Differential Topology Solutions #4

1. Consider the map
$$V \times W \to V \oplus W, \qquad v \times w \to v \oplus w.$$
Is this map bilinear?

   The map is not bilinear. Let $f : V \times W \to V \oplus W$ be the map $f(v, w) = v \oplus w$. Then for any $v_1, v_2 \in V$ and $w \in W$,
$$f(v_1 + v_2, w) = (v_1 + v_2) \oplus w.$$
On the other hand,
$$f(v_1, w) + f(v_2, w) = v_1 \oplus w + v_2 \oplus w = (v_1 + v_2) \oplus 2w.$$
For $w \neq 0$, $2w \neq w$ so that $f(v_1 + v_2, w) \neq f(v_1, w) + f(v_2, w)$.

2. Suppose $v_1, \ldots, v_p \in V$ are linearly dependent vectors. Show that
$$T(v_1, \ldots, v_p) = 0$$
for all $T \in \Lambda^p(V^*)$. Is this true for all $T \in \mathcal{J}^p(V^*)$? If so, prove it, and if not, find a counterexample.

3. Chapter 10, Section 2, #3.

4. Chapter 10, Section 2, #10 (a) and (b).

5. Chapter 10, Section 3, Exercise on p. 165.

The Tensor Product of Vector Spaces. Let V and W be vector spaces over a field (you may assume they are real vector spaces). The tensor product $V \otimes W$ is a vector space equipped with a bilinear map
$$V \times W \longrightarrow V \otimes W, \qquad v \times w \to v \otimes w$$

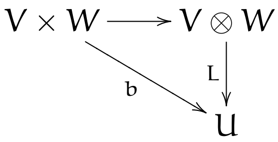
(for every $v \in V, w \in W$) which is *universal* in the following sense. For any bilinear map

$$b : V \times W \longrightarrow U$$

where U is a vector space (over the same field), there is a unique linear map

$$L : V \otimes W \longrightarrow U$$

such that $L(v \otimes w) = b(v, w)$. In other words, any linear map $b : V \times W \longrightarrow U$ factors through the tensor product $V \otimes W$. Another way of saying this is that the diagram



commutes for every vector space U and for every bilinear map b.

6. Find a basis of $V \otimes W$ (given a basis of V and of W). Find the dimension of $V \otimes W$.

   Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{f_1, \dots, f_m\}$ be a basis of W. Let

   $$f : V \times W \to V \otimes W, \qquad f(v, w) = v \otimes w.$$

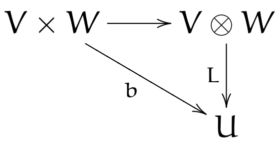
   Claim 1 *The set* $\{e_i \otimes f_j\}$, $i = 1, \dots, n$ *and* $j = 1, \dots, m$ *form a basis of* $V \otimes W$.

   Proof: We form an nm-dimensional vector space U as follows: U is the set of all formal linear combinations (over $\mathbb{R}$) of basis elements g_{ij}, $i = 1, \dots, n$, and $j = 1, \dots, m$. In other words, by definition, g_{ij} forms a basis of U. We show that $U \cong V \otimes W$, under the map $g_{ij} \to e_i \otimes f_j$. Note that *a priori* we don't know whether $e_i \otimes f_j$ span $V \otimes W$, so this map may not be surjective.

   First we display a bilinear map $b : V \times W \to U$. Let $b(e_i, f_j) = g_{ij}$ and extend so that b is bilinear:

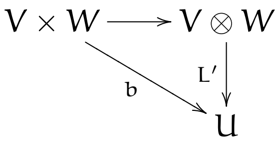
   $$b\left(\sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{m} c_j f_j\right) = \sum_{i,j} a_i c_j g_{ij}.$$

   By the universality property of the tensor product, there is a unique linear map L such that



commutes. We can also find L directly on the image of f: $L(v \otimes w) = b(v, w)$ implies that $L(e_i \otimes f_j) = g_{ij}$.

We show that L is an isomorphism. Clearly b is a surjective map (it hits all the g_{ij}), so that L must be surjective. Suppose that L is not injective. Then $\ker L$ is nontrivial. Choose any nonzero vector $x \in V \otimes W$ in the kernel. (Note that I do not assume that x has the form $v \otimes w$). Clearly x is not in the linear span of $e_i \otimes f_j$, since $L(\sum a_{ij} e_i \otimes f_j) = \sum a_{ij} b(e_i, f_j) = \sum a_{ij} g_{ij} \neq 0$ unless all the $a_{ij} = 0$ (remember we constructed U so that g_{ij} were linearly independent). Now consider the map $L' : V \otimes W$ given by $L'(v \otimes w) = b(v, w)$ and $L'(x) = g_{11}$. This is a linear map, as is easy to check (it is important that x is linearly independent of $e_i \otimes f_j$, otherwise linearity would fail). Also, since $x \notin \ker L'$, $L' \neq L$. Therefore the commutative diagram



commutes. Thus L is not the *unique* linear map making the diagram commute, as is given by the universality property of $V \otimes W$, a contradiction. Therefore $\ker L = 0$ and L is an isomorphism.

