Math 724: Solutions 4

1. Show that $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$ by displaying a map and proving the map is an isomorphism.

2. Show that, if $A \to B \to C \to 0$ is an exact sequence, then

$$0 \to \hom_R(C,N) \to \hom_R(B,N) \to \hom_R(A,N)$$

is also exact.

Let $\beta : B \to C$. But exactness of $A \to B \to C \to 0$, β is surjective. The induced map $\beta^* : \hom_R(C,N) \to \hom_R(B,N)$ is given by

$$\beta^*(\phi) = \phi \circ \beta$$

for every $\phi \in \hom_R(C,N)$. We want to show that β^* is injective. If $\beta^*(\phi) = 0$, then $\phi \circ \beta(b) = 0$ for every $b \in B$. Since β is surjective, for any $c \in C$, there exists $b \in B$ such that $\beta(b) = c$. Thus $\phi(c) = 0$ for every $c \in C$. In other words, ϕ is the 0-homomorphism in $\hom_R(C,N)$. This implies exactness at the first term, $\hom_R(C,N)$.

It remains to show exactness at $\hom_R(B,N)$. Let $\alpha : A \to B$ be the given map, and $\alpha^*(\psi) := \psi \circ \alpha$ for every $\psi \in \hom_R(B,N)$ be the induced map $\hom_R(B,N) \to \hom_R(A,N)$. We need to show that $\ker(\alpha^*) = im(\beta^*)$. If $\psi \in im(\beta^*)$, then $\psi = beta^*(\phi) = \phi\beta$, for some $\phi \in \hom_R(C,N)$. Then $\alpha^*(\psi) = \psi\alpha = \phi\beta\alpha = 0$ since $\beta\alpha = 0$ by the original exactness. Thus $im(\beta^*) \subseteq \ker(\alpha^*)$. Now suppose that $\psi \in \ker(\alpha^*)$. Then $\psi\alpha = 0$. In other words, ψ is 0 on the image of α. By exactness of the original sequence, $im(\alpha) = \ker(\beta)$. Thus ψ is 0 on the kernel of β. Thus we may define $\phi \in \hom_R(C,N)$ by $\phi(c) := \psi(b)$ for any $b \in B$ such that $\beta(b) = c$. Since β is surjective, such a b always exists. Since ψ is 0 on the kernel of β, $\psi(b)$ is independent of choice of b. For if b_1, b_2 both had the property that $\beta(b_1) = \beta(b_2) = c$, then $b_1 - b_2 \in \ker(\beta)$, which implies $\psi(b_1) - \psi(b_2) = \psi(b_1 - b_2) = 0$ since ψ is 0 on the kernel of β. It follows that $\psi(b_1) = \psi(b_2)$. Thus $\phi \in \hom_R(C,N)$ is well-defined. It is easy to check that phi also satisfies $\beta^*(\phi) = \psi$, since by construction $\beta^*(\phi) = \phi\beta = \psi$. Therefore, $\psi \in im(\beta^*)$, and $\ker(\alpha^*) \subseteq im(\beta^*)$. Thus $\ker(\alpha^*) = im(\beta^*)$ and we have shown exactness at $\hom_R(B,N)$.

3. Prove that $\hom_R(\oplus_i M_i, N) = \prod_i \hom_R(M_i, N)$ by exhibiting a map and proving its an isomorphism. We did part of this in class.

Solution. Recall that $\hom_R(\oplus_i M_i, N)$ consists of maps $\phi : \oplus_i M_i \to N$, while $\prod_i \hom_R(M_i,N)$ consists of sequences $(f_1, f_2, \dots)$ where $f_i : M_i \to N$.

For $\phi \in \hom_R(\oplus_i M_i, N)$, we define the map

$$f_i^\phi : M_i \to N \quad \text{by} \quad f_i^\phi(m) = \phi(0, \dots, 0, m, 0, \dots) \text{ for all } m \in M_i,$$

where m occurs in the i^{th} entry in $(0, \ldots, 0, m, 0, \ldots)$. It is easy to show that f_i is a homomorphism of modules if ϕ is.

Then we define

$$\Psi : \hom_R(\oplus_i M_i, N) \longrightarrow \prod_i \hom_R(M_i, N) \text{ by}$$
$$\phi \to (f_1^\phi, f_2^\phi, \ldots).$$

It is easy to show that the image lies in $\prod_i \hom_R(M_i, N)$ and that Ψ is a map of modules as well.

We first show that Ψ is injective. If $\Psi(\phi) = 0$, then $f_i^\phi = 0$ for all i. By construction, this means that $\phi(0, \cdots, 0, m, 0 \cdots) = 0$ for $m \in M_i$ and for all i. Then for any $(m_1, m_2, \ldots, m_k, 0, \ldots) \in \oplus_i M_i$ with $m_i \in M_i$, we have

$$\begin{aligned} \phi(m_1, m_2, \ldots, m_k, 0, \ldots) &= \phi(m_1, 0, \ldots) + \phi(0, m_2, 0, \ldots) + \cdots + \phi(0, \ldots, 0, m_k, 0, \ldots) \\ &= 0 + \cdots + 0 = 0 \end{aligned}$$

(note that we're using the fact that elements of $\oplus_i M_i$ only have finitely many nonzero elements). In other words, $\phi = 0$.

