Math 724: Problem Set 6

Due on 11.9.06

1. Show that $G = \{f_1 = x^3 - 2xy, f_2 = x^2y - 2y^2 + x\}$ is not a Gröbner basis for $I = \langle f_1, f_2 \rangle \subset k[x, y]$ in the grlex ordering.

2. Show that $G = \{y^3 - z^2, x^2 - y, xy - z, xz - y^2\}$ is a Gröbner basis for $I = \langle y - x^2, z - x^3 \rangle \subset k[x, y, z]$ in the grlex ordering..

3. Prove that
$$|Mon\ k[x_1, \ldots, x_n]_{\leq m}| = \binom{m+n}{n}.$$

Solution. We do this by going through a double induction. We start by inducting on m. For $m = 1$, it is clear that
$$|Mon\ kk[x_1, \ldots, x_n]_{\leq 1}| = n + 1 = \binom{1+n}{n},$$
since the only monomials of degree ≤ 1 are $1, x_i, i = 1, \ldots, n$.

Assume true for $m - 1$ (and for all n). Then
$$|Mon\ k[x_1, \ldots, x_n]_{\leq m}| = |Mon\ k[x_1, \ldots, x_n]_{\leq m-1}| + |Mon\ k[x_1, \ldots, x_n]_{=m}| \tag{1}$$
$$= \binom{m-1+n}{n} + ?. \tag{2}$$

We do an induction on n to find the value of $|Mon\ k[x_1, \ldots, x_n]_{=m}|$.

Claim.
$$|Mon\ k[x_1, \ldots, x_n]_{=m}| = \binom{m+(n-1)}{m}.$$

(By the way, where did I get this idea? I just subtracted $\binom{m+n}{n} - \binom{m-1+n}{n}$ since this is what will complete my original inductive claim.)

For $n = 1$, the claim clearly holds, since x_1^m is the only monomial of degree m, and $\binom{m+(n-1)}{m} = 1$.

Assume true for $n - 1$ variables. Then
$$|Mon\ k[x_1, \ldots, x_{n-1}]_{=m}| = \binom{m+(n-2)}{m}.$$

Then we split the monomials into those that have no factor of x_n, and those that do have a factor of x_n. Clearly those that do not have a factor of x_n are in $Mon\ k[x_1, \ldots, x_{n-1}]_{=m}$. Those that do have a factor of x_n are elements of $x_n^k \cdot Mon\ k[x_1, \ldots, x_{n-1}]_{=m-k}$, as k ranges from 1 to m. To count this latter set, we need only count $|Mon\ k[x_1, \ldots, x_{n-1}]_{\leq m-1}|$. Therefore,

$$\begin{aligned}
|Mon\ k[x_1, \ldots, x_n]_{=m}| &= |Mon\ k[x_1, \ldots, x_{n-1}]_{=m}| + |Mon\ k[x_1, \ldots, x_{n-1}]_{\leq m-1}| \\
&= \binom{m+(n-2)}{m} + \binom{(m-1)+(n-1)}{n-1} \\
&= \frac{(m+n-2)!}{m!(n-1)!} + \frac{(m+n-2)!}{(m-1)!(n-1)!} \\
&= \frac{(n-1)(m+n-2)! + m(m+n-2)!}{m!(n-1)!} \\
&= \frac{n-1+m)(m+n-2)!}{m!(n-1)!} \\
&= \binom{m+(n-1)}{m},
\end{aligned}$$

as desired.

Finally, we use this claim to plug into the formula for the first induction (line (2)), and calculate that

$$\binom{m-1+n}{n} + \binom{m+(n-1)}{m} = \binom{m+n}{n},$$

proving the original formula.

4. Calculate the Hilbert polynomial for $I = \langle x^3 - yz^2, y^4 - x^2yz \rangle \subset k[x, y, z]$.

Solution.

