b_1|_W\cdot b_2|_W\cdot\epsilon(\mathcal{O}_{t_1,t_2})|_W\cdot\epsilon(\nu(M^{t_1,t_2},M^{t_1t_2}))|_W\\ &= b_1|_W\cdot b_2|_W\cdot\epsilon(\mathcal{O}_{t_1,t_2}|_W\oplus\nu(M^{t_1,t_2},M^{t_1t_2})|_W). \end{aligned}$$

The equivalence (2.7) of bundles allows us to conclude that this product $b_1\odot b_2$ restricted to a component W of M^T agrees with the product of $i^*_{NK}(b_1)|_W\star i^*_{NK}(b_2)|_W$ as in (2.5), as desired. □

2.3. The case that M is a Hamiltonian T-space

We now turn to the special case that (M,ω) is a Hamiltonian T-space. The motivation for the definition of the $\star$ product on $K_T(M^T)\otimes\mathbb{Z}[T]$ is to provide a target for a localization theoreom. As in the case of ordinary equivariant (rational) cohomology and equivariant K-theory, the fixed points M^T of a Hamiltonian T-space play a special role in inertial K-theory. In particular, we have the following theorem.

Theorem 2.9 *Let (M,ω,Φ) be a Hamiltonian T-space. Suppose there exists a component of Φ which is proper and bounded below, and further suppose that M^T has only finitely many connected components. The map of rings given by*

$$i^*_{NK}:(NK_T(M),\odot)\longrightarrow(K_T(M^T)\otimes\mathbb{Z}[T],\star)$$

is injective.

Proof: We have already shown in Theorem 2.8 that i^*_{NK} is an algebra homomorphism. Thus we need only show injectivity. Since T is abelian, standard symplectic techniques allow us to conclude that $(M^t,\omega|_{M^t},\Phi|_{M^t})$ is also a Hamiltonian T-space for each $t\in M$. Furthermore, the second author and Landweber prove in [19] that, for any Hamiltonian T-space M satisfying the assumptions of our theorem, the inclusion $M^T \hookrightarrow M^t$ induces an injection $K_T(M) \hookrightarrow K_T(M^T)$. Thus each map

$$i_t^* : K_T(M^t) \longrightarrow K_T(M^T) \otimes t$$

is an injection, which implies that the map i_{NK}^* defined by an i_t^* on each component is also injective, as desired. □

It is clear from the proof that the statement holds for any stably complex T-space M with the property that there is an injection

$$K_T(M^t) \longrightarrow K_T(M^T)$$

for every $t \in T$. In the case of equivariant cohomology, these spaces are called **robustly equivariantly injective** in [14]. Thus, Theorem 2.9 holds also in the case that M is robustly equivariantly injective in K-theory.

3. Surjectivity from inertial K-theory to full orbifold K-theory

The main motivation for our definition of the inertial K-theory of a T-space M is that we can exploit it to give explicit computations of the full orbifold K-theory of abelian symplectic quotients. Our methods build on the Kirwan surjectivity techniques in ordinary (non-orbifold) K-theory as developed in [18]. In this section we prove a general surjectivity theorem onto the $\mathsf{K}_{\mathrm{orb}}([M//_\alpha T])$ of abelian symplectic quotients and discuss in some detail the computation of the kernel of the surjection. Indeed, for a wide class of examples, this method yields an explicit description via generators and relations of $\mathsf{K}_{\mathrm{orb}}([M//_\alpha T])$. In Section 4 we will give an explicit illustration of the use of our techniques in the case of weighted projective spaces occurring as symplectic reductions.

We take a moment here to discuss the technical hypotheses on the moment map Φ to be used in this section. A more detailed discussion of these hypotheses may be found in [19, beginning of Section 3]. For the surjectivity theorems (Theorems 3.4 and 3.7) it is only necessary to assume that Φ is proper; this ensures that the tools of equivariant Morse theory may be applied to $\|\Phi\|^2$. However, for the computation of the kernel of the surjective Kirwan map as given in Theorem 3.8, we need the additional technical assumption that there is a component of Φ that is proper and bounded below, and that the fixed point set M^T has only finitely many connected components. In practice this is not a very restrictive condition (see [19, Section 3] for further discussion).

3.1. *Surjectivity to* $\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$

Here we show that under some mild technical assumptions, the inertial K-theory associated to a Hamiltonian T-space (M,ω,Φ) defined in the previous section surjects onto the full orbifold K-theory of the symplectic quotient $[M//_\alpha T] := [\Phi^{-1}(\alpha)/T]$, where α is a regular value of Φ.

