Differential Topology Solution Set #3

Select Solutions

1. Chapter 1, Section 4, #7
2. Chapter 1, Section 4, #8
3. Chapter 1, Section 4, #11(a)-(b)

   #11(a) The $n \times n$ matrices with determinant 1 form a group denoted $SL(n)$. Prove that $SL(n)$ is a submanifold of $M(n)$ and thus is a Lie group.

   Consider the determinant map

$$\det : M(n) \to \mathbb{R}.$$

   If we show that 1 is a regular value of det, then we will have shown that $SL(n)$ is a submanifold, since $\det^{-1}(1) = SL(n)$. To show it's a regular value, we need to show that its derivative is surjective at every point in the level set $SL(n)$.

   We show that $d(\det)_A$ is surjective for all $A \in SL(n)$ by calculating directly. Note that $T_A(M(n)) \cong M(n)$ for any matrix A. We choose a basis E_{ij} for the vector space $M(n)$, where E_{ij} is the matrix with 1 in the ijth entry, and 0s everywhere else.

   We begin by assuming that $A = I$, the identity matrix. Then

$$\begin{aligned} d(\det)_I(E_{ij}) &= \lim_{h \to 0} \frac{\det(I + hE_{ij}) - \det I}{h} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \end{aligned}$$

   (Notice that if $i \neq j$, then $I + hE_{ij}$ has the same determinant as I, since the term hE_{ij} is off-diagonal with just one entry. On the other hand, $\det(I + hE_{ii}) = (1 + h)$ since $I + hE_{ii}$ has 1 in every diagonal entry except the iith, where it has $1 + h$.)

Since $d(\det)_I$ maps to $\mathbb{R}$, we are only required to show that it is nonzero to show that it is surjective. It follows that $d(\det)_I$ is surjective.

Now for generic $A \in SL(n)$, you can do one of two things to prove this: either you can calculate directly as we did when $A = I$, or you can prove that the derivative of the determinant is the trace (as we did in class). Here is the first way. Let $(\hat{A})_{ij}$ indicate the minor of the matrix A with the ith column and jth row eliminated.

$$\begin{aligned} d(\det)_A(E_{ij}) &= \lim_{t\to 0} \frac{\det(A + tE_{ij}) - \det A}{t} \\ &= \lim_{t\to 0} \frac{\det(A + tE_{ij}) - 1}{t} \end{aligned}$$

Note that

$$\begin{aligned} \det(A + tE_{ij}) &= a_{1j} \det((\hat{A})_{1j}) - a_{2j} \det((\hat{A})_{2j}) + \cdots \\ &\qquad \pm (a_{ij} + t) \det((\hat{A})_{ij}) \cdots \pm \det((\hat{A})_{nj}) \\ &= a_{1j} \det((\hat{A})_{1j}) - a_{2j} \det((\hat{A})_{2j}) + \cdots \\ &\qquad \pm a_{ij} \det((\hat{A})_{ij}) \cdots \pm \det((\hat{A})_{nj}) \pm t \det((\hat{A})_{ij}) \\ &= \det A \pm t \det((\hat{A})_{ij}) \\ &= 1 \pm t \det((\hat{A})_{ij}). \end{aligned}$$

We substitute this into the equation for $d(\det)$ and find

$$\begin{aligned} d(\det)_A(E_{ij}) &= \lim_{t\to 0} \frac{1 \pm t \det((\hat{A})_{ij}) - 1}{t} \\ &= \pm \det((\hat{A})_{ij}). \end{aligned}$$

Lastly we need to show that, for any $A \in SL(n)$, there exists some ij such that this determinant is nonzero. But this is clear, for if every $(n-1) \times (n-1)$ minor of A had a zero determinant, then the matrix A would itself have 0 determinant, contrary to its being in $SL(n)$.

#11(b) The tangent space of $SL(n)$ at I may be computed by the kernel of the $d(\det)_I$, since $SL(n)$ is determined by the level set of det at I (i.e. det = 1) We found that

$$d(\det)_I(E_{ij}) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Thus the kernel is the set of all E_{ij} such that $i \neq j$. This is precisely the set of traceless matrices (note that $tr(E_i i) = 1$ for all i).

4. Chapter 1, Section 5, #7

5. Chapter 1, Section 5, #10

6. Chapter 1, Section 6, #6. Read the definition of *contractible* from #4.

7. Chapter 1, Section 6, #7. Show that the antipodal map $x \to -x$ of $S^k \to S^k$ is homotopic to the identity if k is odd.