The first thing is to find a Gröbner basis. Let $f_1 = x^3 - yz^2$ and $f_2 = y^4 - x^2yz$. We note that $\{f_1, f_2\}$ does not form a Gröbner basis of I, since

$$xzf_1 + xf_2 = -y^2z^3 + xy^4 =: f_3$$

and $md(f_3) = (1, 4, 0)$. We note immediately that the leading monomial of f_3, which is xy^4, cannot be in the ideal $\langle x^3, x^2yz \rangle$, which is the ideal generated by the leading monomials of f_1 and f_2 (this latter ideal's elements all have multiples of x^2 in them, and xy^4 does not). This proves that, while $f_3 \in I$, the leading monomials of I do not generate the same ideal as the leading monomials of $\{f_1, f_2\}$, so it can't be a Gröbner basis.

We now add f_3 to the set and continue the process. Is $\{f_1, f_2, f_3\}$ a Gröbner basis of I? It is not, because $y^3f_2 + xzf_3 = y^7 - xy^2z^4 =: f_4$. We claim that $LT(f_4) = xy^2z^4$ is not in the ideal $\langle LT(f_1), LT(f_2), LT(f_3) \rangle$. We can't do this by a simple play on the factors of x^2 as we did above, however we can note that $md(f_4) = (1, 2, 4) \geq md(y^3f_2) = (2, 4, 1)$ and $md(f_4) = (1, 2, 4) \geq md(xzf_3) = (2, 4, 1)$. So, unless there is another way of writing f_4 as a combination of f_1, f_2, f_3, we are done. It is relatively easy to see that f_4 cannot be obtained as another combination of these $f_i, i = 1, 2, 3$.

So we add f_4 to the set and continue. Is $\{f_1, f_2, f_3, f_4\}$ a Gröbner basis of I? It is not, because $y^2 f_4 + z^4 f_3 = y^9 - y^2 z^7 =: f_5$ is in I, and the leading term of f_5 is y^9. Clearly $LT(f_5) \notin \langle LT(f_1), LT(f_2), LT(f_3), LT(f_4)\rangle$, because the other leading terms all have a factor of x, but y^9 does not.

We add f_5 to the set, and after a lot of tedious calculation which I won't write down, you can show that $\{f_1, \dots f_5\}$ is a Gröbner basis for I, because the ideal generated by $LT(I)$ is equal to the ideal generated by the leading terms of $f_1, \dots, f_5$.

Now we are ready to calculate the affine Hilbert polynomial, using this Gröbner basis. Note that we can find right away that the polynomial should be of degree 1. This is because the affine Hilbert polynomial measures the dimension of the affine variety. This is the same as the dimension of variety determined by the leading monomials. We see that $\dim(V(LT(I)) = \dim V(x^3, x^2yz, xy^4, xy^2z^4, y^9) = 1$, since this occurs as the intersection of $x = 0$ and $y = 0$ in $\mathbb{A}^3$.

We can see immediately that this Hilbert polynomial will be of degree 1, since the degree is determined by the terms

$$|[t]\ _m| = \binom{m - |\alpha| + |top_M(t)|}{|top_M(t)|},$$

where $|\alpha|$ is the degree of t. We note that the degree of m in this expression is determined by $|top_M(t)|$. This is the number of variables that can get up to M, where M is the maximum of the degrees occurring in the leading terms of our Gröbner basis. In our case, $M = 9$, since y^9 is in the set of leading terms (and no terms with higher exponents are). But since x^3 is also in the leading terms set, in the monomial $x^i y^j z^k \in Mon\ k[x, y, z] - \langle LT(I)\rangle$, only z can have an exponent higher than 9. This implies that $|top_M(t)| = 1$ (exponent of z equal to 9 or higher) or 0 (none 9 or higher), but never 2.