We begin by recalling the setup. As above, T denotes a compact connected abelian Lie group (i.e. a connected compact torus). Let (M,ω,Φ) be a finite-dimensional Hamiltonian T-space, and assume that the moment map $\Phi : M \to \mathfrak{t}^*$ is proper. Suppose that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ. Then the level set $\Phi^{-1}(\alpha)$ is a submanifold of M, and T acts locally freely on $\Phi^{-1}(\alpha)$. Hence the quotient $[\Phi^{-1}(\alpha)/T]$ is an orbifold. By results of Becerra and Uribe,

$$\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T]) \cong NK_T^{\diamond}(\Phi^{-1}(\alpha)), \tag{3.1}$$

where by the left hand side we mean (by slight abuse of notation) an integral lift of the orbifold K-theory, as in Remark 2.5.

We now show that the right-hand side of (3.1) is computable, using techniques from equivariant symplectic geometry. The inclusion of the level set

$$\iota : \Phi^{-1}(\alpha) \hookrightarrow M$$

is T-equivariant and induces a map in equivariant K-theory,

$$\iota^* : K_T(M) \longrightarrow K_T(\Phi^{-1}(\alpha)).$$

The fixed points sets M^t are also Hamiltonian T-spaces, so we also have

$$\iota^* : K_T(M^t) \longrightarrow K_T(\Phi^{-1}(\alpha)^t).$$

Here by abuse of notation we denote also by ι the inclusions $\Phi^{-1}(\alpha)^t \hookrightarrow M^t$, for all $t \in T$. Hence there exists a map (also denoted ι^*)

$$\iota^* : NK_T(M) = \bigoplus_{t\in T} K_T(M^t) \longrightarrow NK_T(\Phi^{-1}(\alpha)) = \bigoplus_{t\in T} K_T(\Phi^{-1}(\alpha)^t) \tag{3.2}$$

defined in the obvious way on corresponding summands. It is immediate from the definition that the map is an additive homomorphism.

What is not at all obvious is that the map ι^* of (3.2) is also a ring homomorphism with respect to the multiplicative structure $\odot$ on both sides, and furthermore that ι^* provides an effective way of computing $NK_T(\Phi^{-1}(\alpha))$, and hence $\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$. We now address each of these issues in turn.

As observed in [13] and [14], it is not necessarily true that a general T-equivariant map of T-spaces $f : X \to Y$ induces a ring map $f^* : NH_T^*(Y) \to NH_T^*(X)$ on inertial cohomology, with respect to the $\smile$ product in inertial cohomology. Nevertheless, if a T-equivariant inclusion $\iota : X \hookrightarrow Y$ behaves well with respect to the fixed point sets, [14, Proposition 5.1] states that ι^* is a ring homomorphism with respect to the new product in inertial cohomology. We now prove a K-theoretic analogue of this fact.

Proposition 3.1 *Let Y be a stably complex T-space. Let $\iota : X \hookrightarrow Y$ be a T-equivariant inclusion, and suppose also that X is tranvserse to all $Y^t, t \in T$. Then the map $\iota^* : NK_T^\diamond(Y) \to NK_T^\diamond(X)$ induced by inclusion is a ring homomorphism with respect to $\odot$.*

Proof: The argument is nearly the same as that given for equivariant cohomology [14, Proposition 5.1], so we do not fully reproduce it here. The only additional item to check in our K-theoretic setting is that the following diagram in T-equivariant K-theory

$$\begin{array}{ccc} K_T(Y^{t_1,t_2}) & \xrightarrow{(\overline{e}_3^{1,2})_!} & K_T(Y^{t_1t_2}) \\ \Big\downarrow \iota^* & & \Big\downarrow \iota^* \\ K_T(X^{t_1,t_2}) & \xrightarrow{(\overline{e}_3^{1,2})_!} & K_T(X^{t_1t_2}) \end{array} \tag{3.3}$$

commutes, i.e. $\iota^*(\overline{e}_3^{1,2})_! = (\overline{e}_3^{1,2})_!\iota^*$, where by abuse of notation we denote by ι^* both of the inclusions $X^{t_1t_2} \hookrightarrow Y^{t_1t_2}$ and $X^{t_1,t_2} \hookrightarrow Y^{t_1,t_2}$, and also by $\overline{e}_3^{1,2}$ both of the inclusions $X^{t_1,t_2} \hookrightarrow X^{t_1t_2}$ and $Y^{t_1,t_2} \hookrightarrow Y^{t_1t_2}$. Recall that the definition of the pushforward in equivariant K-theory uses the Thom isomorphism with respect to a Spin^c structure on the relevant normal bundles to an inclusion. In this case, both normal bundles in question have natural T-equivariant complex structures, so they have canonical Spin^c structures. Hence, in order to check that (3.3) commutes, it suffices to check that the normal bundle $\nu(Y^{t_1,t_2}, Y^{t_1t_2})$ restricts precisely to the normal bundle $\nu(X^{t_1,t_2}, X^{t_1,t_2})$ via ι^*. This follows from the transversality of X to all fixed points Y^t, for any $t \in T$.

