

SOLUTIONS

Midterm Exam, Math 739

March 18, 2010

Instructions: You may use your textbook and your notes for this exam. You have 1 hour and 15 minutes to take the exam. You must prove all your answers; yes/no questions will be marked wrong if no justification or proof is given. You may attach additional sheets of paper (or use the backs of these pages) if needed. All questions have equal weight.

1. Let $M = \mathbb{R}^3 - 0$ be 3-space minus the origin, with the smooth structure endowed by \mathbb{R}^3 . Let $F: M \rightarrow M$ be given by the smooth map

$$F(x_1, x_2, x_3) = (x_1^2 - x_3^3, x_1x_2, x_1^2 + x_2^2 + x_3^2).$$

- (a) Calculate its total derivative F_* at the point $(1, 2, 3)$ using the coordinates (x_1, x_2, x_3) .
(b) What is $F_* \left(\frac{\partial}{\partial x_1} \right)$?
(c) Is $F_*: T_{(1,2,3)}M \rightarrow T_{F(1,2,3)}M$ an isomorphism?
(d) Is F a diffeomorphism?

(a) F_* at $(1, 2, 3)$ is given by

$$\left(\frac{\partial F^i}{\partial x^j} \right) \Big|_{(1,2,3)} = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} & \frac{\partial F^1}{\partial x^3} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^3} \\ \frac{\partial F^3}{\partial x^1} & \frac{\partial F^3}{\partial x^2} & \frac{\partial F^3}{\partial x^3} \end{pmatrix} \Big|_{(1,2,3)}$$

where $F^1 = x_1^2 - x_3^3$

$F^2 = x_1x_2$

$F^3 = x_1^2 + x_2^2 + x_3^2$

We obtain:

$$F_* = \begin{pmatrix} 2x_1 & 0 & -3x_3^2 \\ x_2 & x_1 & 0 \\ 2x_1 & 2x_2 & 2x_3 \end{pmatrix} \Big|_{(1,2,3)} \\ = \begin{pmatrix} 2 & 0 & -27 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{pmatrix}$$

$$(b) F_* \left(\frac{\partial}{\partial x^1} \right) = \begin{pmatrix} 2 & 0 & -27 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$= 2 \frac{\partial}{\partial y^1} + 2 \frac{\partial}{\partial y^2} + 2 \frac{\partial}{\partial y^3}$$

(c) We simply check if the 3×3 matrix is invertible:

$$\det \begin{pmatrix} 2 & 0 & -27 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{pmatrix} \neq 0$$

\Rightarrow YES, F_* is an isomorphism.

(d) F is not a diffeomorphism. It is not surjective (the 3rd coord is always non-negative).

2. Let $M = S^2 \subset \mathbb{R}^3$ be the two-sphere of radius 1 embedded in \mathbb{R}^3 with its center at 0. Let $f : S^2 \rightarrow \mathbb{R}$ be the smooth function $f(x, y, z) = z$. Show there exists a smooth function $g : S^2 \rightarrow \mathbb{R}$ satisfying

(a) $g(x, y, z) = f(x, y, z)$ if $|z| < \frac{1}{4}$.

(b) $g(0, 0, 1) = g(0, 0, -1) = 0$.

Consider the compact set $A = \{ (x, y, z) \in S^2 \mid |z| \leq \frac{1}{4} \}$.

Let $U \subset S^2$ be $U := \{ (x, y, z) \in S^2 \mid |z| < \frac{1}{2} \}$

Clearly $A \subseteq U$ and $\{ (0, 0, 1), (0, 0, -1) \} \cap U = \emptyset$.

There exists a ^{smooth} bump function h ^{for A} supported on U (see Prop. 2.26). Note that

$$h|_A \equiv 1$$

$$h|_{S^2 \setminus U} \equiv 0.$$

Let $g := h \cdot f$. Note g is smooth as it is the product of two smooth fns.

Furthermore

$$g|_A \equiv f|_A \Rightarrow g \text{ satisfies (a) above.}$$

$$g|_{S^2 \setminus U} \equiv 0 \Rightarrow g \text{ satisfies (b) above.}$$

3. Prove that $M(n, \mathbb{R})$, the set of $n \times n$ real matrices, forms a Lie ^{algebra} ~~group~~ under the bracket operation

$$[A, B] = AB - BA, \quad \text{for all } A, B \in M(n, \mathbb{R}).$$

We need only check the Lie bracket properties hold:

(1) Bilinearity: Let $c_1, c_2 \in \mathbb{R}$. Then $\forall A, B, C \in M(n, \mathbb{R})$

$$\begin{aligned} [c_1 A + c_2 B, C] &= (c_1 A + c_2 B)C - C(c_1 A + c_2 B) \\ &= (c_1 A)C + (c_2 B)C - C(c_1 A) - C(c_2 B) \\ &= (c_1 A)C - C(c_1 A) + (c_2 B)C - C(c_2 B) \\ &= c_1 (AC) - c_1 CA + c_2 BC - c_2 CB \end{aligned}$$

(since mult. by elts in \mathbb{R} commutes w/ matrix multiplication).

$$= c_1 [A, C] + c_2 [B, C] \text{ by distribution.}$$

Similarly,

$$[A, c_1 B + c_2 C] = c_1 [A, B] + c_2 [A, C].$$

(2) Antisymmetry: $[A, B] = AB - BA$
 $= -(BA - AB)$
 $= -[B, A].$

(3) Jacobi Identity:

$$\begin{aligned} [A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\ &= ABC - ACB - (BCA - CBA) + BCA - BAC - (CAB - ACB) \\ &\quad + CAB - CBA - (ABC - BAC) \\ &= 0 \end{aligned}$$

3
(by checking terms).

4. Consider the circle $S^1 \subset \mathbb{R}^2$ as the set of all points $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Show directly that the vector field

$$V = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

on S^1 is left invariant. *Note:* You must do a direct calculation for credit. You may not use the isomorphism between left-invariant vector fields and the tangent space at the identity.

We show left invariant by first calculating what the left S^1 action does as a map on S^1

$$f_\theta : S^1 \rightarrow S^1$$

$$(x, y) \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta x - \sin \theta y \\ \sin \theta x + \cos \theta y \end{pmatrix}$$

Note: The point $(1, 0)$ is moved to $(\cos \theta, \sin \theta)$ under f_θ .

Then $(f_\theta)_*$ at the point (x, y) is given by:

$$\text{Note: } f'_\theta = (f_\theta^1, f_\theta^2) = (\cos \theta x - \sin \theta y, \sin \theta x + \cos \theta y)$$

(as a fn of x, y , not θ).

$$(f_\theta)_* = \begin{pmatrix} \frac{\partial f_\theta^1}{\partial x} & \frac{\partial f_\theta^1}{\partial y} \\ \frac{\partial f_\theta^2}{\partial x} & \frac{\partial f_\theta^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (= f_\theta !!)$$

This is because it's linear

In other words

$$\Rightarrow (f_\theta)_* \left(\frac{\partial}{\partial x} \right) = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}$$

$$(f_\theta)_* \left(\frac{\partial}{\partial y} \right) = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$$

WLOG, we show that for any θ , the vector $V|_{(1,0)} = \frac{\partial}{\partial y}$ maps to

$$V|_{(\cos \theta, \sin \theta)} = \cos \theta \frac{\partial}{\partial y} - \sin \theta \frac{\partial}{\partial x}$$

$$\text{This follows immediately: } f_* (V|_{(1,0)}) = f_* \left(\frac{\partial}{\partial y} \right) = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$$

$$= V|_{f_\theta(1,0)} = V|_{(\cos \theta, \sin \theta)}$$