Now we show the map Ψ is surjective. Choose any $(f_1, \ldots) \in \prod_i \hom_R(M_i, N)$. We need to find $\phi \in \hom_R(\oplus_i M_i, N)$ so that $\Psi(\phi) = (f_1, \ldots)$. Equivalently, we show that $f_i = f_i^\phi$.

Define ϕ as follows:

$$\phi(m_1, \ldots, m_k, 0, \ldots) := f_1(m_2) + \cdots + f_k(m_k).$$

Since k is finite, the sum on the right is finite and ϕ is a well-defined map on $\oplus_i M_i$. It is clearly a homomorphism from $\oplus M_i$ to N. We need only show that $\Psi(\phi) = (f_1, \ldots)$. We know that $\Psi(\phi) = (f_1^\phi, \ldots)$ by construction. But for all i,

$$\begin{aligned} f_i^\phi(m_i) &= \phi(0, \ldots, 0, m_i, 0, \ldots), m_i \in M_i \\ &= f_1(0) + \cdots + f_{i-1}(0) + f_i(m_i) + f_{i+1}(0) \\ &= f_i(m_i). \end{aligned}$$

4. Exercise 2.9.

   Let M be an $R[U^{-1}]$-module. We want to show that U acts by automorphism. Let $\phi_u : M \to M$ given by $\phi_u : m \to um$. This is a map of $R[U^{-1}]$-modules, so we need only show it's an automorphism. But $um = 0$ implies $\frac{1}{u} \cdot (um) = m = 0$, showing that ϕ_u is injective. Let $\phi_{u^{-1}}$ be defined by

   $$\phi_{u^{-1}} : M \to M, \quad m \to \frac{1}{u} \cdot m.$$

   This is well-defined because M is an $R[U^{-1}]$-module. It is clear that $\phi_{u^{-1}} = \phi_u^{-1}$, proving that ϕ_u is an automorphism.

   On the other hand, let M be an R-module such that U acts on M by automorphisms, i.e. for every $u \in U$, we have

   $$\phi_u : M \to M \qquad m \to um$$

an automorphism. (We know that the map looks like this because the U action is induced from the module structure, or the R-action). Then there is an inverse map

$$\phi_u^{-1} : M \to M$$

such that $\phi_u^{-1}(um) = m$ for all $u \in U, m \in M$. Then since ϕ_u^{-1} is a map of R-modules, we have $\phi_u^{-1}(um) = u\phi_u^{-1}(m) = m$. Then M is an $R[U^{-1}]$ module by setting

$$\frac{r}{u} \cdot m := \phi^{-1}(rm) \in M.$$

It follows from the properties of ϕ^{-1} that this satisfies all the properties of an $R[U^{-1}]$-module.

5. Exercise 3.1 p. 109.

6. Exercise 3.4 p. 109. Let R be any ring, and set $S = R[x_1, \ldots, x_r]$. If $f \in S$, write Content(f) for the ideal of R generated by the coefficients of f.

   (a) For $f, g \in S$, show

   $$\text{Content}(fg) \subset \text{Content}(f)\text{Content}(g) \subset rad(\text{Content}(fg)).$$

   Show also that, if Content(f) has a nonzero divisor in R, then f is a nonzero divisor in S.

   Solution. The coefficients of fg are linear combinations of products of coefficients of f and coefficients of g. Since Content(f)Content(g) is generated by products of coefficients of f and those of g, the first inclusion follows immediately.

   For the second inclusion, we begin by explaining the hint in the back of the book. Suppose that, for every P containing Content(fg), we knew that P also contains Content(f)Content(g). Then recall that

   $$rad(\text{Content}(fg)) = \bigcap_{P \supset \text{Content}(fg), P \text{ prime}} P.$$

   (this is true whenever you take a radical, of any ideal). Then since every prime containing Content(fg) also contains Content(f)Content(g), so does the intersection. But the intersection is the radical, which shows that $rad(\text{Content}(fg)) \supset$ Content(f)Content(g), as desired.

   Now we show that, for every P containing Content(fg), P also contains Content(f)Content(g). Since P contains Content(fg), then Content(fg) is 0 in R/P. But since P is prime, R/P is a domain. In particular, R/P has no zero divisors. Then $Content(fg) = 0$ implies $fg = 0$, which implies $f = 0$ or $g = 0$ in the ring $R/P[x_1, \ldots, x_r]$ (since R/P is a domain). This in turn implies that Content$(f) = 0$ or Content$(g) = 0$ in R/P, which implies that Content(f)Content$(g) = 0$ in R/P, which is precisely the statement that Content(f)Content$(g) \subset P$.

   Lastly we show that if Content(f) contains a nonzero divisor, then f is itself a nonzero divisor in S. Suppose f were a zero-divisor. Then there exists $g \in S$, $g \neq 0$ such that $fg = 0$. Then Content$(fg) = 0$, and $rad(\text{Content}(fg)) = 0$. This implies that Content(f)Content$(g) = 0$, which implies that Content(f) consists only of 0-divisors, a contradiction.

(b) If R is Noetherian and f a nonzerodivisor of S, show that Content(f) contains a nonzerodivisor of R.

Solution. If Content(f) contains only zero-divisors, then it annihilates an element $a \in R$ by Corollary 3.2. Since $R \subset S$ as the set of constants, this means that Content(f) annihilates $a \in S$. This in turn implies that $af = 0$, or that f is a zero-divisor in S, a contradiction.