□

Although the transversality hypothesis of Proposition 3.1 is rather restrictive, there is a natural class of examples in which the conditions are satisfied.

Lemma 3.2 *Let (M, ω, Φ) be a Hamiltonian T-space with proper moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ. Then the inclusion of the level set $\iota : \Phi^{-1}(\alpha) \hookrightarrow M$ satisfies the hypotheses of Proposition 3.1.*

This is a purely topological statement and its proof can be found in [14, Theorem 6.4]. We may conclude that we have a ring homomorphism onto the integral full orbifold K-theory

$$\kappa : NK^{\diamond}_T(M) \xrightarrow{\iota^*} NK^{\diamond}_T(\Phi^{-1}(\alpha)) \xrightarrow{\cong} \mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T]). \tag{3.4}$$

We refer to this composition κ as the **orbifold Kirwan map**. We now recall the following, which is the analogue of the Kirwan surjectivity theorem (originally proved for rational Borel-equivariant cohomology) in integral K-theory [18, Theorem 3.1]. We state the theorem only in the special case needed here.

Theorem 3.3 *([18]) Let (M,ω) be a Hamiltonian T-space with proper moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ. Then the map ι^* induced by the inclusion $\iota : \Phi^{-1}(\alpha) \hookrightarrow M$,*

$$\iota^* : K_T(M) \longrightarrow K_T(\Phi^{-1}(\alpha)),$$

is a surjection.

The following is now straightforward.

Theorem 3.4 *Let (M,ω,Φ) be a Hamiltonian T-space with proper moment map $\Phi : M \to \mathfrak{t}^*$. Assume that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ, so that T acts locally freely on $\Phi^{-1}(\alpha)$. Then the orbifold Kirwan map to the orbifold K-theory*

$$\kappa : NK^{\diamond}_T(M) \longrightarrow \mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$$

defined in (3.4) *is a surjective ring homomorphism.*

Proof: Proposition 3.1 and Lemma 3.2 guarantee that κ is a ring homomorphism, so it is enough to check that ι^* is surjective. Since ι^* in (3.2) is defined separately on each summand, the surjectivity follows from surjectivity on each summand. We observe that for a Hamiltonian T-space (M,ω,Φ) as given, for any $t \in T$, the fixed set M^t is itself a Hamiltonian T-space with moment map the restriction $\Phi|_{M^t}$. In particular, since $\Phi^{-1}(\alpha)^t = (\Phi|_{M^t})^{-1}(\alpha)$, Theorem 3.3 implies that each

$$\iota^* : K_T(M^t) \longrightarrow K_T(\Phi^{-1}(\alpha)^t)$$

is surjective, completing the proof. □

Thus, in order to compute $\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$, we must explicitly compute $NK^{\diamond}_T(M)$ and identify the kernel of κ. We discuss the kernel in Section 3.2. As for the domain $NK^{\diamond}_T(M)$, we observe that in fact a different, smaller, ring already surjects onto $NK^{\diamond}_T(\Phi^{-1}(\alpha))$. This is highly relevant for computations, since the domain of the orbifold Kirwan map (3.4) is an infinite direct sum, while the smaller subring is a finite direct sum (and thus more manageable for explicit computations).

The essential idea is similar to those already developed in [14] and subsequently in [13]. Since T acts locally freely on $\Phi^{-1}(\alpha)$ and because $\Phi^{-1}(\alpha)$ is compact (since Φ is proper), there are only finitely many orbit types, and hence only finitely many elements $t \in T$ occur in the stabilizer of a point in $\Phi^{-1}(\alpha)$. In particular, $\Phi^{-1}(\alpha)^t$ is non-empty for only finitely many t. Thus in the codomain of the map (3.2), only finitely many summands are non-zero. It is straightforward to see that the restriction of κ to the direct sum

$$\bigoplus_{t\colon\, \Phi^{-1}(\alpha)^t \neq \emptyset} K_T(M^t) \tag{3.5}$$

itself surjects onto $NK_T^{\diamond}(\Phi^{-1}(\alpha))$. Unfortunately (3.5) is not closed under the $\odot$ multiplication on $NK_T^{\diamond}(M)$. Hence we introduce the **Γ-subring of $NK_T^{\diamond}(M)$**, which is the smallest subring containing (3.5); this will also surject onto $NK_T^{\diamond}(\Phi^{-1}(\alpha))$.

