Differential Topology Solutions #5

1. Chapter 4, Section 4, #3. Let $c : [a, b] \to X$ be a smooth curve, and let $c(a) = p$, $c(b) = q$. Show that if ω is the differential of a function on X, $\omega = df$, then

$$\int_a^b c^*\omega = f(q) - f(p).$$

Solution. We note that $c^*(df) = d(c^*f) = d(f \circ c)$. Therefore

$$\int_a^b c^*\omega = \int_a^b d(f \circ c) = f \circ c(b) - f \circ c(a) = f(q) - f(p).$$

2. Chapter 4, Section 4, #5. A *closed curve* on a manifold X is a smooth map $\gamma : S^1 \to X$. If ω is a 1-form on X, define the *line integral* of ω around γ by

$$\oint_\gamma \omega = \int_{S^1} \gamma^*(\omega).$$

For the case of $X = \mathbb{R}^k$, write $\oint_\gamma \omega$ explicitly in terms of the coordinate expressions of γ and ω.

Solution. The one form ω on $\mathbb{R}^k$ may be written explicitly as $\omega = \sum_{j=1}^k g_j dx_j$, where $x_1, \dots, x_k$ are coordinates on $\mathbb{R}^k$, and g_j are real-valued functions on $\mathbb{R}^k$. In these coordinates, $\gamma = (\gamma_1, \dots, \gamma_k)$, where each $\gamma_i : S^1 \to \mathbb{R}$. Note that

$$\gamma^*(\omega) = \sum_j (g_j \circ \gamma)\gamma^*(dx_j) = \sum_j (g_j \circ \gamma) d\gamma_j.$$

Then

$$\oint_\gamma \omega = \int_{S^1} \gamma^*(\omega) = \sum_j \int_{S^1} (g_j \circ \gamma) d\gamma_j.$$

3. Chapter 4, Section 4, #7. Suppose that the 1-form ω on X is the differential of a function, $\omega = df$. Prove that $\oint_\gamma \omega = 0$ for all closed curves γ on X.

   Solution. By definition, $\oint_\gamma \omega = \int_{S^1} \gamma^*(\omega)$, where $\gamma : S^1 \to X$. Let $S^1(1)$ be the manifold with boundary given by the "upper hemisphere" of S^1 (including endpoints), and $S^1(2)$ be the manifold with boundary given by the "lower hemisphere" of S^1 (including endpoints). Then for any 1-form α,

$$\int_{S^1} \alpha = \int_{S^1(1)} \alpha + \int_{S^1(2)} \alpha. \tag{0.1}$$

   [This follows from the fact that the integrals on any parameterizations of S^1 and the disjoint union $S^1(1) \coprod S^1(2)$ are the same – remember that integration on manifolds is defined in terms of parameterizations, and that what happens on the end points doesn't matter since it's dimension 0 and we're integrating a 1-form.] Substituting $\alpha = \gamma^*\omega = d(f \circ \gamma)$, we obtain:

$$\begin{aligned}
\oint_\gamma \omega &= \int_{S^1(1)} (\gamma^*\omega) + \int_{S^1(2)} (\gamma^*\omega) \\
&= \int_{S^1(1)} d(f \circ \gamma) + \int_{S^1(2)} d(f \circ \gamma) \\
&= (f \circ \gamma(q) - f \circ \gamma(p)) + (f \circ \gamma(p) - f \circ \gamma(q)) \quad \text{By Problem \#1} \\
&= 0
\end{aligned}$$

   where p and q are the two points in the intersection of $S^1(1)$ and $S^1(2)$.

4. Chapter 4, Section 4, #9.

5. Chapter 4, Section 4, #10.

6. Chapter 4, Section 4, #14.

7. Chapter 4, Section 5, #1.

8. Chapter 4, Section 7, #4.

   Solution/Hint. Show that $(\mathrm{curl}\,\overrightarrow{F} \cdot \overrightarrow{n})dA = d\omega$ on S. It follows that the integrals are the same. Then apply Stokes' Theorem.

Additional problems for graduate students, or undergraduate extra credit

9. Chapter 4, Section 7, #8. Let $X = \partial W$, with W compact, and $f : X \to Y$ be a smooth map. Suppose that ω is a closed k form on Y, where $k = \dim X$. Prove that if f extends to all of W, then $\int_X f^*\omega = 0$.

   Solution. Let $F : W \to Y$ be the extension of f. Note that $\partial F = f$. By Stokes' theorem,

   $$\int_X f^*\omega = \int_{\partial W} f^*\omega = \int_W d(f^*\omega) = \int_W f^*(d\omega) = 0.$$

   (Recall that ω closed means that $d\omega = 0$.)

10. Chapter 4, Section 7, #9.