Since $\{g_{ij}\}$ is a basis for U, their image under L^{-1} is a basis for $V \otimes W$. It follows that $e_i \otimes f_j$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$ span $V \otimes W$, and that the dimension of $V \otimes W$ is nm.

Additional problems for graduate students, or undergraduate extra credit

The n*th exterior power* of a vector space V is a vector space $\mathrm{Alt}^n V$, equipped with an alternating multilinear map

$$V \times \cdots \times V \to \mathrm{Alt}^n V, \qquad v_1 \times \cdots \times v_n \to v_1 \odot \cdots \odot v_n,$$

that is universal in the following sense. For any alternating multilinear map $b : V \times \cdots \times V \to U$ (where U is a vector space), there is a unique linear map $L : \mathrm{Alt}^n V \to U$ which takes $v_1 \odot \cdots \odot v_n$ to $b(v_1, \ldots, v_n)$.

7. Show that $\Lambda^n(V^*) \cong \mathrm{Alt}^n V^*$, where V^* is the dual to V (and $\Lambda^n(V^*)$ is the same as we defined in class). *Hint.* Show that $\Lambda^n(V^*)$ has the universality property, or use the universality property of $\mathrm{Alt}^n V^*$ to construct a map between the two spaces and then prove it's an isomorphism.

   We follow the same method as applied in the problem above. We first define a multilinear map
   $$b : V^* \times \cdots \times V^* \to \Lambda^n(V^*)$$
   and then use the universality property of $\mathrm{Alt}^n V^*$. Choose a basis $\{\phi_i\}_{i=1}^k$ of V^*. Let b be given by
   $$b(\phi_{i_1}, \ldots, \phi_{i_n}) = \phi_{i_1} \wedge \cdots \wedge \phi_{i_n},$$
   and extend so that the map is bilinear. Recall that as the $\{i_j\}$ vary on the right hand side, we span $\Lambda^n(V^*)$, as proven in class. It follows that b is surjective.

   [Alternatively define $b(w_1, \ldots, w_n) = w_1 \wedge \cdots \wedge w_n$ and prove it's bilinear – also, mention why it is surjective, since it's not defined on basis elements.]

   By the universality property of the exterior algebra, there is a unique linear map L such that the diagram

   $$\begin{array}{ccc} V^* \times \cdots \times V^* & \longrightarrow & \mathrm{Alt}^n(V^*) \\ & \searrow^{b} & \downarrow L \\ & & \Lambda^n(V^*) \end{array}$$

   commutes. By construction, $L(w_1 \odot \cdots \odot w_n) = w_1 \wedge \cdots \wedge w_n$ for all $w_i \in V^*$. We prove that L is an isomorphism.

   Clearly L is surjective since b is. We need only prove that L is injective. Suppose not. Then let $x \in \ker L$ be nonzero. Note that x must be linearly independent of

$\{\phi_{i_1} \odot \cdots \odot \phi_{i_n}\}$, since the image of these elements is nonzero under L (and L is linear). It follows that we may define L' so that $L' = L$ on the linear span of $\{\phi_{i_1} \odot \cdots \odot \phi_{i_n}\}$, but $L'(x) \neq 0$. Extend L' to all of $\mathrm{Alt}^n(V^*)$ by linearity. The map L' is linear, and it also makes the above diagram commute, violating the uniqueness of L. Therefore L must be injective. It follows that $L : \mathrm{Alt}^n(V^*) \to \Lambda^n(V^*)$ is an isomorphism.

8. Show that $\Lambda^n(V)$ can be constructed as a quotient of $V \otimes V \otimes \cdots \otimes V$ (n times). In other words, there is a surjective map

$$L : V^{\otimes n} \longrightarrow \Lambda^n(V).$$

Write down the map, show it's surjective, and find its kernel.

For intuitive purposes, we start with $n = 2$. Let $e_1, \dots e_k$ be a basis of V. A basis of $V \otimes V$ is given by $e_i \times e_j$. Consider the map $L(e_i \otimes e_j) = e_i \wedge e_j$, and extend the map linearly to all of $V \otimes V$. Clearly L is surjective, since $V \wedge V$ has a basis given by $e_i \wedge e_j$ with $i < j$. The map is linear because it is defined on a basis of $V \otimes V$ and extended linearly. We need only find the kernel. Since $\dim V \otimes V = k^2$ and $\dim \Lambda^2(V) = \binom{k}{2}$, the dimension of the kernel is $k^2 - \binom{k}{2} = \frac{k^2+k}{2}$.