Recall that if a torus T acts locally freely on a space Y, then by definition the stabilizer group $\mathrm{Stab}(y)$ of any point $y \in Y$ is finite; we call $\mathrm{Stab}(y)$ a **finite stabilizer group**. Similarly, given a finite stabilizer group $\mathrm{Stab}(y)$, we call an element $t \in \mathrm{Stab}(y)$ a **finite stabilizer element**. Let Γ denote the subgroup of T generated by all finite stabilizer elements, called the **finite stabilizer subgroup** of T associated to Y. For any subgroup Γ of T we may define the following.

Definition 3.5 Let Y be a stably complex T-space and let Γ be a subgroup of T. Then

$$NK_T^{\Gamma}(Y) := \bigoplus_{t \in \Gamma} K_T(Y^t) \tag{3.6}$$

is a subring of $NK_T^{\diamond}(Y)$, called the **Γ-subring** of $NK_T^{\diamond}(Y)$.

Remark 3.6 The Γ-subring is closed under the $\odot$ multiplication. This follows immediately from Definition 2.3 and the comment after it, together with the fact that Γ is a subgroup and thus closed under multiplication.

In the case of the level set $\Phi^{-1}(\alpha)$ of a Hamiltonian T-action (M,ω,Φ) with a proper moment map Φ, this associated subgroup Γ will be a finite subgroup of T.[3] Hence the direct sum in (3.6) is also finite. The preceding discussion establishes the following.

Theorem 3.7 *Let (M,ω,Φ) be a Hamiltonian T-space with proper moment map $\Phi: M \to \mathfrak{t}^*$. Assume that $\alpha \in \mathfrak{t}^*$ is a regular value of Φ, so that T acts locally freely on $\Phi^{-1}(\alpha)$. Let Γ be the finite stabilizer subgroup of T associated to $\Phi^{-1}(\alpha)$. Then*

[3] Although we do not make it explicit in the notation, this subgroup Γ depends on the choice of level set $\Phi^{-1}(\alpha)$. In [14] the subgroup Γ is chosen such that κ^{Γ} is surjective for any choice of level set, so our choice differs slightly from that of Goldin, Holm, and Knutson.

the restriction κ^Γ *of the orbifold Kirwan map* (3.4) *to the* Γ*-subring* (3.6)*,*

$$\kappa^\Gamma : NK_T^\Gamma(M) \longrightarrow NK_T^\diamond(\Phi^{-1}(\alpha)) \cong \mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T]),$$

is a surjective ring homomorphism.

3.2. *The kernel of the orbifold Kirwan map*

Theorem 3.7 shows that the inclusion of the level set $\Phi^{-1}(\alpha) \hookrightarrow M$ induces a surjective ring homomorphism κ^Γ from the Γ-subring $NK_T^\Gamma(M)$ to $NK_T(\Phi^{-1}(\alpha)) \cong \mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$. So to compute explicitly the ring $\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$, it then remains to compute the domain $NK_T^\Gamma(M)$ and the kernel $\ker(\kappa^\Gamma)$ of the orbifold Kirwan map.

We begin with some comments on the computation of the domain $NK_T^\Gamma(M)$. In a large class of examples, the T-equivariant K-theory of the original Hamiltonian T-space M is well-known to be explicitly computable. For example, following the Delzant construction of symplectic toric orbifolds, the original Hamiltonian space M is just $\mathbb{C}^N$, an affine space equipped with a linear T-action, so in particular is T-equivariantly contractible. In this case, then, the domain $NK_T^\Gamma(M = \mathbb{C}^N)$ is simply a direct sum of $|\Gamma|$ copies of $K_T(\mathrm{pt}) \cong R(T)$, as an $R(T)$-module.

Another important class of examples are **GKM spaces**. Suppose that the original Hamiltonian T-space (M,ω,Φ) is **GKM** in the sense of [17] or [19]. Each fixed point set M^t is then also GKM. A specific class of examples are the homogeneous spaces G/T of compact connected Lie groups G with maximal torus T, considered as a Hamiltonian T-space with respect to the natural left action of the maximal torus T (see e.g. [14, Lemma 8.2]). In this situation, the results of Section 2.3 imply that the natural restriction

$$i_{NK}^* : (NK_T^\diamond(M), \odot) \longrightarrow (K_T(M^T) \otimes \mathbb{Z}[T], \star)$$

is injective. Furthermore, the image of this injection can be explicitly and combinatorially described using GKM ("Goresky-Kottwitz-MacPherson") theory. Indeed, GKM-type techniques in equivariant cohomology were already used in [14, Section 8] in order to give explicit computations associated to flag manifolds in inertial cohomology; the K-theoretic methods using GKM theory in K-theory (as in [17, 19], and references therein) are analogous.

