Math 724: Solution Set 2

1. Show that $k[x_1, \ldots, x_n]$ is

   (a) a Noetherian ring,

   (b) not a Noetherian module (a Noetherian module is one in which all submodules are finitely generated), and

   (c) a finitely generated k-algebra.

   Solution. We essentially did this in class. It is a Noetherian ring because k is, and we apply the Hilbert Basis theorem inductively. It is not a Noetherian module because it is not itself finitely generated as a module. Assume it were finitely generated. Then let l be the highest degree any term in any of the generating polynomials has. Since we are considering this as a module over k, there are no polynomials of degree higher than l generated by these. But since we can find a polynomial of arbitrarily high degree, we have a contradiction.

   Another way of saying this is that $k[x_1, \ldots, x_n]$ is an infinite dimensional vector space, so it cannot have a finite basis. (A basis for a vector space is a module basis!)

   It is a finitely generated k-algebra, because it is generated by $x_1, \ldots, x_n$.

2. Show that prime ideals are radical.

   Solution. Let $I \subset R$ be a prime ideal. Let $f^n \in I$ for some $f \in R$ and $n \in \mathbb{Z}^+$. Then I prime implies either $f \in I$ or $f^{n-1} \in I$, since $f^n = f(f^{n-1})$. If $f \in I$, then we're done. Otherwise, we continue inductively. Eventually we have $f^2 = f \cdot f \in I$, implying $f \in I$. Since this holds for all $f \in R$ with $f^n \in I$, I is radical.

3. Exercise 1.6, p. 48

   Solution.

   a Fix a monomial A. All monomials B such that $A > B$ have the same or smaller degree than A. Then since there are finitely many variables $x_1, \ldots, x_r$, we need only show that there are finitely many sequences $(m_1, \ldots, m_r)$ whose sum is smaller or equal to $\deg A$. Since $m_1, \ldots, m_r$ all satisfy $0 \leq m_i \leq \deg A$, we know that the number of choices of sequences is clearly less than $(\deg A + 1)^r$. This is a finite number. (Also, this is a bad bound on how many monomials are actually smaller in lexicographic order than A, but it's an easy proof.)

(a) Let p be invariant under Σ. Choose a monomial of highest degree in p, written $ax_1^{m_1}\cdots x_r^{m_r}$ with $a \in k$. Since p is invariant, p also has terms $ax_{\sigma(1)}^{m_1}\cdots x_{\sigma(r)}^{m_r}$ for any $\sigma \in \Sigma$. We reorder such a monomial by putting x_1 first, x_2 second, etc. If $\sigma(s) = 1$, then $s = \sigma^{-1}(1)$, which means that the exponent of x_1 is $m_{\sigma^{-1}(1)}$ These monomials equal $ax_1^{m_{\sigma^{-1}(1)}}\cdots x_r^{m_{\sigma^{-1}(1)}}$. Since we do this for any $\sigma \in \Sigma$, there is a term with exponents $n_1 \geq \cdots \geq n_r$. All of these terms are of highest degree, so we need only compare the exponents. If a term has any $n_i < n_j$ with $i < j$, then there is a permutation switching x_i and x_j so that the new exponents will have the relationship that $n_i' \geq n_j'$. In the lexicographic ordering, $x_i^{n_i}x_j^{n_j} < x_i^{n_i'}x_j^{n_j'}$, so the terms with exponents $n_1 \geq n_2 \cdots \geq n_r$ will be first in lexicographic ordering.

(b) This is just a calculation.

(c) Suppose that there exists some $(\mu_1, \ldots, \mu_r)$ such that $m_i = \sum_{j\geq i}^r \mu_j = 0$ for all i. Then it follows that $m_r = \mu_r = 0$, which implies $m_{r-1} = \mu_{r-1} + \mu_r = \mu_{r-1} = 0$. Inductively, we see that $\mu_i = 0$ for all i. This implies the map is an injection. By part (c), this implies that a monomial $x_1^{m_1}\ldots x_r^{m_r}$ is the leading term of a unique product of $f_1, \ldots, f_r$.

(d) Let $p \in S^\Sigma$. Then p can be written as a sum of homogeneous terms $p = \sum g_i$, where $g_i \in S^\Sigma$. Moreover, we choose g_i to be the sum of all the monomials in p with the given degree. Since g_i is homogeneous and invariant, it has leading term given by $a_1x_1^{m_1}\ldots x_r^{m_r}$ with $m_1 \geq \ldots m_r$. By part (d), there is a unique product of f_i's which equal this monomial, so that g_i has the same leading term as $a_1 f_1^{\mu_1}\ldots f_r^{\mu_r}$. Since both g_i and $a_1 f_1^{\mu_1}\ldots f_r^{\mu_r}$ are invariant, so is their difference. Since they are both homogeneous of degree $\deg g_i$, so is their difference. Thus

$$h_i := g_i - a_1 f_1^{\mu_1}\ldots f_r^{\mu_r}$$

is a homogeneous polynomial, of degree $\deg g_i$. Let $a_2x_1^{n_1}\ldots x_r^{n_r}$ be the initial term of h_i. Then $n_1 \leq m_1$ and if they are equal, then $n_2 \leq m_2$, etc. In other words, there exists some index j such that $n_i = m_i$ for $i < j$, and $n_j < m_j$. In other words, h_i is *strictly* lower in lexicographic ordering than the initial term of $f_1^{\mu_1}\ldots f_r^{\mu_r}$. By part (a), this inductive process has only finitely many steps, showing that eventually we find

$$h_l = g_i - \sum_j a_j f_1^{\mu_1^j}\cdots f_r^{\mu_r^j} = 0.$$

This implies that $g_i = \sum_j a_j f_1^{\mu_1^j}\cdots f_r^{\mu_r^j}$ is a linear combination of products of f_i's, and then in turn that p is also a linear combination of the elementary symmetric functions.

4. Exercise 1.9, p. 49

Solution. Let P be prime. Since P is radical, it corresponds to an algebraic set $X = Z(P)$ by Nullstellensatz. Suppose $X = X_1 \cup X_2$, where X_1 and X_2 are also algebraic sets, neither one equal to X itself. Then $J_1 = I(X_1)$ and $J_2 = I(X_2)$ are radical ideals, neither one equal to P. Furthermore, $J_i \supset P$ for $i = 1, 2$ since a function which is zero on X is clearly zero on X_i. Let $a \in J_1$ but not in P. Let $b \in J_2$ but not in P. Then ab is zero on X, but neither a nor b is. This implies that $ab \in P$, but neither $a \in P$ nor $b \in P$, contradicting the fact that P is prime.