Math 724: Solution Set 1

1. Is the ideal $(3x, 7)$ in $\mathbb{Z}[x]$ maximal? Prime? Principal? Prove your answer.

   Solution. Note that $(3x, 7) = (x, 7)$ since $x = 7(x) - 3x(2)$ (implying $x \in (3x, 7)$, but clearly $(3x, 7) \subset (x, 7)$). This ideal is maximal because $\mathbb{Z}[x]/(x, 7) \cong \mathbb{Z}/(7) \cong \mathbb{Z}_7$. Since it is maximal is it prime.

   We show it is not principal. Suppose $(3x, 7) = (p)$ for some $p \in \mathbb{Z}[x]$. Then p must have minimal degree in $(3x, 7)$, or elements of smaller degree in $(3x, 7)$ would not be in (p). Thus since 7 is of degree 0, the smallest degree possible, we know that $\deg(p) = 0$ so p is a constant. If $p \neq 7$, then $7 \notin (p)$ since 7 is prime. But if $p = 7$, then $3x \notin (p)$, since $\mathbb{Z}[x]$ is a unique factorization domain, 7 is prime, and 3 is not a multiple of 7.

2. Let M be an R-module, and $I = \operatorname{ann} M$. Show that I is an ideal, and that M can be made into a well-defined R/I-module under the operation $(r + I)m = rm$. Note that you have to prove this R/I action is well-defined (among other things). If I is maximal, what does that imply about M?

   Solution. To show that I is an ideal, we note first that it is a subgroup, since for any $a, b \in I$ and $m \in M$, we have
   $$(a + b)m = am + bm = 0 + 0 = 0$$
   implying $a + b \in I$. In addition, $am = 0$ implies $-am = 0$ so every element a has an inverse $-a$ in I. Clearly $0 \in I$, so it's a subgroup.

   To show it's an ideal we need only show that for all $r \in R$ and $a \in I$, we have $ra \in I$. To that end, let $m \in M$. Then $ram = r(am) = r0 = 0$.

   We now show that this can be made into an R/I-module. We begin by showing that the operation $(r + I)m = rm$ is well-defined. Let $r' + I = r + I$. Then $r' - r \in I$. We need to show that $rm = r'm$. But
   $$rm = rm + (r' - r)m = (r + (r' - r))m = r'm,$$
   as desired. This product makes it into a module since for all $r + I, s + I \in R/I$ and $m, n \in M$,

   (a) $1m = (1 + I)m = m$

   (b) $(rs)m = (rs + I)m = [(r + I)(s + I)]m = (r + I)sm = r(sm)$

   (c) $r(m + n) = (r + I)(m + n) = (r + I)m + (r + I)n = rm + rn$

   (d) $(r + s)m = (r + s + I)m = (r + I + s + I)m = (r + I)m + (s + I)m = rm + sm.$

   If I is maximal, then R/I is a field, implying M is a vector space over R/I.