Now we turn to the computation of the kernel of the orbifold Kirwan map κ. First observe (as in the proof of Theorem 3.4) that for each $t \in \Gamma$, M^t is itself a Hamiltonian T-space, with moment map the restriction of Φ to M^t. Thus, for each $t \in \Gamma$, the map induced by inclusion

$$\kappa_t := \iota^* : K_T(M^t) \longrightarrow K_T(\Phi^{-1}(\alpha)^t)$$

is precisely the ordinary (non-orbifold) Kirwan map for the Hamiltonian T-space $(M^t, \omega|_{M^t}, \Phi|_{M^t})$. The kernel of the ordinary Kirwan map for an abelian symplectic quotient has been explicitly described in [19] following previous work in cohomology of Tolman and Weitsman [29]. More specifically, [19, Theorem 3.1] gives a list of ideals in $K_T(M^t)$ which generates $\ker(\kappa_t)$; for reference, we include the statement below. We fix once and for all a choice of inner product on $\mathfrak{t}^*$, with respect to which we define the norm-square $\|\Phi\|^2 : M \to \mathbb{R}$ and identify $\mathfrak{t} \cong \mathfrak{t}^*$.

Theorem 3.8 *([19, Theorem 3.1]) Let (M, ω, Φ) be a Hamiltonian T-space. Suppose there exists a component of Φ which is proper and bounded below, and further suppose that M^T has only finitely many connected components. Let*

$$Z := \{\Phi(C) \mid C \text{ a connected component of } \mathrm{Crit}(\|\Phi\|^2) \subseteq M\} \subseteq \mathfrak{t}^* \cong \mathfrak{t} \qquad (3.7)$$

be the set of images under Φ of components of the critical set of $\|\Phi\|^2$. For $\xi \in \mathfrak{t}$, define

$$\begin{aligned} M_\xi &:= \{x \in M \mid \langle \mu(x), \xi \rangle \leq 0\}, \\ \mathcal{K}_\xi &:= \{\alpha \in K_T^*(M) \mid \alpha|_{M_\xi} = 0\}, \quad \text{and} \\ \mathcal{K} &:= \sum_{\xi \in Z \subseteq \mathfrak{t}} \mathcal{K}_\xi. \end{aligned}$$

Then there is a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow K_T^*(M) \xrightarrow{\kappa} K_T^*(\Phi^{-1}(0)) \longrightarrow 0,$$

where $\kappa : K_T^*(M) \to K_T^*(\Phi^{-1}(0))$ is the Kirwan map.

Remark 3.9 Although the statement of [19, Theorem 3.1] explicitly refers to the symplectic quotient $M//_\alpha T$, it is straightforward to see that the theorem in fact holds at the level of the T-equivariant K-theory of the level set $\Phi^{-1}(0)$. Moreover, although [19, Theorem 3.1] is (for convenience) stated only for the 0-level set $\Phi^{-1}(0)$, since T-moment maps are determined only up to an additive constant, it is straightforward to see that the analogous statement also holds for the case of non-zero regular values $\alpha \in \mathfrak{t}^*$ and its corresponding level set $\Phi^{-1}(\alpha)$.

Using Theorem 3.8 and localization techniques as described in [19, Section 2], an explicit list of generators for the kernel of κ_t may be constructed for each $t \in \Gamma$. Since

$$\ker(\kappa^\Gamma) = \bigoplus_{t \in \Gamma} \ker(\kappa_t),$$

the kernel of the restricted orbifold Kirwan map κ^Γ is therefore obtained by computing each $\ker(\kappa_t)$ separately. This completes the explicit description of

$\mathsf{K}_{\mathrm{orb}}([\Phi^{-1}(\alpha)/T])$ in terms of generators and relations for a wide class of abelian symplectic quotients.

4. Example: the full orbifold K-theory of weighted projective spaces

We now give a complete description of the ring structure of the full orbifold K-theory of weighted projective spaces obtained as symplectic quotients of $\mathbb{C}^n$ by S^1, closing with the worked example of $\mathbb{P}(1,2,4)$. Similar and related results, in ordinary (i.e. non-orbifold) cohomology and ordinary K-theory, are contained in [2, 3, 4, 24, 27]. We do not require any mention of a stacky fan as in [8] or of labelled polytopes as in [26]. The technique here is different from that in [6] because we use the fact that these spaces occur as symplectic quotients. We also avoid using the Chern character isomorphism, allowing us to obtain results over $\mathbb{Z}$. In particular, we prove in Proposition 4.1 that $\mathsf{K}_{\mathrm{orb}}([M])$ has no additive torsion for those weighed projective spaces $[M]$ obtained as symplectic quotients by a (connected) circle.