   If k is odd, then $S^k \subset \mathbb{R}^{k+1}$ where $k+1$ is even. Then the matrix

$$A_t = \begin{pmatrix} \cos \pi t & -\sin \pi t & 0 & 0 & 0 & 0 \\ \sin \pi t & \cos \pi t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \pi t & -\sin \pi t \\ 0 & 0 & 0 & 0 & \sin \pi t & \cos \pi t \end{pmatrix}$$

   provides a homotopy from the identity map to the antipodal map. More specifically, let $F : S^k \times I \to S^k$ be given by

$$F(x, t) = A_t(x)$$

   for any $x \in S^k$. Note that $A_t(x) \in S^k$ as well, since $A_t(A_t)^T = I$ (so in particular, unit-length vectors get mapped to unit-length vectors).

8. Chapter 1, Section 7, #4

9. Chapter 1, Section 7, #6. Prove that S^k is simply connected if $k > 1$.

   Recall that S^k is simply connected if every map $f : S^1 \to S^k$ is homotopic to a constant map. If $k > 1$, then there exists a point $p \in S^k$ such that $p \notin f(S^1)$. Rotate the sphere in $\mathbb{R}^{k+1}$ so that p is the North Pole, stereographically project every point in $S^k - \{p\}$ into the k-plane perpendicular to the vector p. This is a diffeomorphism ϕ from $S^k - \{p\}$ to the k-plane $\mathbb{R}^k$. Then $\phi \circ f : S^1 \to \mathbb{R}^k$ is a map of S^1 into a contractible space, so it is clearly homotopic to a constant map $S^1 \to q$, where $q \in R^k$. We write this homotopy explicitly:

$$F : S^1 \times I \to \mathbb{R}^k, \quad F(x, 0) = \phi \circ f(x), \quad F(x, 1) = q.$$

Then we apply ϕ^{-1} to obtain a homotopy

$$G : S^1 \times I \to S^k, \quad G(x, 0) = f(x), \quad G(x, 1) = \phi^{-1}(q).$$

from $f(x)$ to the constant map $S^1 \to \phi^{-1}(q)$.

10. Chapter 1, Section 7, #8 (just do a couple)

11. Chapter 1, Section 8, #1

12. Chapter 1, Section 8, #7. Show that if k is odd, there exists a vector field $\vec{v}$ on S^k having no zeros.

    We follow the hint. At a point $(x_1, x_2, \ldots, x_{k+1}) \in S^k \subset \mathbb{R}^{k+1})$, let

    $$\vec{v}(x_1, x_2, \ldots x_{k+1}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{k+1}, x_k).$$

    Note that $\vec{v}(x)$ is tangent to S^k at x, since

    $$\begin{aligned} \vec{v}(x) \cdot x &= (-x_2, x_1, -x_4, x_3, \ldots, -x_{k+1}, x_k) \cdot (x_1, x_2, \ldots x_{k+1}) \\ &= -x_2 x_1 + x_1 x_2 - \cdots - x_{k+1} x_k + x_k x_{k+1} = 0. \end{aligned}$$

    Furthermore, since $x \neq 0$, it follows that $\vec{v}(x) \neq 0$ on S^k.

Additional problems for graduate students, or undergraduate extra credit

13. Oops.

14. Let $m^*(A)$ be the measure of A as defined in class. In other words, for all $\varepsilon > 0$, there is a countable set of rectangles $\{S_i\}$ such that $A \subset \cup_i S_i$ and

$$\sum_{i=1}^{\infty} \mathrm{vol}(S_i) - \varepsilon < m^*(A) \leq \sum_{i=1}^{\infty} \mathrm{vol}(S_i)$$

Find a (Cantor-like) subset A of the unit interval $[0,1]$ with the properties:

(a) A is constructed by removing a countable number of intervals from $[0,1]$ (open and/or closed)

(b) Between any two points $p, q \in A$, there is a point $b \in (p,q)$ not contained in A.

(c) $m^*(A) \neq 0$ and $m^*(A) < 1$.

Generalize this to obtain a set with these properties with arbitrary measure in $(0,1)$. *Hint.* You will need to use convergence of series.

15. Find an example of a k-dimensional manifold such that $T(M)$ is not diffeomorphic to $M \times \mathbb{R}^k$, and prove your answer.

Let $M = S^2$, and suppose $T(S^2)$ were diffeomorphic to $S^2 \times \mathbb{R}^2$. Choose a nonzero vector $v \in \mathbb{R}^2$, such as $v = (1,0)$. Then the global map

$$s : S^2 \to S^2 \times \mathbb{R}^2 \cong T(S^2)$$

given by $s(x) = (x, v)$ gives a nonzero smooth vector field on S^2 (the vector field is constantly valued v). This implies that the antipodal map on S^2 is homotopic to the identity map (see Exercise #8 in Chapter 1, Section 8). We showed already that if k is odd that the antipodal map on S^k is homotopic to the identity; we need to show that if $k = 2$, then the antipodal map on S^2 is *not* homotopic to the identity.

I did not see any way to prove this without introducing more sophisticated techniques which we will see in Chapter 3, so just stating that there is no nonvanishing vector field on S^2 (mentioned in Exercise #7 of Chapter 1, Section 8) is sufficient for this problem.