Math 724: Problem Set 8

Due on 12.12.06

1. Suppose that $\{M_n, \theta_n\}$, $\{N_n, \eta_n\}$ and $\{K_n, \kappa_n\}$ are inverse systems of modules. Assume that the diagram below is commutative with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M_n \arrow[r] \arrow[d, "\theta_n"] & N_n \arrow[r] \arrow[d, "\eta_n"] & K_n \arrow[r] \arrow[d, "\kappa_n"] & 0 \\ 0 \arrow[r] & M_{n-1} \arrow[r] & N_{n-1} \arrow[r] & K_{n-1} \arrow[r] & 0. \end{tikzcd}$$

   (a) Show that the sequence
   $$0 \to \varprojlim M_n \to \varprojlim N_n \to \varprojlim K_n$$
   is exact.

   (b) Show that if θ_n is surjective for all $n \geq 0$, that the sequence is also right exact, i.e.
   $$0 \to \varprojlim M_n \to \varprojlim N_n \to \varprojlim K_n \to 0$$
   is exact.

2. Let R be a ring (you may assume Noetherian). Let $\mathfrak{m}$ be a maximal ideal of R.

   (a) Show that $R_\mathfrak{m}$, the localization of R at $\mathfrak{m}$, is a local ring. Is the localization R_P of R at any prime ideal P local? Prove your answer.

   (b) Let $\hat{R}_\mathfrak{m}$ denote the completion of R with respect to the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^i\}$ of R. Write down the natural map $R \to \hat{R}_\mathfrak{m}$ and prove that it factors through the natural localization map $R \to R_\mathfrak{m}$. As part of your proof, demonstrate a commutative diagram that illustrates what it means to "factor through the localization map". Make sure to prove that all your maps are well-defined!

3. Let R be a ring filtered by ideals $R = I_0 \supseteq I_1 \supseteq \cdots$. Similar to the I-adic topology defined in class, we can define a topology on R as follows: the base of open sets around 0 consist of I_i for all i, and the the base of open sets around $r \in R$ consist of $r + I_i$ for all i.

   (a) Prove that the operation $+$ is continuous under this topology, i.e. that the map $R \times R \to R$ given by $(r, s) \to r + s$ is continuous in this topology. (You need to show that the inverse image of all open sets are open – and open sets on $R \times R$ are given by the product of the topology on R with itself.)

   (b) Prove that any ideal I of R containing I_j for some j is an open set. *Hint:* consider the set of cosets of I_j in I

   (c) Prove that multiplication is continuous in this topology, i.e. $R \times R \to R$ given by $(r, s) \to r \cdot s$ is a continuous map.