Recall that such a weighted projective space is specified by an integer vector in $\mathbb{Z}^{n+1}_{>0}$:

$$b = (b_0, b_1, b_2, \cdots, b_n), \quad b_k \in \mathbb{Z}, b_k > 0.$$

The vector b determines an action of S^1 on $\mathbb{C}^{n+1}$, defined by

$$t \cdot (z_0, z_1, \ldots, z_n) := (t^{b_0} z_0, t^{b_1} z_1, \ldots, t^{b_n} z_n),$$

for $t \in S^1$ and $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$. An S^1-moment map for this action is given by

$$\Phi(z_0, z_1, \ldots, z_n) = -\frac{1}{2} \sum_{k=0}^{n} b_k \|z_k\|^2.$$

This is clearly proper and its negative bounded below. Moreover, since the b_k are positive, the only S^1-fixed point in $\mathbb{C}^{n+1}$ is the origin $\{0\} \in \mathbb{C}^{n+1}$; in particular, $(\mathbb{C}^{n+1})^{S^1}$ has only finitely many connected components. Any negative moment map value is also regular, so we may define the **weighted projective space** $\mathbb{P}^n_b$ (sometimes also denoted $\mathbb{P}^n(b)$) as the orbifold arising as a symplectic quotient

$$\mathbb{P}^n_b := \mathbb{C}^{n+1} /\!/_\alpha S^1 = [\Phi^{-1}(\alpha)/S^1]$$

for a regular (negative) value α. The differential structure does not change as α varies, though the symplectic volume does. We note that $\mathbb{P}^n_b$ may be a non-effective orbifold: if the integers b_i are not relatively prime, i.e. $g = \gcd(b_0, b_1, \ldots, b_n) \neq 1$, then there is a global stabilizer isomorphic to the cyclic group $\mathbb{Z}_g$. Effective or not, however, all the hypotheses of the theorems in Section 3 are satisfied for these weighted projective spaces.

We begin by computing the finite stabilizer subgroup $\Gamma \subseteq S^1$ for $\Phi^{-1}(\alpha)$. Throughout, we will denote by $\mathbb{Z}_\ell$ the cyclic subgroup in S^1 given by the ℓ-th roots of unity $\{e^{2\pi i s/\ell} \mid s = 0,\ldots,\ell-1\}$ in S^1. Given a non-zero vector $z = (z_0, z_1, \ldots, z_n)$ in $\mathbb{C}^{n+1}$, the stabilizer subgroup of z in S^1 is precisely

$$\Gamma_z := \bigcap_{z_k \neq 0} \mathbb{Z}_{b_k} \subseteq S^1.$$

In particular, this implies that Γ is generated by the subgroups $\mathbb{Z}_{b_k}$ for each k, $0 \leq k \leq n$, so $\Gamma = \mathbb{Z}_\ell \subseteq S^1$ where

$$\ell = \operatorname{lcm}(b_0, b_1, \ldots, b_n).$$

Let $\zeta_s := e^{2\pi i s/\ell} \in S^1$. Then, by definition, the Γ-subring of the inertial K-theory of the S^1-space $\mathbb{C}^{n+1}$ is additively defined to be

$$NK^{\Gamma}_{S^1}(\mathbb{C}^{n+1}) := \bigoplus_{s=0}^{\ell} K_{S^1}((\mathbb{C}^{n+1})^{\zeta_s}) \cong \bigoplus_{s=0}^{\ell} K_{S^1}(\mathrm{pt}) = \bigoplus_{s=0}^{\ell} R(S^1) = \bigoplus_{s=0}^{\ell} \mathbb{Z}[u, u^{-1}],$$

where $R(S^1)$ denotes the representation ring of S^1 and we use u as the variable in $R(S^1)$. For the first isomorphism above, we use the fact that the S^1-action on $\mathbb{C}^{n+1}$ is linear, so any fixed-point set for any group element is an S^1-invariant affine subspace of $\mathbb{C}^{n+1}$, hence S^1-equivariantly contractible to a point.

We denote by α_s the element in $NK^{\Gamma}_{S^1}(\mathbb{C}^{n+1})$ which is the identity $1 \in K_{S^1}(\mathrm{pt}) \cong R(S^1)$ in the summand corresponding to ζ_s and is 0 elsewhere. These are clearly additive K_{S^1}-module generators of $NK^{\Gamma}_{S^1}(\mathbb{C}^{n+1})$. Hence, in order to determine the ring structure of the Γ-subring of the inertial K-theory, it suffices to calculate the products $\alpha_s \odot \alpha_t$, for $0 \leq s, t < \ell$, in inertial K-theory. Since $\mathbb{C}^{n+1}$ with the given S^1-action is a Hamiltonian S^1-space with a component of the moment map which is proper and bounded below (and the fixed point set has finitely many connected components), from Section 2 we know that the map ι^*_{NK} is injective (and in fact, in this case, is an isomorphism). Hence we may use for our computations the $\star$ product on inertial K-theory as defined in Section 2, instead of the $\odot$ product.

