Differential Topology Solution Set #2

Select Solutions

1. Show that X compact implies that any smooth map $f : X \to Y$ is proper. Recall that a space is called compact if, for every cover $\{U\ \ \}$ by open sets such that

   $$X = \cup\ \ U$$

   there is a finite subcover. This means that there is a finite subset $U_{\ \ i}$ of all the U such that

   $$X = \cup_{i \in I} U_{\ \ i}$$

   and I is a finite set.

   Alternatively, you may use the definition that X is compact if and only if it is closed and bounded.

   Let $Z \subset Y$ be a compact set. We need to prove that $f^{-1}(Z)$ is compact. It is sufficient to prove that $f^{-1}(Z)$ is closed and bounded. It is obviously bounded, since any subset of a compact set X is bounded (since X is). We show it is closed.

   Since Z is compact, it is closed. The map $f : X \to Y$ is continuous, and so the inverse image of closed sets are closed. Therefore, $f^{-1}(Z)$ must also be closed.

2. Chapter 1, Section 3, #2

3. Chapter 1, Section 3, #4 Construct a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a diffeomorphism onto its image.

   Let $f(x, y) = (e^y \cos x, e^y \sin x)$. We begin by simplifying our life and writing $f(x, y)$ in polar coordinates (r, θ). We obtain:

   $$f(x, y) = (e^y, x).$$

   This function is globally defined. Clearly f is smooth (it is infinitely differentiable in all directions). We check that it is also locally invertible, with a smooth inverse.

Choose a point (x, y). Let an open neighborhood be given by $U = \{(a, b) : x - \frac{}{2} < a < x + \frac{}{2}, y - 1 < b < y + 1\}$. Then clearly U contains (x, y). The image of U under f consists of

$$V = \{(s, t) \text{ such that } e^{y-1} < s < e^{y+1} \text{ and } x - \frac{\pi}{2} < t < x + \frac{\pi}{2}\}.$$

In the neighborhood V, the inverse map is

$$f^{-1}(s, t) = (t, \ln s).$$

Note that t is the unique value mod 2π such that $x - \frac{}{2} < t < x + \frac{}{2}$. Notice also that $\ln s$ is always well-defined, because $s > 0$ in the image (indeed, $(0, 0)$ is not in the image of f at all). These observations imply the inverse map is well-defined. Also, f^{-1} is infinitely differentiable in both directions as well – as long as $s \neq 0$, which it is not. It follows that f is locally a diffeomorphism.

On the other hand, f is clearly not a diffeomorphism onto its image: in particular, it is not 1-1 on its image since there are many values of (e^y, x) that are equivalent if we do not restrict the domain of x. (Remember we're in polar coordinates, so $(e^y, x) = (e^y, x + 2k\pi)$ for any integer $k \in \mathbb{Z}$.)

4. Chapter 1, Section 3, #7(a).

5. Chapter 1, Section 4, #1

6. Chapter 1, Section 4, #2(a)-(b)

   (a) If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

   Suppose f is a submersion that is not surjective. Let $y_0 = f(x_0)$ be a point in the image of X, and y_1 be a point not in $f(X)$. Since Y is connected, let $s(t)$ be a smooth path in Y such that $s(0) = y_0$ and $s(1) = y_1$. Since X is compact, the inverse image $f^{-1}(s(t))$ is closed in X (of course, $f(f^{-1}(s(t))) \neq s(t)$ for all t since not all of $s(t)$ lies in $f(X)$).

   Let $g(t)$ be a curve in X such that $f(g(t)) = s(t)$ for $0 \leq t \leq t'$, where $f^{-1}(s(t' + \epsilon))$ is empty for small enough ϵ. We let $x' = g(t')$, and $y' = f(x')$. Then for any parametrization ϕ of Y around y', (i.e. $\phi : U \to V$, where $U \subset \mathbb{R}^l$ open and V an open neighborhood of y' in Y,) $f \circ \phi^{-1}$ is not surjective since $\phi^{-1}(s(t' + \epsilon))$ is not in the image of f, but is in the parametrization neighborhood.

   But this contradicts the local submersion theorem, which says that there exist local coordinates around x' and y' such that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_l)$, the canonical submersion, where $k \geq l$ are the dimensions of X and Y, respectively.

(b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

If there were, then by part (a) there would be a surjective map from a compact set onto $\mathbb{R}^n$ for some n. Consider the projection $\pi : \mathbb{R}^n \to \mathbb{R}^1$ to the first coordinate. Then $\pi \circ f : X \to \mathbb{R}^1$ is a smooth surjective function. Every smooth function on a compact set X has a maximum and a minimum, so $\pi \circ f$ cannot be surjective, a contradiction.