To actually calculate the Hilbert polynomial, we need to figure out the size of $Mon\ k[x, y, z] - \langle LT(I)\rangle$. These are the monomials not divisible by any of $x^3, x^2yz, xy^4, xy^2z^4, andy^9$. We obtain the monomials:

$$\begin{aligned}
&t = x^2, x^2z, x^2z^2, \dots, x^2z^k, x^2y, x^2y^3, x^2y^3, \text{where degree of } x \text{ is } 2\\
&\ x, xz, xz^2, xz^3, \dots, xz^k, xy, xy^2, xy^3, xyz, xyz^2, \dots, xyz^k, xy^2z, xy^2z^2, xy^2z^3, \quad \deg x = 1\\
&\ 1, z, z^2, \dots, z^k, y, y^2, \dots, y^8, yz, yz^2, \dots, yz^k, \dots, y^8z^k, \quad \deg x = 0.
\end{aligned}$$

Now we need to look at R_9, which is the set of these no exponent exceeds 9. We find that $top_9(t) = 1$ when $t = x^2z^9, xz^9, xyz^9, z^9, yz^9, y^2z^9, y^3z^9, y^4z^9, y^5z^9, y^6z^9, y^7z^9, y^8z^9$, and otherwise $|top_9(t)| = 0$. Thus we find for all $m \geq 3 \cdot 9 = 27$, that

$$\begin{aligned}
h(m) &= |Mon\ k[x, y, z]_{\leq m} - \langle LT(I)\rangle|\\
&= \sum_{t \in R_9, |top_9(t)|=1} \binom{m - deg(t) + 1}{1} + \sum_{t \in R_9,\ |top_9(t)|=0} 1\\
&= (m - 10) + (m - 9) + (m - 10) + (m - 8) + m - 9 + m - 10 + m - 11 + m - 12\\
&+ m - 13 + m - 14 + m - 15 + m - 16 + C
\end{aligned}$$

where C is the number of t's whose exponents are strictly below 9 (I didn't bother to count them up). Simplify the expression and you're done.

5. The *Krull-dimension* of the ring R is the supremum of the lengths of chains of prime ideals in R. Let $I \subset k[x_1, x_2, x_3]$ be a monomial ideal. Describe the possible Krull dimensions of $k[x_1, x_2, x_3]/I$ for different I (prove your answer). Describe how this relates with the varieties that I represent.

   Solution. Let $I = (x_1)$. Then $V(I)$ is the 2-plane where $x_1 = 0$. The Krull dimension of $R = k[x_1, x_2, x_3]$ is 2, since

$$(0) \subset (x_2) \subset (x_2, x_3)$$

   is a chain of primes of length 2, but there is no longer chain. The same calculation can be done for $I = (x_i)$ for $i = 2, 3$.

   Let $I = (x_1, x_2)$. Then $V(I)$ is 1-dimensional, the intersection of two 2-planes, $x_1 = 0$ and $x_2 = 0$. In other words, it's the x_3-axis. A chain of primes of maximal length is

$$(0) \subset (x_3).$$

   Note that $(x_1, x_2, x_3) = (x_1, x_3) = (x_2, x_3) = (x_3)$ in R/I. The calculation is similar for $I = (x_1, x_3)$ or $I = (x_2, x_3)$.

   Let $I = (x_1 x_2)$. Then $V(I)$ is 2-dimensional, the union of the two planes where $x_1 = 0$ or $x_2 = 0$. The corresponding maximal chain is length 2:

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3).$$

   Note that (0) is not a prime ideal in R/I. Similarly for I generated by any one monomial that is the produce of any of the x_i's.

   If $I = (x_1, x_2 x_3)$, then $V(I)$ is one-dimensional, since we much have that $x_2 = 0$ or $x_3 = 0$ AND $x_1 = 0$. This is an intersection of two-planes, and gives us the x_3-axis, union the x_2-axis. The corresponding chain of primes is

$$(x_2) \subset (x_2, x_3),$$

   showing the Krull dimension is 1. Note that $(x_1) = (0)$ is not prime in R/I.

   Lastly consider ideals of the form $I = (x_1 x_2 . x_1 x_3)$. Then this is the intersection of the algebraic set given by $x_1 = 0$ or $x_2 = 0$, and that given by $x_1 = 0$ or $x_3 = 0$. This simplifies to $x_1 = 0$, OR $x_2 = x_3 = 0$. Thus we get the union of the algebraic set given by $x_1 = 0$ (a 2-plane), and the x_1-axis. It is 2-dimensional (taking the maximum of dimension of the pieces in the union). The corresponding maximal chain of prime ideals is

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3).$$

   Again, note that (0) is not prime.