For any integer $s \in \mathbb{Z}$, let $[s]$ denote the smallest non-negative integer congruent to s modulo ℓ. Also let $\langle q \rangle := q - \lfloor q \rfloor$ denote the fractional part of any rational number $q \in \mathbb{Q}$. In our case, the logweight of an element $\zeta_s \in \Gamma$ acting on the k-th coordinate may be explicitly computed to be

$$a_k(\zeta_s) = \frac{[b_k s]}{\ell} = \left\langle \frac{b_k s}{\ell} \right\rangle.$$

Hence ζ_s acts on the k-th coordinate as $e^{2\pi i a_k(\zeta_s)}$. By the formula for the $\star$ product in Definition 2.6, we immediately obtain the relation

$$\alpha_s \star \alpha_{s'} = \alpha_{[s+s']} \left(\prod_{k=0}^{n} \left(1-u^{-b_k}\right)^{a_k(\zeta_s)+a_k(\zeta_{s'})-a_k(\zeta_s\zeta_{s'})} \right) \tag{4.1}$$

among the generators of the twisted sectors, where we have used that the S^1-equivariant K-theoretic Euler class of the S^1-equivariant bundle $\mathbb{C}_\lambda \to \mathrm{pt}$ of S^1-weight $\lambda \in \mathbb{Z}$ is

$$\epsilon_{S^1}(\mathbb{C}_\lambda) = 1-u^{-\lambda} \in R(S^1) = \mathbb{Z}[u,u^{-1}].$$

Hence the Γ-subring may be described as

$$NK_{S^1}^{\Gamma}(\mathbb{C}^{n+1}) \cong \mathbb{Z}[u,u^{-1}][\alpha_0,\ldots,\alpha_\ell] \Big/ \mathcal{I}, \tag{4.2}$$

where $\mathcal{I}$ is the ideal generated by the relations (4.1) for all $0 \le s,s' \le \ell-1$, i.e.

$$\mathcal{I} := \left\langle \alpha_s\alpha_{s'} - \left(\prod_{k=0}^{n} \left(1-u^{-b_k}\right)^{a_k(\zeta_s)+a_k(\zeta_{s'})-a_k(\zeta_s\zeta_{s'})} \right) \alpha_{[s+s']} \,\middle|\, 0 \le s,s' \le \ell-1 \right\rangle. \tag{4.3}$$

In order to obtain the orbifold K-theory of the symplectic quotient $\mathbb{P}_b^n$, we must now compute the kernel of the K-theory Kirwan map for each sector, i.e.

$$\kappa_s : K_{S^1}((\mathbb{C}^{n+1})^{\zeta_s}) \to K_{S^1}((\Phi^{-1}(-1))^{\zeta_s})$$

for each $0 \le s \le \ell-1$. (Here we have chosen to reduce at the regular value -1.) As mentioned above, since each $(\mathbb{C}^{n+1})^{\zeta_s}$ is itself a Hamiltonian S^1-space and its symplectic quotient by this S^1 is again a toric variety, we may apply [19, Theorem 3.1] to obtain

$$\ker(\kappa_s) = \left\langle \prod_{k:a_k(\zeta_s)=0,0\le k\le n} (1-u^{-b_k}) \right\rangle. \tag{4.4}$$

Here we use the fact that the k-th coordinate line in $\mathbb{C}^{n+1}$ is fixed by ζ_s if and only if $a_k(\zeta_s) = 0$, and that in this case (where we have just an S^1-action, not a larger-dimensional torus) the negative normal bundle with respect to Φ is in fact all of the tangent space to the fixed point $\{0\}$ in $(\mathbb{C}^{n+1})^{\zeta_s}$.

Hence we conclude that the full orbifold K-theory of the weighted projective space $\mathbb{P}_b^n$ is given by

$$\mathsf{K}_{\mathrm{orb}}(\mathbb{P}_b^n) = \mathbb{Z}[u,u^{-1},\alpha_0,\ldots,\alpha_\ell] \Big/ \mathcal{I}+\mathcal{J},$$

where $\mathcal{I}$ is the ideal in (4.3) and

$$\mathcal{J} = \left\langle \prod_{k:a_k(\zeta_s)=0, 0\leq k\leq n} (1-u^{-b_k}) \middle| 0 \leq s \leq \ell - 1 \right\rangle.$$

Simple algebra then shows that the full orbifold K-theory of a weighted projective spaces is torsion-free.