7. Chapter 1, Section 4, #3

8. Chapter 1, Section 5, #1

9. Chapter 1, Section 5, #2

10. Chapter 1, Section 5, #4

Additional problems for graduate students, or undergraduate extra credit

11. Chapter 1, Section 3, 7(b)
12. Chapter 1, Section 3, #10
13. Chapter 1, Section 4, #7
14. Chapter 1, Section 4, #12
15. Suppose that X is given by

$$X = \{\begin{pmatrix} e^{i \ \ _1} & 0 \\ 0 & e^{i \ \ _2} \end{pmatrix} \text{ such that } \theta_i \in \mathbb{R}\}.$$

Show that the subset

$$X' = \{\begin{pmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{pmatrix}, \theta \in \mathbb{R}\}$$

is a submanifold of X. Do this by finding a smooth map $f : X \to \mathbb{C}$ and showing that there is a regular value y of f such that $X = f^{-1}(y)$.

Hint. Think about determinants.

Let $f : X \to \mathbb{C}$ be the determinant map. In other words,

$$f(\begin{pmatrix} e^{i \ \ _1} & 0 \\ 0 & e^{i \ \ _2} \end{pmatrix}) = e^{i(\ \ _1 + \ \ _2)}.$$

Then f is a smooth map. Its image is the circle $S^1 \subset \mathbb{C}$ given by the set of e^{i} (elements of unit length in $\mathbb{C}$). Furthermore, $e^{i(\ \ _1 + \ \ _2)} = 1$ if and only if $i(\theta_1 + \theta_2) = 0$ or $2ki\pi$ for some $k \in \mathbb{Z}$. In other words, $\theta_1 + \theta_2 = 2k\pi$, or $\theta_2 = 2k\pi - \theta_1$. But then since $e^{2k \ \ - \ \ _1} = e^{- \ \ _1}$, we find that $f^{-1}(1) = X'$.

We need only show that 1 is a regular value for f. For any matrix $A \in X'$, we calculate the derivative $df_A : T_A(X) \to T_{\det A} S^1 = \mathbb{R}^1$.

Note that $T_A(X)$ is isomorphic to $\mathbb{R}^2$, since the parameters θ_1 and θ_2 are real numbers. (You can also check this with any local parametrization of X at some point $A \in X'$.) Just to be explicit, I will find two curves $c_1(t)$ and $c_2(t)$ in X whose derivatives at $t = 0$ will form a basis for the tangent space at some $A \in X'$. For the sake of clarity, write an element of X as $(e^{i \ \ _1}, e^{i \ \ _2})$ instead of the matrix form. Let

$$c_1(t) = (e^{i(t+ \ \ )}, e^{i(t- \ \ )}), \quad c_2(t) = (e^{i(t+ \ \ )}, e^{i(-t- \ \ )}).$$

Notice that $c_1(t)$ and $c_2(t)$ are in X for all time. At $t = 0$, they pass through the point $A = (e^{i} \ , e^{-i} \ ) \in X'$, as desired. We differentiate at 0, and find that

$$c_1'(0) = (ie^{i} \ , ie^{-i} \ ) \quad c_2'(0) = (ie^{i} \ , -ie^{-i} \ ).$$

In particular, the (real) linear span of these two vectors is any **real** linear combination of these two linearly independent vectors. Now any element B in $T_A(X)$ is of the form

$$B = ac_1'(0) + bc_2'(0),$$

where a and b are real numbers. To be even more explicit,

$$B = ((a+b)ie^{i} \ , (a-b)ie^{-i} \ ) = \begin{pmatrix} (a+b)ie^{i} & 0 \\ 0 & (a-b)ie^{-i} \end{pmatrix}.$$

Now that we have identified $T_A(X)$ with a copy of $\mathbb{R}^2$, it makes sense to take the derivative $df_A(B)$ in the usual way. Note that f is defined on the set of all 2×2 matrices with complex entries, so we can evaluate expressions such as $f(A + tB)$, even if $A + tB$ does not live in the copy of $\mathbb{R}^2$ that we just identified. The sum is occurring in the space $\mathbb{C}^4$ identified with $M(2, \mathbb{C})$, in which both A and B live.

For any $B \in T_A(X)$,

$$df_A(B) = \lim_{t \to 0} \frac{f(A + tB) - f(A)}{t} = \lim_{t \to 0} \frac{\det(A + tB) - 1}{t}.$$

We calculate the determinant of $A + tB$ directly, where $A = (e^{i} \ , e^{-i} \ )$. We obtain

$$\begin{aligned} f(A + tB) &= f((e^{i} \ + t(a+b)ie^{i} \ , e^{-i} \ + t(a-b)ie^{-i} \ )) \\ &= \det \begin{pmatrix} e^{i} \ + t(a+b)ie^{i} & 0 \\ 0 & e^{-i} \ + t(a-b)ie^{-i} \end{pmatrix} \\ &= 1 + t(a+b)i + t(a-b)i + t^2(a^2 - b^2). \end{aligned}$$

We now finish the calculation:

$$df_A(B) = \lim_{t \to 0} \frac{1 + t(a+b)i + t(a-b)i + t^2(a^2 - b^2) - 1}{t} = (a+b)i + (a-b)i = 2ai.$$

This is clearly nonzero if $a \neq 0$. Since the target space is one-dimension, this implies that df_A is surjective for all points $A \in X'$. It follows that 1 is a regular value, and that X' is therefore a submanifold of X.