We note immediately that $\ker L$ contains elements of the form $e_i \otimes e_i$. There are k of these elements, and they are linearly independent in $V \otimes V$. Furthermore, $L(e_i \otimes e_j) = -L(e_j \otimes e_i)$ implies that $e_i \otimes e_j + e_j \otimes e_i$ is in the kernel, for all i, j. There are $\binom{k}{2}$ elements of this form. They are clearly linearly independent of all the $e_i \otimes e_i$, and they are also linearly independent of each other. Thus we have found $k + \binom{k}{2} = \frac{k^2+k}{2}$ linearly independent elements in the kernel. Since this is the dimension of the kernel, we have found a basis of the kernel, and hence the whole kernel. In other words,

$$K = \mathrm{span}\{e_i \otimes e_i, e_i \otimes e_j - e_j \otimes e_i\}, i, j = 1, \dots, k.$$

We now do the general case. A basis of $V^{\otimes n}$ is given by $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$, $i_j = 1, \dots k$. Consider the map $L(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_1} \wedge \cdots \wedge e_{i_n} \in \Lambda^n(V)$, and extend by linearity to $V^{\otimes n}$. Clearly L is surjective, since a basis of $\Lambda^n(V)$ is given by those $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with increasing indices. The map is linear because it is defined on a basis of $V^{\otimes n}$ and extended linearly. We need only find the kernel. Since $\dim V^{\otimes n} = k^n$ and $\dim \Lambda^n(V) = \binom{k}{n}$, the kernel is dimension $k^n - \binom{k}{n}$.

Clearly basis elements of $V^{\otimes n}$ with repeating indices are in the kernel. It is also clear that two different basis elements of $V^{\otimes n}$ with permuted indices map to the same elements (up to sign) in $\Lambda^n(V)$. Thus the kernel contains all elements of the form

$$\ker L \supset \{e_{i_1} \otimes \cdots \otimes e_{i_n} - (-1)^{\mathrm{sgn}(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}, e_{j_1} \otimes \cdots \otimes e_{j_n}\}$$

where $j_1, \dots, j_n$ not all distinct, and σ is any permutation in S_n of the numbers $i_1, \dots, i_n$. These elements are all linearly independent in $V^{\otimes n}$, which can be verified by a tedious induction on n argument, for $n \geq 2$. (For the case that $n = 1$, these two different kinds of elements of the kernel overlap - check it out!). Now we count the number of these elements. The elements of repeating indices could be counted directly by summing over the number of repeats. However, it is more useful for our purposes if we express this number as a difference. The number of basis elements with repeating indices is k^n minus the number of basis elements in $V^{\otimes n}$ with distinct indices. The number of basis elements with distinct indices is $\binom{k}{n} n!$ (where the $n!$ term comes from the fact that order is important). Therefore,

$$\begin{aligned} &\#\{e_{j_1} \otimes \cdots \otimes e_{j_n}\} \quad \text{where } j_1, \dots, j_n \text{ not all distinct} \\ &= k^n - \binom{k}{n} n!. \end{aligned}$$

On the other hand, the number of (distinct, up to sign)elements in the set $\{e_{i_1} \otimes \cdots \otimes e_{i_n} - (-1)^{sgn(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}\}$ is obtained by first fixing $i_1, \dots, i_n$ to be strictly increasing (since they are distinct), and then considering all the permutations σ might be. We must exclude the identity permutation, since the element we obtain using it is 0. We obtain

$$\begin{aligned} &\#\{e_{i_1} \otimes \cdots \otimes e_{i_n} - (-1)^{sgn(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}\} \\ &= \binom{k}{n} \cdot (n! - 1) \quad \text{number of increasing index sequences times} \\ &\qquad \text{number of nontrivial permutations} \end{aligned}$$

It is left only to show that the dimension of the span of the vectors we found (contained in the kernel) is indeed the dimension of the kernel itself. In other words, we need to prove that

$$k^n - \binom{k}{n} = k^n - \binom{k}{n} n! + \binom{k}{n}(n! - 1).$$

But this is obvious. It follows now that

$$\ker L = span\{e_{i_1} \otimes \cdots \otimes e_{i_n} - (-1)^{sgn(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}, e_{j_1} \otimes \cdots \otimes e_{j_n}\}$$

where $j_1, \dots, j_n$ not all distinct, and σ is any permutation in S_n of the numbers $i_1, \dots, i_n$.