Proposition 4.1 *The full orbifold K-theory ring of a weighted projective space $\mathbb{P}^n_b$ obtained as a symplectic quotient does not contain (additive $\mathbb{Z}$) torsion.*

Proof: It is sufficient to check that each summand

$$K^*_{S^1}((\Phi^{-1}(1))^{\zeta_s})$$

is torsion-free over $\mathbb{Z}$. This piece is precisely the ring

$$A = \frac{\mathbb{Z}[u,u^{-1}]}{\left\langle \prod_{k:a_k(\zeta_s)=0, 0\leq k\leq n} (1-u^{-b_k}) \right\rangle}. \tag{4.5}$$

Now suppose that $f \in A$ is a torsion class: that is, there is an integer $m \geq 2$ satisfying $m \cdot f = 0$ in A. Let $F \in \mathbb{Z}[u,u^{-1}]$ be a representative of f. Then mF must be in the ideal in the denominator of (4.5). But $\mathbb{Z}[u,u^{-1}]$ is a unique factorization domain, and m is not a unit in $\mathbb{Z}[u,u^{-1}]$. Thus, since

$$mF = \tau \cdot \prod (1-u^{-b_k}),$$

unique factorization implies that τ is multiple of m. Thus we deduce that F itself is in the ideal, and hence $f = 0$ in A. □

4.1. A worked example: $\mathbb{P}^2_{(1,2,4)}$

We now illustrate the computations above using the specific weighted projective space $\mathbb{P}^2_{(1,2,4)}$, the orbifold that is the symplectic quotient of $\mathbb{C}^3$ equipped with the S^1-action

$$t \cdot (z_0, z_1, z_2) := (tz_0, t^2 z_1, t^4 z_2).$$

for $t \in S^1, (z_0,z_1,z_2) \in \mathbb{C}^3$. We will denote the weight spaces of this $\mathbb{C}^3$ by $\mathbb{C}_{(1)}, \mathbb{C}_{(2)}, \mathbb{C}_{(4)}$ respectively. In this case $\ell = \text{lcm}(1,2,4) = 4$ so $\Gamma \cong \mathbb{Z}_4$, generated by $e^{i2\pi/4} = i \in S^1$. The following chart contains the necessary information for computing the inertial K-theory of $\mathbb{C}^3$.

s	0	1	2	3
ζ_s	1	i	-1	$-i$
$(\mathbb{C}^3)^{\zeta_s}$	$\mathbb{C}^3$	$\mathbb{C}_{(4)}$	$\mathbb{C}_{(2)} \oplus \mathbb{C}_{(4)}$	$\mathbb{C}_{(4)}$
$a_1(\zeta_s)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$a_2(\zeta_s)$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$a_3(\zeta_s)$	0	0	0	0
generator of ζ_s sector	α_0	α_1	α_2	α_3

(4.6)

Using the formula (4.1) given above, we immediately conclude that the product structure in the inertial K-theory of $\mathbb{C}^3$ is given by the following multiplication table. Recall that α_0, being the generator of the untwisted (identity) sector, is the multiplicative identity in the ring, so we need not include it in this table.

	α_1	α_2	α_3
α_1	$(1-u^{-2})\alpha_2$	α_3	$(1-u^{-1})(1-u^{-2})\alpha_0$
α_2		$(1-u^{-1})\alpha_0$	$(1-u^{-1})\alpha_1$
α_3			$(1-u^{-1})(1-u^{-2})\alpha_2$

(4.7)

Let $\mathcal{I}$ be the ideal generated by the product relations in (4.7). Then we have

$$NK_{S^1}^{\Gamma}(\mathbb{C}^3) \cong \mathbb{Z}[u,u^{-1}][\alpha_0,\alpha_1,\alpha_2,\alpha_3] \Big/ \mathcal{I} + \langle \alpha_0 - 1 \rangle.$$

We may also explicitly compute the kernels of the K-theory Kirwan maps κ_s for $0 \leq s \leq 3$, according to (4.4). We have

$$\begin{aligned} \ker(\kappa_0) &= \langle \alpha_0(1-u^{-1})(1-u^{-2})(1-u^{-4}) \rangle, \\ \ker(\kappa_1) &= \langle \alpha_1(1-u^{-4}) \rangle, \\ \ker(\kappa_2) &= \langle \alpha_2(1-u^{-2})(1-u^{-4}) \rangle, \\ \ker(\kappa_3) &= \langle \alpha_3(1-u^{-4}) \rangle. \end{aligned}$$

Let $\mathcal{J}$ be the ideal generated by $\ker(\kappa_s)$ for all s, $0 \leq s \leq 3$. We conclude that

$$\mathsf{K}_{\mathrm{orb}}(\mathbb{P}^2_{1,2,4}) \cong \mathbb{Z}[u,u^{-1}][\alpha_0,\alpha_1,\alpha_2,\alpha_3] \Big/ \mathcal{I} + \langle \alpha_0 - 1 \rangle + \mathcal{J}. \tag{4.8}$$

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